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Electron in the classical external field of a plane electromagnetic wave: High-energy limits

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It is shown that Volkov's exact solution of the Dirac equation for an electron in the classical external field of an arbitrary plane electromagnetic wave can be cast, by means of an exact Mendlowitz-type unitary transformation, in a form in which positive- and negative-helicity states are separately described by two-component spinors. High-energy limits are then obtained by a mere expansion.

I. INTRODUCTION

It has been known for a long time that the problem of an electron in the external field of a plane electromagnetic wave can be solved exactly, both classically and quantum mechanically. The problem of finding the exact solution of the Lorentz-force equation was solved by Frenkel[1] in 1925, and the exact solution of Dirac's equation was first obtained by Volkov[2] in 1935. The advent of powerful laser sources has renewed interest in this old problem. As an unfocussed laser beam can be represented as the classical field of a plane electromagnetic wave to a good approximation, many authors have investigated the scattering of an intense laser radiation by free electrons within the framework of classical electrodynamics.

For the same reason, those quantum effects that occur in the interaction of free or weakly bound electrons with the beam emanating from a laser are generally calculated within the framework of the first-quantized theory. Interesting new results have been obtained in this context. A generalization of the Klein-Nishina cross section of the Compton effect and the cross section of the stimulated and inverse bremsstrahlung multiphoton processes have been derived within the semiclassical approach using the Volkov-state technique.[3]

Both processes, which are of interest for laser-fusion schemes, exhibit an explicit dependence on the external-field intensity. However, it should be borne in mind that they have been calculated assuming the light as a monochromatic wave (or a square pulse) which is, in principle, incompatible with scattering boundary conditions since such a wave is of infinite extent in space and time. In fact, it is not certain that these results would not be drastically altered by taking explicitly into account the switching on and switching off of the light, that is, by assuming a plane-wave packet rather than a monochromatic wave in order to satisfy asymptotic boundary conditions.[4]

Actually, even with plane-wave packets, the semiclassical external-field theory is not free of difficulties owing to the time dependence of the fields. This is clearly apparent when one wishes to get the second-order nonrelativistic approximation of the Volkov solution.[5] Owing to time dependence, the Foldy-Wouthuysen transformation is no longer unitary, and the more the fields vary quickly, the more the results differ from the correct results. Hence, when dealing with relativistic corrections, the semiclassical external-field theory seems free of ambiguities only for slowly varying fields.

In this paper, we are interested in the opposite limit of Volkov states, that is, the high-energy approximation where the rest energy of the particle is small with respect to its kinetic energy. This problem is relevant not only to the interaction between a high-energy electron and the field of an arbitrary plane electromagnetic wave, but also to an arbitrary constant and uniform field since for an ultrarelativistic particle any constant and uniform electromagnetic field is equivalent to a crossed field[6] (a field where the electric and magnetic fields $\vec{E}$ and $\vec{H}$ are constant and such that $\vec{E}^2-\vec{H}^2=0$, $\vec{E}\cdot\vec{H}=0$). Now, a crossed field is a special case of a plane wave. If $A(n\cdot x)$ and n denote the potential and wave vector ($n^2=0$), respectively, of an arbitrary plane electromagnetic wave,[7] the four-vector $A^\mu=j^\mu\, n\cdot x$ (where j^μ is such that $j\cdot n=0$) can represent the potential of a crossed field. Thus, the high-energy limit of the Volkov solution will also represent the state of an ultrarelativistic electron interacting with an arbitrary constant and uniform electromagnetic field by means of a particular choice of the potential. Furthermore, as the intensity of light is relative to a Lorentz frame, this problem is also relevant to that of an electron with a moderate initial velocity interacting with an ultrastrong light beam.

To derive the high-energy approximation, we encounter a situation somewhat similar to the nonrelativistic case. In the last case, we had to find a change of representation by means of which a Volkov state is *exactly* transformed into a two-component spin state. The nonrelativistic limits were then obtained by a mere expansion to the desired order in $1/m$. Now, as an ultrarelativistic particle is essentially in a helicity state, our present task is thus to derive a unitary transformation which will transform *exactly* a given Volkov state ψ_p describing the state of an electron of given initial four-momentum p^μ in the field of a plane wave, into two two-component states describing separately positive- and negative-helicity states of the interacting particle. The relativistic limits of ψ_p will then be obtained by a mere expansion to the desired order in $m/|\vec{p}|$. In order to derive the sought-after transformation, we find it necessary to summarize some results previously obtained in this context.

II. BEHAVIOR OF AN ELECTRON IN THE EXTERNAL FIELD OF A PLANE ELECTROMAGNETIC WAVE

In this problem an essential part is played by the operator[8, 9]

$$M^\mu{}_\rho = \delta^\mu_\rho + \nu(n^\mu j^\rho - j^\mu n^\rho) + \tfrac{1}{2}\nu^2 n^\mu n_\rho \ , \tag{2.1}$$

where n^μ is the null vector of the plane wave and j^μ a four-vector such that $n\cdot j=0$ and $j^2=-1$. ν is an arbitrary parameter.

The operator M exhibits remarkable properties. Let us summarize some of them. Since

$$M_\mu{}^\rho g_{\rho\sigma} M^\sigma{}_\alpha = g_{\mu\alpha} \ , \tag{2.2}$$

where $g^{\mu\nu}$ is the metric tensor ($g^{00}=-g^{11}=-g^{22}=-g^{33}=1$), M is the matrix of a Lorentz transformation. In fact, this Lorentz transformation leaves the tensor of the plane-wave field

$$F^{\mu\rho} = n^\mu \frac{dA^\rho}{d(n\cdot x)} - n^\rho \frac{dA^\mu}{d(n\cdot x)}$$

unaltered,[9] since

$$M^\mu{}_\rho M^\alpha{}_\sigma F^{\rho\sigma} = F^{\mu\alpha} \ . \tag{2.3}$$

In particular, this is a *Lorentz transformation which gives no Doppler shift* since, if $k^\mu = \omega n^\mu$ is the wave vector of a monochromatic wave of frequency ω, we have

$$M^\mu{}_\rho k^\rho = k^\mu \ . \tag{2.4}$$

If we replace in Eq. (2.1) the constant parameter ν by the space-time-dependent function

$$\nu = -(e/n\cdot p)(-A^2)^{1/2} \tag{2.5}$$

and the four-vector j^μ by

$$j^\mu = (-A^2)^{-1/2} A^\mu \tag{2.6}$$

one obtains a Lorentz-type matrix $M^\mu{}_\rho(p, A)$ which remains an operator of symmetry for the free field. In addition, it describes now the motion of an electron with initial four-momentum p in the plane-wave field. The vector

$$p'^\mu = M^\mu{}_\alpha(p, A)p^\alpha = p^\mu + eA^\mu - \left(\frac{eA\cdot p}{n\cdot p} + \frac{e^2A^2}{2n\cdot p}\right) n^\mu \tag{2.7}$$

is the exact solution of the Lorentz force equation

$$\frac{dp'^\mu}{d\tau} = -(e/m) F^{\mu\alpha} p'_\alpha \ , \tag{2.8}$$

where τ denotes the proper time.

Furthermore, if $\vec{\rho}_0$ denotes the initial electron spin vector and $\vec{\rho}$ the spin vector of the interacting electron, it can be shown[8, 9] that the matrix $M(p, A)$ gives also the motion of the electron spin ($g=2$) as described by the Thomas-Bargmann-Michel-Telegdi equation[10]:

$$\begin{aligned}\frac{d\vec{\rho}}{dt} = &-(e/2m)[g-2+2(m/p'^0)]\vec{\rho}\times\vec{H} \\ &-(e/2m)[(g-2)p'^0(p'^0+m)^{-1}](\vec{v}'\cdot\vec{H})\vec{v}'\times\vec{\rho} \\ &-(e/2m)[g-2p'^0(p'^0+m)^{-1}]\vec{\rho}\times(\vec{E}\times\vec{v}') \ ,\end{aligned} \tag{2.9}$$

where $\vec{v}'$ is the velocity of the interacting electron. To this end, the Lorentz-type operator M is decomposed into a pure Lorentz boost L which is such that

$$L^\mu{}_\alpha p^\alpha = p'^\mu \tag{2.10}$$

and a pure rotation R in the three-dimensional space. The three-vector $\vec{\rho} = R\vec{\rho}_0$ is the exact solution[8, 9] of Eq. (2.9) when $g=2$.

The matrix M can also be generalized[8] to the quantum case by taking

$$\hat{\nu} = -(e/in\cdot\partial)(-A^2)^{1/2} \ . \tag{2.11}$$

As $n\cdot\partial$ commutes with $n\cdot x$ and, therefore, also with A^μ, all properties of the symmetry mentioned above are valid even in the quantum case. A remarkable fact is that the quantum Lorentz-type operator $\hat{M}$ can now be used to transform the field-free Dirac equation

$$(i\gamma\cdot\partial - m)\varphi = 0 \tag{2.12}$$

into the Dirac equation with field of a plane electromagnetic wave

$$[\gamma_\mu(i\partial^\mu + eA^\mu) - m]\psi = 0 \tag{2.13}$$

giving the general solution ψ in the form

$$\psi = \hat{U}\varphi \ , \tag{2.14}$$

where $\hat{U}$ is a metric-unitary operator which is unequivocally determined by means of the quantum operator $\hat{M}$. When the initial state φ is a free-electron state φ_p of given four-momentum p^μ, $\hat{U}$ becomes diagonal and the Volkov solution ψ_p is obtained as

$$\psi_p = U_p \varphi_p = T_p e^{-ie\lambda_p}\varphi_p \ , \qquad (2.15)$$

where

$$\lambda_p = -(n\cdot p)^{-1} \times \int_{-\infty}^{n\cdot x} [p\cdot A(n\cdot x) + \tfrac{1}{2}eA^2(n\cdot x)]\, d(n\cdot x) \qquad (2.16)$$

where

$$T_p(n\cdot x) = 1 - (e/2n\cdot p)\gamma\cdot n\gamma\cdot A \ . \qquad (2.17)$$

The spinor T_p is simply the spinor image of the classical Lorentz-type operator $M(p,A)$, that is,

$$T_p^{-1}\gamma^\mu T_p = M^\mu{}_\alpha(p,A)\gamma^\alpha \ , \quad \gamma^0 T_p^\dagger \gamma^0 = T_p^{-1} \ . \qquad (2.18)$$

The classical character of the Volkov solution ψ_p is best shown in the following fashion. Let us take $\varphi = \varphi_p$ in Eq. (2.12). Obviously, one obtains

$$(\gamma\cdot p - m)\varphi_p = 0 \ . \qquad (2.19)$$

As $M(p,A)$ is a Lorentz-type operator, Eq. (2.19) can also be written in the form

$$(\gamma'\cdot p' - m)\varphi_p = 0 \ , \qquad (2.20)$$

where p'^μ is given by Eq. (2.7) and

$$\gamma'_\mu = M_\mu{}^\beta(p,A)\gamma_\beta \ . \qquad (2.21)$$

Then using Eqs. (2.15)–(2.18) we see that the Volkov solution ψ_p satisfies the following algebraic equation:

$$(\gamma\cdot p' - m)\psi_p = 0 \qquad (2.22)$$

The strong resemblance between Eqs. (2.19) and (2.22) shows that the Volkov solution describes an electron which is moving in a plane-wave field according to the Lorentz force equation, while its spin precesses according to the Thomas-Bargmann-Michel-Telegdi equation.[10] In other words, in the frame transformed from the initial frame by means of the "variable" Lorentz transformation $M^{-1}(p,A)$, the electron is merely described by the field-free Dirac equation (in the external-field approximation) while the plane-wave field is left unaltered by this change of frame.

It is the physical meaning of the Furry-type change of representation $\psi' = \hat{U}\psi$ introduced in Ref. 8. When all relativistic and spin effects are neglected, that is, when the assumptions about the frequency and strength of the wave are no more than the conditions of validity of the Schrödinger equation, this method is called in the literature the "space translation method" and was introduced by Henneberger[11] as a useful tool to calculate multiphoton processes that occur in a laser beam.

III. VOLKOV HELICITY STATES

As is well known, the field-free Dirac equation can be cast into a form well adapted for discussing the nonrelativistic limits, by means of a unitary transformation called the Foldy-Wouthuysen[12,13] transformation. The physical meaning of this transformation is that an electron state of given four-momentum p^μ is transformed into a two-component spin state.

There exists also a unitary transformation of the field-free Dirac equation which leads to a form appropriate for obtaining the high-energy limits. This transformation was first discovered by Mendlowitz[14] and discussed by Cini and Touschek[15] and by Bose, Gamba, and Sudarshan.[16] In fact, the Mendlowitz and Foldy-Wouthuysen transformations can be shown to be special cases of a general class of unitary transformations.[17]

As for the Foldy-Wouthuysen transformation, the Mendlowitz transformation has an interesting physical meaning in that positive- and negative-helicity states are separately described by two-component wave functions. The use of the Foldy-Wouthuysen and Mendlowitz transformations is often considered as a transition to the Pauli and Weyl equations, respectively.

The nonrelativistic limits of the Volkov problem were considered in a previous paper.[5] In particular, it was shown that the Foldy-Wouthuysen method yields incorrect results since the transformation is not unitary within the Volkov-state space with scattering boundary conditions. The correct nonrelativistic limits were then obtained via a closed-form unitary (but noncanonical) transformation. It was shown that the unitarily equivalent nonrelativistic limit at order $1/m^2$ of the Volkov state ψ_p is a solution of the generalized Pauli equation (obtained via the traditional Foldy-Wouthuysen method) modified by the additional anti-Hermitian term $-i(e/8m^2)\vec{\sigma}\cdot\partial\vec{H}/\partial t$.

In the same way, it is possible to derive a closed-form Mendlowitz-type unitary transformation which will transform the Volkov solution ψ_p of Eq. (2.13) into two-component spinors describing separately positive- and negative-helicity states of an electron in the field of a plane electromagnetic wave of arbitrary shape.

Our starting point is the algebraic equation (2.22) which we rewrite in the form

$$p'^0\psi_p = (\gamma^0\vec{\gamma}\cdot\vec{p}' + \gamma^0 m)\psi_p \ . \qquad (3.1)$$

Equations (3.1) [or (2.22)] and (2.7) involve the relation

$$p'^2 = p^2 = m^2 , \tag{3.2}$$

which tells us that the four-vectors p^μ and p'^μ are, respectively, the four-momentum of the free electron and of the particle in the field. Decoupling is thus assumed. In other words, Eq. (3.1) is a direct consequence of the asymptotic boundary condition $A \to 0$ when $n \cdot x \to \pm\infty$.

The familiar form of Eq. (3.1) suggests to us the Mendlowitz-type unitary transformation

$$\psi_p \to \psi'_p = C_{p'}\psi_p , \tag{3.3}$$

where

$$C_{p'} = \exp\left(-\frac{1}{2}\frac{\vec{\gamma}\cdot\vec{p}\,'}{|\vec{p}\,'|}\tan^{-1}(m/|\vec{p}\,'|)\right) . \tag{3.4}$$

We have thus, taking into account Eqs. (3.1) and (3.2),

$$\begin{aligned}\psi'_p &= \left(\cos[\tfrac{1}{2}\tan^{-1}(m/|\vec{p}\,'|)] - \sin[\tfrac{1}{2}\tan^{-1}(m/|\vec{p}\,'|)]\frac{\vec{\gamma}\cdot\vec{p}\,'}{|\vec{p}\,'|}\right)\psi_p \\ &= [2p'^0(p'^0 + |\vec{p}\,'|)]^{-1/2}\left(p'^0 + |\vec{p}\,'| - m\frac{\vec{\gamma}\cdot\vec{p}\,'}{|\vec{p}\,'|}\right)\psi_p \\ &= \frac{p'^0}{|\vec{p}\,'|}[2p'^0(p'^0 + |\vec{p}\,'|)]^{-1/2}(p'^0 + |\vec{p}\,'| - m\gamma^0)\psi_p .\end{aligned} \tag{3.5}$$

In explicit form, using Eq. (2.15), one has

$$\psi'_p = (p'^0/2p^0|\vec{p}\,'|)[p^0p'^0(p'^0 + |\vec{p}\,'|)(p^0+m)]^{-1/2}e^{-ip\cdot r}e^{-ie\lambda_p} \times \begin{pmatrix}(p'^0 + |\vec{p}\,'| - m)\{(p^0+m)[1 + i(e/2n\cdot p)\vec{\sigma}\cdot\vec{n}\times\vec{A}\,] + (e/2n\cdot p)\vec{\sigma}\cdot\vec{A}\vec{\sigma}\cdot\vec{p}\}W \\ (p'^0 + |\vec{p}\,'| + m)\{[1 + i(e/2n\cdot p)\vec{\sigma}\cdot\vec{n}\times\vec{A}]\vec{\sigma}\cdot\vec{p} + (p^0+m)(e/2n\cdot p)\vec{\sigma}\cdot\vec{A}\}W\end{pmatrix} , \tag{3.6}$$

where W is a constant spinor such that $W^*W = 1$.

Let us now consider the matrix

$$S_p = [(p^0+m)(p'^0+m)]^{-1/2}\{(p^0+m)[1 + i(e/2n\cdot p)\vec{\sigma}\cdot\vec{n}\times\vec{A}] + (e/2n\cdot p)\vec{\sigma}\cdot\vec{A}\;\vec{\sigma}\cdot\vec{p}\} , \tag{3.7}$$

which is unitary since

$$S_p^\dagger S_p = S_p S_p^\dagger = 1 . \tag{3.8}$$

In Ref. 5, it was shown that the motion of electron spin in the field of a plane electromagnetic wave is described by the quaternionic equation

$$\vec{\sigma}\cdot\vec{\rho} = S_p\vec{\sigma}\cdot\vec{\rho}_0 S_p^{-1} \tag{3.9}$$

where $\vec{\rho}_0$ and $\vec{\rho}$ denote, respectively, the spin vectors of the field-free electron and of the interacting particle. As we are looking for helicity electron states, we may expect the S_p matrix to play a role in the explicit form of the state ψ'_p.

If we use Eqs. (3.7) and (3.8) and some trivial identities such as

$$\begin{aligned}&(\vec{n}\times\vec{A})\cdot\vec{p}\,\vec{\sigma}\cdot\vec{n}\times\vec{A} = \vec{A}^2\vec{\sigma}\cdot\vec{p} - \vec{A}^2\vec{n}\cdot\vec{p}\,\vec{\sigma}\cdot\vec{n} - \vec{A}\cdot\vec{p}\,\vec{\sigma}\cdot\vec{A} , \\ &(\vec{n}\times\vec{A})\cdot\vec{p}\,\vec{\sigma}\cdot(\vec{A}\times\vec{p}) = \vec{A}^2\vec{p}^2\,\vec{\sigma}\cdot\vec{n} - (\vec{A}\cdot\vec{p})^2\vec{\sigma}\cdot\vec{n} + \vec{A}\cdot\vec{p}\,\vec{n}\cdot\vec{p}\,\vec{\sigma}\cdot\vec{A} - \vec{n}\cdot\vec{p}\,\vec{A}^2\vec{\sigma}\cdot\vec{p} , \\ &(p'^0 + |\vec{p}\,'| - m)(p'^0+m) = |\vec{p}\,'|(p'^0 + |\vec{p}\,'| + m) , \\ &p'^0 + |\vec{p}\,'| + m = [2(p'^0 + |\vec{p}\,'|)(p'^0+m)]^{1/2} ,\end{aligned}$$

and if we choose now the Kramers representation[18] which is obtained from the standard representation by means of the unitary matrix

$$K = \frac{1}{\sqrt{2}}(1 + \gamma^5\gamma^0) , \tag{3.10}$$

the state ψ'_p given by Eq. (3.6) takes the form

$$\psi'_p = \psi'_{p,+} + \psi'_{p,-} \tag{3.11}$$

where

$$\begin{aligned}\psi'_{p,+} &= \frac{1}{\sqrt{2}}(1 + \gamma^5\gamma^0)\tfrac{1}{2}(1 - \gamma^5)\psi'_p \\ &= \tfrac{1}{2}(p'^0/p^0)^{1/2}e^{-ip\cdot r}e^{-ie\lambda_p}\left(1 + \frac{\vec{\sigma}\cdot\vec{p}\,'}{|\vec{p}\,'|}\right)S_p\begin{pmatrix}W\\0\end{pmatrix}\end{aligned} \tag{3.12}$$

and

$$\psi'_{p,-} = \frac{1}{\sqrt{2}}(1+\gamma^5\gamma^0)\tfrac{1}{2}(1+\gamma^5)\psi'_p$$

$$= \tfrac{1}{2}(p'^0/p^0)^{1/2} e^{-ip\cdot r} e^{-ie\lambda_p}\left(-1+\frac{\vec{\sigma}\cdot\vec{p}'}{|\vec{p}'|}\right) S_p \begin{pmatrix} 0 \\ W \end{pmatrix} \quad (3.13)$$

Hence, we have

$$\frac{\vec{\sigma}\cdot\vec{p}'}{|\vec{p}'|}\psi'_{p,+} = \psi'_{p,+}, \quad (3.14)$$

$$\frac{\vec{\sigma}\cdot\vec{p}'}{|\vec{p}'|}\psi'_{p,-} = -\psi'_{p,-}. \quad (3.15)$$

The two-component wave functions $\psi'_{p,+}$ and $\psi'_{p,-}$ describe, respectively, the positive- and negative-helicity states of the electron in the plane-wave field. It is thus justified to call them "Volkov helicity states."

As expected, when the field is switched off, these states become identical to those states φ'_p obtained from the free electron state φ_p by using the diagonal form of the Mendlowitz transformation[14]:

$$\varphi'_p = C_p \varphi_p, \quad (3.16)$$

where

$$C_p = \exp\left(-\frac{1}{2}\frac{\vec{\gamma}\cdot\vec{p}}{|\vec{p}|}\tan^{-1}(m/|\vec{p}|)\right). \quad (3.17)$$

This allows us to get a description of the electron helicity motion in a particular simple form.

To this end, we look for an operator U'_p which will be such that

$$\psi'_p = U'_p \varphi'_p. \quad (3.18)$$

Using Eqs. (2.15), (3.3), and (3.13) we have

$$\psi'_p = C_{p'}\psi_p = C_{p'}U_p\varphi_p = C_{p'}U_pC_p^{-1}\varphi'_p, \quad (3.19)$$

hence

$$U'_p = C_{p'}U_pC_p^{-1}. \quad (3.20)$$

Thus, as φ'_p and ψ'_p are, respectively, eigenvectors of the helicity operators $\vec{\sigma}\cdot\vec{p}/|\vec{p}|$ and $\vec{\sigma}\cdot\vec{p}'/|\vec{p}'|$, $|\vec{p}'|$, the behavior of the helicity of the electron in the field of a plane wave is therefore described by the relation

$$\frac{\vec{\sigma}\cdot\vec{p}'}{|\vec{p}'|} = U'_p\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}U'^{-1}_p. \quad (3.21)$$

IV. HIGH-ENERGY LIMITS

As the Volkov solution is now transformed into two two-component helicity states, high-energy limits can be obtained via a mere expansion in $|\vec{p}|^{-1}$. We confine ourself to the extreme relativistic limit where the rest mass of the free particle can be neglected with respect to its kinetic energy.

In this case we have $|\vec{p}| \simeq p^0$ and, owing to (3.2), we also have[19] $|\vec{p}'| \simeq p'^0$. It is then easy to verify that Eq. (3.21) reduces to

$$\frac{\vec{\sigma}\cdot\vec{p}'}{|\vec{p}'|} = S_p\frac{\vec{\sigma}\cdot\vec{p}'}{|\vec{p}|}S_p^{-1}, \quad (4.1)$$

where S_p is given by (3.7). This is to be compared with the relation obtained in Ref. 5:

$$\vec{\sigma}\cdot\vec{p} = S_p\vec{\sigma}\cdot\vec{p}_0S_p^{-1}, \quad (4.2)$$

which describes the behavior of the electron spin vector in the plane-wave field. As shown in Ref. 5, Eq. (4.2) is the exact solution of the Thomas-Bargmann-Michel-Telegdi (TBMT) equation. Hence, the helicity of an ultrarelativistic electron evolves in the field of a plane wave according to the TBMT equation.[10]

In this limit, the positive- and negative-helicity states $\psi'_{p,+}$ and $\psi'_{p,-}$ given by Eqs. (3.12) and (3.13) become

$$\psi'^E_{p,+} = \left(1+\frac{ie}{2n\cdot p}\vec{\sigma}\cdot\vec{n}\times\vec{A}+\frac{e\vec{\sigma}\cdot\vec{A}}{2n\cdot p}\right) \times e^{-ie\lambda_p}e^{-ip\cdot r}\frac{1}{2}\left(1+\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\right)W, \quad (4.3)$$

$$\psi'^E_{p,-} = \left(1+\frac{ie}{2n\cdot p}\vec{\sigma}\cdot\vec{n}\times\vec{A}-\frac{e\vec{\sigma}\cdot\vec{A}}{2n\cdot p}\right) \times e^{-ie\lambda_p}e^{-ip\cdot r}\frac{1}{2}\left(-1+\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\right)W. \quad (4.4)$$

As expected, $\psi'^E_{p,+}$ and $\psi'^E_{p,-}$ are now exact solutions of the Weyl-type equations[18] where minimum coupling is applied:

$$i\frac{\partial}{\partial t}\psi'^E_{p,+} = \vec{\sigma}\cdot(-i\vec{\nabla}+e\vec{A})\psi'^E_{p,+}, \quad (4.5)$$

$$i\frac{\partial}{\partial t}\psi'^E_{p,-} = -\vec{\sigma}\cdot(-i\vec{\nabla}+e\vec{A})\psi'^E_{p,-}, \quad (4.6)$$

or of the second-order equation:

$$\left(i\frac{\partial}{\partial t}+\vec{\sigma}\cdot(-i\vec{\nabla}+e\vec{A})\right)\left(i\frac{\partial}{\partial t}-\vec{\sigma}\cdot(i\vec{\nabla}+e\vec{A})\right)\psi'^E_p = 0,$$

that is,

$$\left(i\frac{\partial}{\partial t}\right)^2\psi'^E_p = [(-i\vec{\nabla}+e\vec{A})^2+e\vec{\sigma}\cdot(\vec{H}+i\vec{E})]\psi'^E_p, \quad (4.7)$$

which is, in fact, the Feynman-Gell-Mann equation[20] where the mass term has been neglected.

V. CONCLUSION

Assuming asymptotic boundary conditions, Volkov's exact solution of Dirac's equation with field of an arbitrary plane electromagnetic wave satisfies an algebraic equation formally analogous to the matrix equation satisfied by the corresponding field-free solution. We have used this property to derive an exact Mendlowitz-type unitary transformation which casts the Volkov solution into two two-component helicity states. High-energy limits can then be obtained by merely expanding these helicity states to the desired order in $m/|\vec{p}|$. At the extreme relativistic limit, these states become exact solutions of a Weyl-type equation. Incidentally, let us remark that divergent terms seem to occur at this limit if $\vec{n}\cdot\vec{p}=|\vec{p}|$ (the electron initial velocity and wave vector are collinear). However, it can be observed that, in this configuration, the extreme relativistic limit cannot be carried out owing to the asymptotic condition. Actually, an interaction between a plane electromagnetic wave and an electron moving with the velocity of light in the same direction cannot occur if the particle was free in the remote past. The difficulty appears, however, when periodic boundary conditions are substituted for the asymptotic conditions. This is another example of those difficulties which occur with monochromatic waves, in which case the electron is eternally coupled to the external field.

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[1]J. Frenkel, Z. Phys. 32, 27 (1925).

[2]D. M. Volkov, Z. Phys. 94, 250 (1935). See also V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Relativistic Quantum Theory* (Pergamon, Oxford, 1971), part 1, Sec. 40.

[3]A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. 46, 776 (1963) [Sov. Phys.—JETP 19, 529 (1964)]; I. I. Goldman, Phys. Lett. 8, 103 (1964); L. S. Brown and T. W. B. Kibble, Phys. Rev. 133, A705 (1964); H. Brehme, Phys. Rev. C 3, 837 (1971). A complete set of references up to 1966 can be found in the review paper by J. H. Eberly, in *Progress in Optics*, edited by E. Wolf (North-Holland, Amsterdam, 1968), Vol. VII, p. 361. More recent papers are cited in Ref. 8.

[4]R. A. Neville and F. Rohrlich, Phys. Rev. D 3, 1692 (1971); J. F. Dawson and Z. Fried, *ibid.* 1, 3363 (1970).

[5]J. Kupersztych, Phys. Rev. Lett. 42, 483 (1979).

[6]A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. 46, 1768 (1964) [Sov. Phys.—JETP 19, 1191 (1964)].

[7]Throughout the paper we take $\hbar=c=1$. $-e$ is the electron charge. For simplicity, the four-potential of the arbitrary plane electromagnetic wave is $A^{\mu}(n\cdot x) = (0,\vec{A})$, with $\vec{n}\cdot\vec{A}=0$. We have $n\cdot x=t-\vec{n}\cdot\vec{x}$ with $\vec{n}^2=1$. We take the γ matrices in the standard (Pauli) representation.

[8]J. Kupersztych, Phys. Rev. D 17, 629 (1978).

[9]J. Kupersztych, Nuovo Cimento 31B, 1 (1976).

[10]V. Bargmann, L. Michel, and V. L. Telegdi, Phys. Rev. Lett. 2, 435 (1959). See also H. C. Corben, *Classical and Quantum Theories of Spinning Particles* (Holden-Day, San Francisco, 1968).

[11]W. C. Henneberger, Phys. Rev. Lett. 21, 217 (1968).

[12]L. L. Foldy and S. A. Wouthuysen, Phys. Rev. 78, 29 (1950).

[13]J. D. Bjørken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), Chap. 4.

[14]H. Mendlowitz, Phys. Rev. 102, 527 (1956).

[15]M. Cini and B. Touschek, Nuovo Cimento 7, 422 (1958).

[16]S. K. Bose, A. Gamba, and E. C. G. Sudarshan, Phys. Rev. 113, 1661 (1959).

[17]W. Hanus, H. Janyszek, and A. Rakowski, Acta Phys. Pol. 34, 1071 (1968); H. A. Cohen, Am. J. Phys. 39, 409 (1971).

[18]See Ref. 13, Sect. 10.12.

[19]This result exhibits the satisfactory fact that the ultrarelativistic limit is independent from the explicit value of the field strength, in agreement with the above remark that the field intensity is relative to a Lorentz frame. Therefore, when the external field is switched off, the electron remains an ultrarelativistic particle.

[20]R. P. Feynman and M. Gell-Mann, Phys. Rev. 109, 193 (1958); L. M. Brown, *ibid.* 111, 957 (1958).