Monopole dissociation in the early Universe

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It has recently been shown that topological excitations of metastable phases such as non-Abelian monopoles can dissociate and decay through the radial expansion of the excitation core. In this paper it is indicated that, according to some grand unified field theories, the monopole dissociation process might occur in the early Universe. In particular, the case of an SU(5)-symmetric grand unified field theory which breaks spontaneously to SU(4)×U(1) symmetry and then to SU(3)×SU(2)×U(1) symmetry is studied, and it is demonstrated that monopoles that are created in the transition from SU(5) symmetry to SU(4)×U(1) symmetry generally dissociate *after* the SU(4)×U(1) phase becomes metastable with respect to the SU(3)×SU(2)×U(1) phase but *before* there is any significant spontaneous production of bubbles associated with barrier penetration from the SU(4)×U(1) phase to the SU(3)×SU(2)×U(1) phase. Thus, the monopole dissociation process does indeed occur in such models and the monopoles may act as seeds for the inhomogeneous phase transition from SU(4)×U(1) symmetry to SU(3)×SU(2)×U(1) symmetry. If the transition occurs inhomogeneously, the transition temperature is higher and less latent heat is released in the transition.

I. INTRODUCTION

If a system is homogeneous and isotropic, the decay of a metastable phase to a stable one is mediated by the spontaneous formation of bubbles of stable phase in a background of metastable phase. Once formed, the bubbles grow radially until they coalesce, completing the transition from the metastable phase to the stable phase. The energy released in the collision of the bubble walls is converted into the latent heat of the first-order transition. Because the probability of formation of the bubbles is uniform in space, the transition that occurs in this manner is referred to as a *homogeneous* transition in spite of the fact that different regions of space are converted to the stable phase at different times.

If a system contains impurities, the transition process may be significantly modified. If there is a greater probability of bubble formation near the impurities (for a given temperature), the transition may be completed by the bubbles induced by the impurities rather than those produced spontaneously. The presence of impurities therefore alters the temperature at which large numbers of bubbles may be formed and, consequently, alters the amount of supercooling and the amount of latent heat released in the process. Because the probability of bubble formation is not uniform in space in the case of impurities (it depends on the positions of the impurities), the process is referred to as an *inhomogeneous* transition.

In a recent paper,[1] henceforth referred to as Ref. 1, it was demonstrated that topological excitations, such as 't Hooft-Polyakov monopoles[2] in non-Abelian gauge theories or vortices in two-dimensional theories, can act as natural impurity sites in first-order phase transitions. The excitations act as impurity sites when their cores (for geometrical and topological reasons) are constrained to contain the stable phase of the system. If the phase which contains the excitations is "supercooled" sufficiently (as a metastable state), the excitations become unstable and the cores grow radially thereby converting the metastable phase to the stable phase contained in the growing cores. Because the cores of the excitations already contain stable phase, no activation energy is required to produce an initial bubble of stable phase and the monopole dissociation process usually dominates the bubble nucleation process in converting the metastable phase to the stable phase.

In this paper it is examined whether or not, assuming that grand unified field theories of elementary-particle interactions are correct, monopole dissociation phenomena may have occurred in the early history of the Universe. The particular case that is studied is that of the SU(5)-symmetric grand unified field theory of Georgi and Glashow[3] with a cubic interaction term for the adjoint scalar field. This theory undergoes a series of phase transitions as a function of temperature so that at high temperatures SU(5) symmetry is manifest, at a temperature of 10^{14-15} GeV (for a wide range of parameters) a phase transition takes place in which the symmetry is spontaneously broken to SU(4)× U(1) and the adjoint scalar field obtains a vacuum expectation value, and at somewhat lower temperatures a second phase transition takes place in which the symmetry is reduced to SU(3)×SU(2)×U(1). [At much lower temperatures, around 10^2 GeV or so, an additional phase transition takes place which reduces the symmetry to $SU(3)_{color}\times U(1)_{em}$, but this last transition will be ignored in this paper.]

In the transition from SU(5) symmetry to SU(4)×U(1) symmetry, many monopoles of the SU(4)×U(1) phase are expected to be created when bubbles of SU(4)×U(1) phase formed in the SU(5)-symmetric metastable phase coalesce; the expectation values of the scalar field are uncorrelated from bubble to bubble, and when the bubbles coalesce, bubble knots form which, after the completion of the transition, lead to SU(4) ×U(1) (4-1) monopoles.

In Sec. II, the most important ideas and conclusions of Ref. 1 are reviewed. In Sec. III, the SU(5) model that will be studied is introduced and a convenient parametrization scheme is proposed. The phases of the theory as a function of the parameters are examined and the temperature dependence of the theory is discussed. In order to determine whether monopole dissociation can occur, it is necessary to determine whether, as the temperature decreases, the monopole dissociates before bubbles are spontaneously formed and complete the transition. In Sec. IV, the spontaneous decay from the SU(4)×U(1) (4-1) phase to the SU(3)×SU(2)×U(1) (3-2-1) phase is studied using the semiclassical techniques developed by Coleman[4] for studying zero-temperature decay and extended for studying finite-temperature decay. The value of the temperature for which the decay occurs is determined as a function of the parameters of the theory. In Sec. V, the SU(4)×U(1) monopoles that are formed in the initial phase transition are studied, and an ansatz for the most stable monopole in the (4-1) phase is proposed. According to the ansatz, the core of the monopole has a value which coincidentally lies near an SU(3)×SU(2)×U(1) minimum. As the temperature decreases to a point where the SU(4)×U(1) phase is metastable compared to the SU(3)× SU(2)×U(1) stable phase, the monopoles become unstable and dissociate through radial expansion of the cores since (as discussed in Ref. 1) it is energetically favorable for the core to grow to create a greater region of (3-2-1) phase. Some general analytic considerations of the stability of the (4-1) monopole are also discussed in Sec. V. In Sec. VI, the numerical analysis of the stability of the (4-1) monopole is presented for a wide range of parameters. The results show that the monopoles do dissociate at a temperature which is greater than that necessary for bubble nucleation. In Sec. VII, the consequences of these results is discussed.

II. PRINCIPLES OF MONOPOLE DISSOCIATION

The circumstances that suggest that topological excitations, such as non-Abelian monopoles, may dissociate and act as seeds for a first-order phase transition can be best understood by studying the example (considered in Ref. 1 in detail) of an SO(3)-symmetric theory in 3+1 dimensions defined by the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \tfrac{1}{2}(D_\mu \phi^a)^2 - \lambda V(\phi^a\phi^a), \qquad (2.1)$$

where ϕ^a, $a=1, 2, 3$ is a scalar field in the vector representation of SO(3). For the case of a spontaneous-symmetry-breaking potential, such as

$$V(\phi^a\phi^a) = \beta(\phi^a\phi^a - a^2)^2 \qquad (2.2)$$

[see Fig. 1(a)], 't Hooft and Polyakov[2] showed that there can exist nonsingular, finite-energy configurations which have the property that there is a net magnetic flux emerging from a surface enclosing the core but of much larger radius than the core. The standard ansatz for describing such an excitation that is spherically symmetric

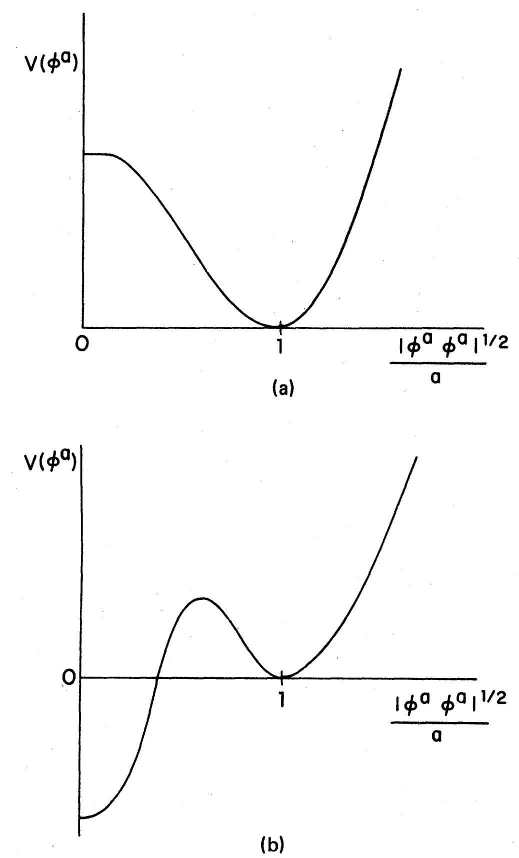

FIG. 1. The cross sections of two SO(3)-symmetric potentials for a scalar field in the vector representation ϕ^a: (a) the case of a stable spontaneous-symmetry-breaking potential; (b) the case of a metastable spontaneous-symmetry-breaking potential. In either case, the core of the monopole is forced to approach the value $|\phi| = 0$ as $r \to 0$.

and time independent is[5]

$$e\phi^a = \hat{x}_a h(r) a\,, \qquad A^a_\mu = \epsilon_{\mu ab}\hat{x}_b\left(\frac{1-K(r)}{er}\right), \tag{2.3}$$

where e is the gauge coupling constant, $\hat{x}_i$ is a unit vector in space, and $h(r)$ and $K(r)$ are functions of a radial coordinate only. The energy functional may be reexpressed in terms of the radial functions

$$E = \frac{4\pi}{e^2} M_W \int_0^\infty dr \left[(K')^2 + \frac{1}{2}\,\frac{(1-K^2)}{r^2} + \tfrac{1}{2} r^2 (h')^2 + h^2K^2 + \lambda V(h)\right]; \tag{2.4}$$

the first two terms are associated with the kinetic energy of the gauge field T_A; the second two terms are associated with the kinetic energy of the scalar field T_ϕ; and the last term is associated with the potential energy of the scalar field V_ϕ. In order for the solution to be of finite energy, it is necessary that

$$h(r) \to 1, \quad K(r) \to 0 \quad \text{as } r \to \infty\,. \tag{2.5}$$

In order for the solution to be nonsingular at short distances it is necessary that

$$h(r) \to 0, \quad K(r) \to 1 \quad \text{as } r \to 0\,. \tag{2.6}$$

Thus, the core is at an SO(3) symmetry point even though the monopole is an excitation of an SO(3)-symmetry-breaking potential.

If $V(h)$ is positive semidefinite, as is the case in Fig. 1(a), all the terms in the expression for E are positive semidefinite and E is bounded below by zero. The monopole solution always exists and represents the minimum of the energy functional for functions obeying the boundary conditions. The limit $\lambda \to 0$ is of interest in this case because (1) the exact solution for $h(r)$ and $K(r)$ are known (they were found by Prasad and Sommerfield[6] and hence this solution is referred to as the Prasad-Sommerfield monopole in this paper), (2) the monopole in this limit can be shown to be stable, and (3) the mass of the monopole in this limit is a lower bound on possible masses for monopoles for $\lambda > 0$ when V is positive definite and, furthermore, the mass in this limit is a lower bound on $T_A + T_\phi$ for all $h(r)$ and $K(r)$ that obey the required boundary conditions.

The case of interest is when the symmetry-breaking phase is metastable rather than stable, and so $V(\phi)$ is not positive semidefinite, as shown in Fig. 1(b):

$$V(\phi) = \beta\phi^a\phi^a(\phi^a\phi^a - a^2)^2 + \delta\phi^2 + c\,, \tag{2.7}$$

where c is chosen so that V for the metastable minimum is zero. There is no simple argument for this case that shows that a monopole solution exists. However, assuming such a solution does exist, one can employ the same ansatz, Eq. (2.3), as before and in order for the solution to be of finite energy and nonsingular, $h(r)$ and $K(r)$ are forced to obey the same boundary conditions. As a result, the core of the monopole near $h(r) = 0$ is forced to lie in the stable phase, while at large distances from the core the monopole solution has $h(r) = 1$ in the metastable phase. As the difference in the energy density between the metastable phase and the stable phase ϵ is increased, the monopole can gain energy by allowing the core to grow radially leading to a larger volume of stable phase in the core. As was shown in Ref. 1, for ϵ smaller than some finite critical value ϵ_c, the monopole remains stable because the gain in energy from the growth of the core ($-\epsilon R^3$) is compensated by a loss in energy associated with the core wall (σR^2); for $\epsilon \geq \epsilon_c$, the core wall energy is not sufficient to stabilize the solution and if ϵ is varied from $\epsilon < \epsilon_c$ to $\epsilon > \epsilon_c$, the previously stable monopole becomes unstable and the core grows radially indefinitely. As it grows, an increasingly greater volume of stable phase is created. If there is a sufficient density of such monopoles, the cores will coalesce and convert the total system to the stable phase.

Thus, there exists a region of values $\epsilon_c > \epsilon > 0$ for which the system is metastable and the monopoles are a stable excitation of this phase. Because a region of stable phase automatically lies in the core, there is no activation energy associated with this kind of "nucleation process," and the decay of the monopole being considered is purely classical. Therefore, if one compares this process with the process of bubble nucleation triggered by quantum fluctuations of the vacuum, one expects monopole dissociation to occur at a value $\epsilon = \epsilon_c$ that is too small to have any significant quantum bubble nucleation; this was in fact established in Ref. 1. One can also compare the efficiency of monopole dissociation and nucleation with the thermal production of bubbles and, even though the absence of an activation energy in the case of monopole dissociation suggests that the first process should *dominate* (that is, as ϵ increases, it occurs and completes the transition before there is any significant amount of bubble production), the argument is not as compelling and a detailed dynamical calculation is required. One of the goals of this paper is to show the results of such calculations for a physically interesting case of phase transitions of grand unified field theories in the conditions expected to be found in early cosmology.

III. THE SU(5) MODEL

To test whether monopole dissociation might have occurred in the early Universe, a particular model, the SU(5) model of Georgi and Glashow[3] in which the grand unified symmetry is broken by a scalar field ϕ in the adjoint representation, will be studied. The phase diagram for the model has been studied independently by Guth and Weinberg,[7] who, in turn, relied on the results of Li,[8] and so this section will be appropriately abbreviated. However, because the parametrization to be used in this paper must be somewhat different from that used in the other papers, it is necessary to elaborate to some degree.

The model that will be considered is described by a Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \tfrac{1}{2}(D_\mu\phi^a)^2 - V(\phi)\,, \qquad D_\mu = \partial_\mu + gA_\mu x\,, \tag{3.1}$$

where $V(\phi)$ is the most general renormalizable potential describing the self-interaction of the adjoint-representation Higgs field (the fundamental Higgs field plays no role at the high temperatures to be considered in this paper and will be ignored in this analysis)

$$V(\phi) = -\frac{\mu^2}{2}\,\mathrm{Tr}\phi^2 + \frac{a}{4}(\mathrm{Tr}\phi^2)^2 + \frac{b}{2}\mathrm{Tr}\phi^4 + \frac{c}{3}\mathrm{Tr}\phi^3\,. \tag{3.2}$$

The interesting range of parameters will correspond to $a > b$ and the calculations will be done using the tree approximation to the potential. In order for the approximation to be feasible, it is necessary that the parameters of the theory be small compared to unity for perturbation theory to be valid and yet large compared to $(g^2/4\pi)^2$ so that the higher-order loop corrections can be neglected, i.e.,

$$1 \gg a > b \gg (g^2/4\pi)^2\,, \qquad 10^{-1} > a, b > 10^{-3}\,. \tag{3.3}$$

In order to most easily compare results for monopole dissociation with those for bubble nucleation, it is useful to reparametrize the theory in terms of dimensionless variables:

$$\phi' = \frac{b}{c}\,\phi\,, \quad \beta = \frac{\mu^2 b}{c}\,, \qquad A'^a_\mu = gA^a_\mu\,, \quad \gamma = \frac{a}{b} + \frac{7}{15}\,. \tag{3.4}$$

The energy for the system can then be expressed (in the $A^a_0 = 0$ gauge) as

$$E = \frac{1}{g^2}\int d^3r\left[\tfrac{1}{2}(F^a_{0i})^2 + \tfrac{1}{2}(F^a_{ij})^2 + \frac{1}{2}\,\frac{c^2g^2}{b^2}\,(D_i\phi)^2 + r^2\,\frac{c^4g^2}{b^2}\left(-\frac{\beta}{2}\,\mathrm{Tr}\phi^2 + \tfrac{1}{4}(\gamma - \tfrac{7}{15})(\mathrm{Tr}\phi^2)^2 + \tfrac{1}{2}\,\mathrm{Tr}\phi^4 + \tfrac{1}{3}\,\mathrm{Tr}\phi^3\right)\right], \tag{3.5}$$

where the primes on the variables have been dropped. Since the radial coordinates still have dimensions, it is useful to replace r by

$$x = r\left(\frac{cg}{b}\right), \tag{3.6}$$

so that

$$E = \frac{1}{g^2}\left(\frac{cg}{b}\right)\int d^3x\left[\tfrac{1}{2}(F^a_{0i})^2 + \tfrac{1}{2}(F^a_{ij})^2 + \tfrac{1}{2}(D_i\phi)^2 + x^2\left(\frac{b}{g^2}\right)\left(-\frac{\beta}{2}\mathrm{Tr}\phi^2 + \tfrac{1}{4}(\gamma - \tfrac{7}{15})(\mathrm{Tr}\phi^2)^2 + \tfrac{1}{2}\,\mathrm{Tr}\phi^4 + \tfrac{1}{3}\,\mathrm{Tr}\phi^3\right)\right]. \tag{3.7}$$

Finally, if in a given phase ϕ obtains a vacuum expectation value proportional to δ (dimensionless), it is sometimes useful to scale this parameter out of the problem via new variables

$$\phi' = \frac{\phi}{\delta}\,, \quad A'_\mu = \frac{A_\mu}{\delta}\,, \quad x' = x\delta\,, \tag{3.8}$$

in which case (again, the primes have been dropped),

$$E = \frac{1}{g^2}\left(\frac{cg}{b}\right)\delta\int d^3x\left[\tfrac{1}{2}(F^a_{0i})^2 + \tfrac{1}{2}(F^a_{ij})^2 + \tfrac{1}{2}(D_i\phi)^2 + \frac{b}{g^2}\left(-\frac{\beta}{2}\,\frac{1}{\delta^2}\,\mathrm{Tr}\phi^2 + \tfrac{1}{4}(\gamma - \tfrac{7}{15})(\mathrm{Tr}\phi^2)^2 + \tfrac{1}{2}\,\mathrm{Tr}\phi^4 + \frac{1}{3}\,\frac{1}{\delta}\,\mathrm{Tr}\phi^3\right)\right]. \tag{3.9}$$

The first step in the analysis is to identify the possible phases of the theory as a function of the zero-temperature parameters. For this analysis it is necessary to examine the extrema of the potential which, according to Eq. (3.7), is proportional to

$$V(\phi)\propto -\frac{\beta}{2}\,\mathrm{Tr}\phi^2+\tfrac{1}{4}(\gamma-\tfrac{7}{15})(\mathrm{Tr}\phi^2)^2+\tfrac{1}{2}\,\mathrm{Tr}\phi^4+\tfrac{1}{3}\,\mathrm{Tr}\phi^3\,. \tag{3.10}$$

The phase transitions occur when $\beta>0$ is positive and, in this case, γ must be positive in order for the energy to be bounded below. The possible global maxima of the potential are given as follows:

(1) $\phi=0$: SU(5) symmetry is manifest; this is a local minimum for $\beta<0$.

(2) $$\phi=\delta_{4-1}\begin{bmatrix}1&&&&0\\&1&&&\\&&1&&\\&&&1&\\0&&&&-4\end{bmatrix}\quad\text{where } \delta_{4-1}=\frac{1}{(\gamma+\frac{5}{6})}\left\{\frac{3}{40}+\left[\left(\frac{3}{40}\right)^2+\frac{\beta}{20}\left(\gamma+\frac{5}{6}\right)\right]^{1/2}\right\};$$

SU(5) is broken down to SU(4)× U(1) symmetry and the phase is a local minimum for

$$-\frac{9}{80(\gamma+\frac{5}{6})}<\beta<\tfrac{5}{4}(\gamma+\tfrac{7}{30})\,.$$

(3) $$\phi=\delta_{3-2-1}\begin{bmatrix}1&&&&0\\&1&&&\\&&1&&\\&&&-\frac{3}{2}&\\0&&&&-\frac{3}{2}\end{bmatrix}\quad\text{where } \delta_{3-2-1}=\left(\frac{2\beta}{15\gamma}\right)^{1/2}\left[\left(1+\frac{1}{120\beta\gamma}\right)^{1/2}+\frac{1}{(120\beta\gamma)^{1/2}}\right];$$

SU(5) is broken down to SU(3)×SU(2)×U(1) symmetry and this phase is a local minimum for $\beta>\frac{15}{32}(\gamma-\frac{4}{15})$ for $\gamma>\frac{2}{15}$. The three regions of parameter space overlap one another and in Fig. 2 is shown the phase diagram (originally produced by Guth and Tye[9]) of global minima of the theory.

In order for the model to have any chance of being a description of our universe, it is necessary that for temperatures near zero the parameters of the theory lie in the SU(3)×SU(2)×U(1) (3-2-1) sector of the phase diagram. At temperatures large compared to all masses in the theory, the finite-temperature effective potential is obtained from the zero-temperature effective potential by replacing the parameter β in the theory by

$$\beta(T)=\beta-\frac{b\sigma}{c^2}\,T^2\,,$$

where

$$\sigma=\tfrac{1}{60}(130a+94b+75g^2)\,. \tag{3.11}$$

If T is large, $\beta(T)$ is negative and the SU(5) phase is the global maximum. As T decreases, the system moves to the right horizontally in Fig. 2 until at T near zero one arrives in the 3-2-1 sector at the zero-temperature values of β and γ. The detailed description of the passing from high

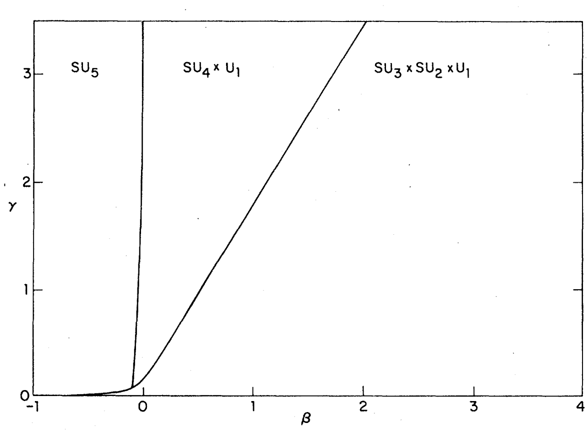

FIG. 2. The global minima of the SU(5) potential as a function of the parameters of the theory. Increasing the temperature T corresponds approximately to decreasing β while keeping γ constant.

T to low T is beyond present analytical capabilities; it will be *assumed* that changes in T correspond to purely horizontal movements on the diagram, as was effectively assumed by Guth and Weinberg,[7] but a better treatment of the corrections to these effects might prove useful.

Under these assumptions, if the system proceeds from high T to low T and $\gamma > \frac{4}{15}$, the system will undergo a sequence of phase transitions from SU(5) to SU(4)×U(1) (5 → 4-1) and from SU(4)×U(1) to SU(3)×SU(2)×U(1) (4-1 → 3-2-1). The nature of these transitions is determined by where the zero-temperature values of the parameters (β, γ) lie in the 3-2-1 sector.

In Fig. 3 is shown an analysis of the 3-2-1 sector that will be referred to repeatedly during the course of the paper. As T decreases from a very high value, $\beta(T)$ first crosses a line at $\beta(T_{c1}) = 0$ ($T_{c1} \sim 10^{14}$ GeV) and (if the system is homogeneous) the system supercools to some temperature $T \sim 0.1\ T_{c1}$, at which point bubbles of 4-1 phase nucleate; they coalesce to form monopoles and latent heat which drives the temperature back up to near T_{c1}. The system continues to cool until $\beta(T)$ approaches line 1 in Fig. 3. Since (β, γ) at $T = 0$ lie in the 3-2-1 sector where the SU(5) phase is not even a local minimum, it is clear that this first phase transition from SU(5) to SU(4)×U(1) must be complete before line 1 is crossed and $(\beta(T), \gamma)$ enter the 3-2-1 sector.

Line 1 in Fig. 3 corresponds to the values of the parameters for which the 4-1 minimum has the same energy density as the 3-2-1 minimum. If T continues to decrease and $\beta(T)$ continues to move to the right in the figure, the 4-1 phase becomes metastable compared to the 3-2-1 phase, but the barrier between the two phases prevents the transition from occurring immediately. Instead, the system supercools in the 4-1 phase and (ignoring the monopoles for the moment) $\beta(T)$ continues to move to the right in the diagram until bubbles are nucleated and the second transition is complete. Line 5 in Fig. 3 represents the values of the parameters for which the barrier between the 4-1 phase and the 3-2-1 phase is no longer present and the 4-1 phase is no longer even a local minimum. Therefore, the transition must be complete for values of $(\beta(T), \gamma)$ to the right of line 5, and, if the zero-temperature values of β and γ lie to the right of line 5, the two transitions are completed at finite values of T on the order of $T_c = 10^{14}$ GeV or so.

More interesting possibilities occur if the zero-temperature values of the parameters (β, γ) lie in the region between lines 1 and 5. If the values lie too close to line 1 (again, ignoring monopoles)

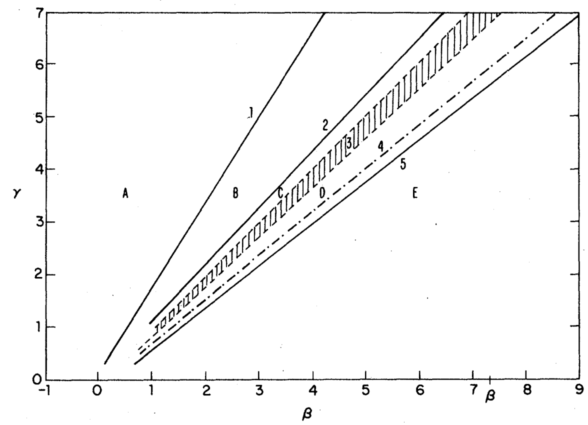

FIG. 3. The early-Universe scenarios as a function of the zero-temperature SU(5) parameters. See the text (Sec. III) for details.

even as T approaches zero, the probability of bubble nucleation is incredibly small because the barrier is high and the transition can, at best, be completed very slowly. When the expansion rate of the Universe is taken into account, it can be shown that the transition can never complete itself. If the zero-temperature values lie close to line 5, as T approaches zero the barrier can become small enough that bubble nucleation becomes highly probable and the transition can complete itself. By computing the bubble-nucleation probability as a function of the parameters in Sec. IV, one can determine approximately the boundary between these two possibilities. In Sec. VI, the values of the parameters for which monopole dissociation takes place will be determined (classical dissociation ignoring the thermal fluctuations). In order for monopole dissociation to be possible, the monopoles must dissociate for a value of T that is too high for bubble nucleation to occur prolifically and complete the transition.

IV. BUBBLE NUCLEATION

In order to determine the temperature at which bubble nucleation and the spontaneous transition from the 4-1 to the 3-2-1 phase occur, it is necessary to know the finite-temperature effective potential as a function of T. Since this is not known for all values of T, two separate computations, one for T much greater than all the masses in the theory and one for $T = 0$, must be performed and the results for intermediate values can be found by interpolation. The calculation has been done independently and somewhat more generally by Guth and Weinberg[7] and so the discussion in this section will be abbreviated.

For the case of zero temperature, the decay rate of the metastable phase is given by the imaginary part of the energy and the probability of bubble formation per unit time per unit volume is of the form

$$\Gamma \sim R \exp(-A), \tag{4.1}$$

where A is the Euclidean action corresponding to the tunneling solution with least action and R is expected to be on the order of T_c^4. Coleman[4] showed that the solution of least action corresponds to solving the Euler-Lagrange equations of the theory in four-dimensional Euclidean space with the boundary condition that the fields approach the false vacuum at $r = (x^2 + t^2)^{1/2} \to \infty$. It will be assumed that the lowest action solutions are those of highest symmetry [O(4) invariant and of greatest simplicity]; the solution considered will have ϕ a diagonal matrix for all r and a trivial gauge field. The field equations that define the solution are therefore reduced to

$$\frac{d^2\phi(r)}{dr^2} + \frac{3}{r}\frac{d\phi(r)}{dr} = \frac{\partial V}{\partial \phi}, \tag{4.2}$$

where

$$\phi \to \phi_\infty = \delta_{4-1}\begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & -4 \end{bmatrix} \text{ as } r \to \infty .$$

Two possibilities for $\phi(0)$ remain:

$$\phi \sim \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & -\frac{3}{2} & \\ 0 & & & & -\frac{3}{2} \end{bmatrix} \text{ or } \begin{bmatrix} 1 & & & & 0 \\ & -\frac{3}{2} & & & \\ & & \frac{3}{2} & & \\ & & & 1 & \\ 0 & & & & 1 \end{bmatrix},$$

but the numerical computations show that the former case leads to solutions of lower action than the latter. Two independent components of ϕ remain and the equations of motion can be solved numerically by varying $\phi(0)$ until a value is found for which $\phi(r) \to \phi_\infty$ as $r \to \infty$.[10] The results for a value of $b = g^2/3 \sim 0.1$ are shown in Fig. 4; this value lies at one end of the allowed range according to Eq. (3.3) and (in that range) provides the lowest values of A for a given choice of parameters β and γ. As expected, for values of β/γ near line 1, the action for bubble nucleation is infinite and for β/γ near line 5 the action is zero.

The high-temperature nucleation rate can be found by determining the imaginary part of the free energy and leads to a bubble nucleation rate

$$\Gamma(T) \sim R \exp[-E(T)/T], \tag{4.3}$$

where $E(T)$ is the energy (computed from the finite-T effective potential) of the bubble of critical size. If T is larger than all the masses in the theory, the high-T effective potential is obtained from the zero-T potential by replacing β with $\beta(T)$ as defined in Eq. (3.11) and by computing paths periodic in Euclidean time with period $1/T$. For larger T, this amounts to reducing the dimensionality of the problem to three and the Euler-Lagrange equations become

$$\frac{d^2\phi(r)}{dr^2} + \frac{2}{r}\frac{d\phi(r)}{dr} = \frac{\partial V}{\partial \phi}, \tag{4.4}$$

where now r is the three-dimensional radial coordinate. Using the same assumptions and methods of solution as were used for the zero-temperature problem, $E(T)$ was computed for large T. The value of $E(T)$ was found to be infinite for $\beta(T)/\gamma$ to the left of line 1 in Fig. 3 and to decrease rapidly as $\beta(T)/\gamma$ moves to the right of line 1.

If an exact computation of the bubble action as a function of T were possible, one would expect it to interpolate between the results of high T and zero T. The effects of temperature on the action, though, should decrease rapidly as T decreases below T_c since the energy density,

$$\rho = \rho_0 + aT^4, \tag{4.5}$$

where $\rho_0 \sim T_c^4$ is the energy density difference

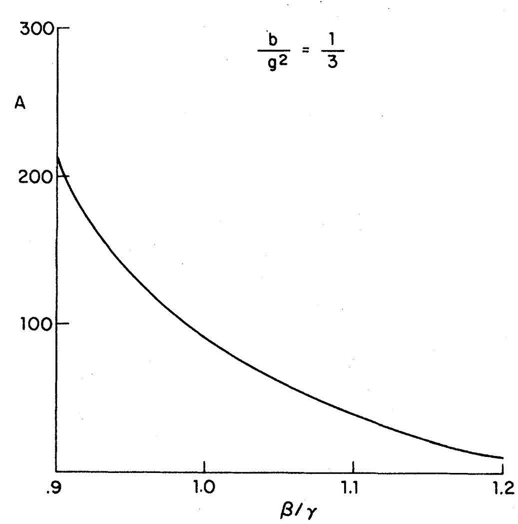

FIG. 4. The zero-temperature action for the O(4)-invariant bubble as a function of the SU(5) parameters for $b/g^2 = \frac{1}{3}$.

between the 4-1 and 3-2-1 phases and the second term is the contribution of the massless particles (which dominate the energy density at these temperatures), has negligible dependence on the temperature for $T \lesssim T_c/4$. Therefore, if $E(0)$ is the bubble energy evaluated at $T=0$ using the high-T approximation, $A(T)=E(T)/T$ in the exact solution should lie in the range

$$\frac{E(0)}{\frac{1}{4}T_c} > A(T) > \frac{E(0)}{T_c} \tag{4.6}$$

for values $T_c/4 < T < T_c$. (This argument is derived from Ref. 7.) In Fig. 5 the evaluation of this range of values as a function of the parameters is shown and it appears that $A(T)$ can be as much as an order of magnitude less than the value of A computed using the zero-temperature methods for a given choice of parameters. An exact solution would show that as T decreases below T_c (line 1), $A(T)$ decreases rapidly to a minimum at $T \gtrsim T_c/4$ and $A(T) \gtrsim 0.1A$ and then increases to A as thermal effects drop off rapidly beyond this point. If one believes these approximations, one can now compute the values of the parameters for which bubble nucleation is possible.

First, however, one must take account of the gravitational effects because the completion of the phase transition depends on the nucleation rate being fast compared to the expansion rate of the Universe. Following the "standard model,"[7] a homogeneous isotropic universe will be assumed that is described by a Robertson-Walker metric

$$ds^2 = dt^2 - R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right), \tag{4.7}$$

where $k=1,\ 0,\ -1$, according to whether the Universe is closed, flat, or open. Substituting the form for the metric in Einstein's equations one obtains

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3M_P^2}\rho - \frac{k}{R^2}, \tag{4.8}$$

where $M_P = 1.2\times 10^{19}$ GeV is the Planck mass; ρ is the energy density

$$\rho = \rho_0 + \left(\frac{\pi^2}{30}\right)\eta T^4, \tag{4.9}$$

where η is the number of effectively massless degrees of freedom (the number of massless bosons plus $\frac{7}{8}$ the number of massless fermions $\sim 10^2$), and ρ_0 is the difference in the energy density between the metastable phase and the stable phase. As T decreases, the energy density is dominated by ρ_0 and $R(t)$ grows exponentially with time. Typically, supercooling below $T \lesssim \frac{1}{3}T_c$ indicates the onset of exponential growth.

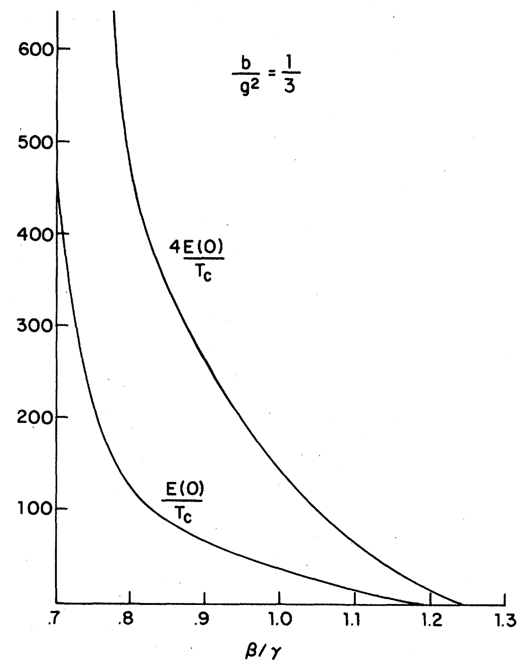

FIG. 5. The possible range for the minimum action $A(T)$ as a function of the SU(5) parameters for $b/g^2=\frac{1}{3}$. The minimum value of $A(T)$ should lie somewhere between the two curves.

Taking account of the expansion of the Universe, the fraction of space remaining in the old phase at time t is[9]

$$p(T) = \exp\left\{-\int_{t_0}^{t} dt_1 f(t_1) R^3(t_1)\frac{4\pi}{3}\left[\int_{t_1}^{t} dt_2 \frac{1}{R(t_2)}\right]^3\right\}, \tag{4.10}$$

where $f(t)$ is the rate of bubble nucleation per unit time per unit physical volume and

$$f(t) \propto \exp[-A(T(t))]. \tag{4.11}$$

The equation can be reexpressed in terms of temperature if one assumes the Universe is expanding adiabatically (setting $k=0$):

$$\left(\frac{\dot{T}}{T}\right)^2 = \frac{8\pi\rho_0}{3M_P^2}\left[1 + \frac{\pi^2}{30}\eta\left(\frac{T^4}{\rho_0}\right)\right] = h^2(T). \tag{4.12}$$

Then,

$$\rho(T) = \exp\left\{-\xi\int_{T}^{T_c} dT_1 \frac{e^{-A(T)}}{T_1^4 h(T_1)}\left[\int_{T}^{T_1} dT_2 \frac{1}{h(T_2)}\right]^3\right\}, \tag{4.13}$$

where

$$\xi = \tfrac{4}{3}\pi T_c^4, \quad T_c \simeq 10^{14}\ \text{GeV},$$

and

$$h(T) \approx \left(\frac{8\pi}{3M_P^2}\rho_0\right)^{1/2} \quad \text{for small } T$$

$$\simeq 10^9.$$

Combining the various terms leads one to the conclusion that $p(T)$ decreases to zero if $A(T) \lesssim \ln 20$. Since $A(T)$ decreases to its minimum value at $T_m \gtrsim T_c/4$, the result implies that bubbles are only produced in numbers sufficient to reduce $p(T)$ to zero when $A(T_m) \lesssim \ln 20$. Otherwise bubble nucleation occurs too rarely to compensate for the expansion rate of the region exterior to the bubbles and supercooling continues down to $T \to 0$. Using the values obtained in Figs. 4 and 5 as a guide, the values of $\beta(T)$ and γ at which bubble nucleation takes place can be determined. For $b = 0.1$, line 4 in Fig. 3 corresponds to the values for which A, evaluated using the O(4)-invariant bubble, is equal to ln20 and the region labeled 3 represents the range [according to Eq. (4.6)] for which $A(T_m)$ computed approximately using the O(3)-invariant bubble is equal to ln20. The parameters and approximations have been made as optimistically as possible in the sense that lines 3 and 4 are as far to the left as possible [within the conceivable accuracy of the approximations according to Eq. (3.3)].

To reiterate, ignoring the possible effects of monopoles, one can make the following conclusions: If the zero-temperature parameters, β and γ, lie to the left of line 3, bubbles are never produced in significant enough numbers to complete the transition. Whatever bubbles are produced never coalesce and the Universe (exterior to the bubbles) supercools forever. If the parameters lie to the right of line 3, bubbles can be produced at $T \sim T_c/4$ in sufficient numbers to complete the transition.

V. SU(4)×U(1) MONOPOLES

In the transition from the SU(5) phase to the SU(4)×U(1) phase, bubbles of 4-1 phase (inside of which ϕ obtains a vacuum expectation value) are formed spontaneously in the SU(5) background. Different bubbles, however, can have different expectation values of ϕ and so, when the bubble walls coalesce, the disagreement in values will lead to singularities referred to as bubble knots (this description is, strictly speaking, not gauge invariant; see Ref. 11). Once the system thermalizes, the knots become monopoles or dyons; they begin to annihilate in pairs and/or decay to the ground state which is what will be referred to as the 4-1 monopole.

The lowest-lying monopole excitation will be assumed to be that which contains the minimum magnetic charge, zero (4-1) electric charge, and maximum residual symmetry.[12] At large distances from the core of the monopole, the scalar field ϕ must approach a 4-1 minimum point in order for the monopole to be a finite-energy solution; for example,

$$\phi \to \phi_0 = \delta_{4-1} \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & -4 \end{pmatrix} \quad \text{as } r \to \infty .$$

In order for the monopole solution to be spherically symmetric, it must be invariant under $\vec{L} + \vec{T}$, a combination of ordinary spatial rotations $\vec{L}$ and rotations in an SU(2) embedding in SU(5) generated by $\vec{T}$. For every such embedding, one can write the most general spherically symmetric ansatz for ϕ and $\vec{A}_a$ using the methods of Goldhaber and Wilkinson.[13] For any matrix $\tilde{Q}$ which satisfies

$$[\tilde{Q}, \phi_0] = 0 , \tag{5.1}$$

the configuration

$$\Phi(r) = \phi_0 , \quad \vec{A}(r) = \frac{1}{g} \tilde{Q} \vec{A}_D \tag{5.2}$$

is a solution of the field equations corresponding to a Dirac monopole for

$$\vec{A}_D = \hat{\phi}(1 - \cos\theta)/r \sin\theta .$$

The requirement that the string be unobservable implies

$$\exp(4\pi i \tilde{Q}) = 1 . \tag{5.3}$$

Since $\tilde{Q}$ commutes with ϕ_0 which breaks the symmetry down to SU(4)×U(1), $\tilde{Q}$ must be a linear combination of the U(1) charge Q and the SU(4) color charge C:

$$\tilde{Q} \sim MQ + C \tag{5.4}$$

and it is clear from the dependence of $\vec{A}$ on $\tilde{Q}$ that M is proportional to the magnetic charge.

The Dirac-type solution can be transformed to a string-free spherically symmetric 't Hooft-Polyakov monopole by a gauge transformation if and only if there is another SU(2) embedding in SU(5) generated by $\vec{I}$ satisfying

$$\tilde{Q} = I_3 - T_3 , \quad [\vec{I}, \tilde{Q}] = 0 , \quad [\vec{I}, \phi_0] = 0 . \tag{5.5}$$

The form of the gauge transformation is

$$\begin{aligned} \Lambda(\hat{r}) &= \Omega(\hat{r}) \omega^{-1}(\hat{r}) , \\ \Omega(\hat{r}) &= e^{-i\phi T_3} e^{-i\phi T_2} e^{i\phi T_3} , \\ \omega(\hat{r}) &= e^{-i\phi I_3} e^{-i\phi I_2} e^{i\phi I_3} , \end{aligned} \tag{5.6}$$

and the spherically symmetric equivalent form of the solution is

$$\vec{A} = \frac{1}{gr}[\vec{I}(r) - \vec{T}] \times \vec{r},$$
$$\Phi(\hat{r}) = \Omega(\hat{r})\phi_0\Omega^{-1}(\hat{r}), \tag{5.7}$$
$$\vec{I}(\hat{r}) = \Lambda(\hat{r})\vec{I}\Lambda^{-1}(\hat{r}).$$

An immediate consequence is that, since $[\vec{I}, \phi_0] = 0$, $\vec{I}$ is restricted to act on the upper 4×4 components of ϕ only; therefore, the 5-5 component of $\tilde{Q}$,

$$\tilde{Q}_{55} = -(T_3)_{55}, \tag{5.8}$$

which gives the U(1) charge part of $\tilde{Q}$ only, is determined by $\vec{T}$ alone.[13] In order to have a topologically nontrivial solution, T must be in the $\frac{1}{2}$, 1, $\frac{3}{2}$, or 2 representation of SU(2). (The representation need not necessarily be reducible either.) Therefore, either the 5-5 component of T is nonzero and the minimum magnetic charge corresponds to

$$\vec{T} = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & T_{1/2} \end{bmatrix},$$

where $T_{1/2}$ is a 2×2 matrix, or T has a zero 5-5 component and is of the form

$$\vec{T} = \begin{pmatrix} T_4 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\vec{T}_4$ is a 4×4 matrix. In the first case, for $\vec{I} = 0$,

$$\Phi(r) = g(r)\begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & -\frac{3}{2} & \\ 0 & & & & -\frac{3}{2} \end{bmatrix} + \tfrac{5}{2}f(r)\hat{r}\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \vec{\sigma} \end{bmatrix} \tag{5.9}$$

[where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$] is the simplest ansatz and the residual symmetry of the theory, defined by

$$[\Gamma, \phi(r)] = 0 \quad \text{for all } r, \tag{5.10}$$

is SU(3)×U(1)×U(1). In the latter case, the residual symmetry is reduced. This sort of (crude) analysis suggests that the lowest-lying monopole corresponds to the ansatz given by Eq. (5.9). The ansatz for the vector field is

$$\vec{A} = \frac{[1 - K(r)]}{gr}\hat{r}\times\begin{bmatrix} 0 & & & 0 \\ & 0 & & \\ & & 0 & \\ 0 & & & \vec{\sigma} \end{bmatrix}. \tag{5.11}$$

For the solution to be nonsingular, $f(r)\to0$ and $K(r)\to1$ as $r\to0$, but there is no constraint on $g(r)$. For the solution to be of finite energy, $f(r)$ and $g(r)$ must approach δ_{4-1}, and $K(r)$ must approach 0 as $r\to\infty$ so $\Phi(r)$ approaches a 4-1 minimum.

The ansatz in Eqs. (5.9) and (5.11) was determined solely by considerations of symmetry without regard for whether the 4-1 phase is stable or metastable. However, it is coincidentally true that, as $r\to0$,

$$\Phi(r) \propto \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & -\frac{3}{2} & \\ 0 & & & & -\frac{3}{2} \end{bmatrix},$$

so that, if $g(r)\to\delta_{3-2-1}$ as $r\to0$, the core of the monopole lies near the 3-2-1 minimum. If the temperature decreases such that the system crosses to the right of line 1 in Fig. 3, the 4-1 phase becomes metastable compared to the stable 3-2-1 phase and the cores of the monopoles, the remnant of the previous transition, are naturally occupied with stable phase and can undergo the same sort of dissociation instability discussed for simpler models in Sec. II.

This result is amusing since, at first sight, it appears impossible that the phenomena discussed in Sec. II might occur in the early Universe since the models of Sec. II always involved a transition from lower manifest symmetry to higher symmetry—just the opposite of the case considered in this section. However, when dealing with symmetry groups larger than SO(3), there are many possible topological excitations of a system since (for three spatial dimensions) the stability of the excitation only depends on the SU(2) embeddings in the group and not on the whole group. In this case, an SU(2) subgroup previously broken in the 4-1 phase is restored in the 3-2-1 phase (even though the total symmetry is reduced) and so the phenomena of Sec. II can occur in the early Universe.

To analyze the stability of the 4-1 monopole as a function of the parameters of the theory, it is useful to reexpress the energy functional by substituting the functions of the radial variable

$$F(x)=\frac{f(x)}{\delta_{4-1}}\,,\quad G(x)=\frac{g(x)}{\delta_{4-1}}\,,\quad \text{and } K(x)$$

into the expression of Eq. (3.9). After some manipulations one obtains

$$E=\left(\frac{4\pi}{g}\right)\left(\frac{cg}{b}\right)\delta_{4-1}\int_0^\infty dx\left[(K')^2+\frac{1}{2}\,\frac{(1-K^2)}{r^2}+\frac{7.5}{2}r^2(G')^2+\frac{12.5}{2}r^2(F')^2+12.5F^2K^2+x^2\frac{b}{g^2}U(G,F)\right], \tag{5.12}$$

where the prime indicates derivative with respect to x and where

$$U(G,F)=-\frac{\beta}{2}\left(\frac{1}{\delta_{4-1}}\right)^2(7.5G^2+12.5F^2)+\tfrac{1}{4}(\gamma-\tfrac{7}{15})(7.5G^2+12.5F^2)^2$$
$$+\tfrac{1}{2}(13.125G^4+78.125F^4+168.75G^2F^2)+\frac{1}{3}\,\frac{1}{\delta_{4-1}}(-3.75G^3-56.25G^2F). \tag{5.13}$$

$G(x)$, $F(x)$, and $K(x)$ must obey the boundary conditions

$$\left.\begin{matrix}G(x),\ F(x)\to 1\\ K(x)\to 0\end{matrix}\right.\quad \text{as } x\to\infty \tag{5.14}$$

in order for the solution to be of finite energy and

$$\left.\begin{matrix}F(x)\to 0\\ K(x)\to 1\end{matrix}\right.\quad \text{as } x\to 0 \tag{5.15}$$

in order for the solution to be nonsingular. Note that there is no boundary condition on $G(x)$ as $x\to 0$, so ϕ need not be zero in the core. The equations which the functions must satisfy are

$$G''+\frac{2}{x}G'=\frac{1}{7.5}\,\frac{b}{g^2}\,\frac{\partial U(G,F)}{\partial G}\,, \tag{5.16a}$$

$$F''+\frac{2}{x}F'=\frac{1}{6.25}\,\frac{b}{g^2}\,\frac{\partial U(G,F)}{\partial F}+\frac{2K^2F}{x^2}\,, \tag{5.16b}$$

$$K''=\frac{K(K^2-1)}{x^2}+12.5KF^2\,. \tag{5.16c}$$

If $G(x)$ were constrained to approach δ_{3-2-1} as $x\to 0$, the thin-wall analysis that was applied to the simpler models of Sec. II (in Ref. 1) could be immediately extended to this ansatz for the 4-1 monopole. The fact that $G(x)$ need not equal δ_{3-2-1} means that the energy-density difference between the core and the exterior of the monopole is not simply the energy-density difference between the 4-1 and 3-2-1 minima, but is somewhat less. The value that should be substituted for ϵ in the thin-wall argument is

$$\epsilon'=\frac{b}{g^2}[U(G(\infty),F(\infty))-U(G(0),F(0))]<\epsilon\,. \tag{5.17}$$

However, this fact only serves to reinforce the central conclusion of the thin-wall analysis that there must exist a finite range of values $\epsilon_c>\epsilon'>0$ for which (classically) stable monopole solutions in the metastable phase can be found. (If anything, the fact that $\epsilon'<\epsilon$ appears to make ϵ_c larger, since ϵ' must equal the old value of ϵ_c before the monopole dissociates.)

In order to have some notion on what determines ϵ_c it is useful to analyze the possible modes of instability of the monopole, extending an argument that was posed in Ref. 1. The idea is to analyze the stability of a monopole solution $(\phi(x),\ A_\mu^a(x))$ under dilations

$$\begin{aligned}\phi(x)&\to\phi(\lambda x)\,,\\ A_\mu^a(x)&\to\lambda A_\mu^a(\lambda x)\,.\end{aligned} \tag{5.18}$$

The energy functional consists of three sets of terms corresponding to the kinetic energy of the gauge field T_A, the kinetic energy of the scalar field T_ϕ, and the potential energy of the scalar field V_ϕ, and under dilations these transform according to

$$\begin{aligned}T_A&\to\lambda T_A\,,\\ T_\phi&\to\lambda^{-1}T_\phi\,,\\ V_\phi&\to\lambda^{-3}V_\phi\,.\end{aligned} \tag{5.19}$$

If $(\phi(x),\ A_\mu^a(x))$ are solutions of the equation in motion, the energy

$$E=T_A+T_\phi+V_\phi \tag{5.20}$$

must be stable under variations, i.e.,

$$E'|_{\lambda=1}=T_A-T_\phi-3V_\phi=0\,, \tag{5.21}$$

$$E''|_{\lambda=1}=2T_\phi+12V_\phi>0\,. \tag{5.22}$$

For the models considered in Sec. II and Ref. 1, as $\lambda\to 0$ one approaches the Prasad-Sommerfield limit which, as argued in Sec. II, yields a mass for the monopole solution that is a lower bound on the vector and scalar kinetic energy terms in the theory:

$$\int_0^\infty dr\left[(K')^2+\frac{1}{2}\frac{(1-K^2)}{r^2}+\tfrac{1}{2}r^2(h')^2+h^2K^2\right]=T_A^{\rm PS}+T_\phi^{\rm PS}\geq 1\,. \quad (5.23)$$

[The constant of proportionality associated with Eq. (2.4) is just the Prasad-Sommerfield mass itself, and the Prasad-Sommerfield solution yields unity for the integral in the expression.] Equation (5.23) is a bound for any functions $K(x)$ and $h(x)$ that obey the boundary conditions, Eqs. (2.5) and (2.6). The purpose of normalizing the functions $F(x)$ and $K(x)$ in the way that was done in Eq. (5.12) is that they obey the very same boundary conditions. The first five terms in Eq. (5.12), which are equal to T_A+T_ϕ for the 4-1 monopole, consist only of terms that are positive semidefinite. Therefore, fixing the value of the coefficient of the integral in Eq. (5.12) to be unity, it is clear that

$$T_A+T_\phi=\int dx\left[(K')^2+\frac{1}{2}\frac{(1-K^2)}{r^2}+\frac{7.5}{2}x^2(G')^2+\frac{12.5}{2}x^2(F')^2+K^2F^2\right]$$
$$\geq\int_0^\infty dx\left[(K')^2+\frac{(1-K^2)}{2x^2}+\tfrac{1}{2}r^2(F')^2+F^2K^2\right]=T_A^{\rm PS}+T_\phi^{\rm PS}\geq 1\,. \quad (5.24)$$

This bound is the same as Eq. (5.23) that was used for the models in Ref. 1, and so Eqs. (5.20)–(5.22) and (5.24) can be manipulated in the same way as Ref. 1 to draw the same conclusions:

(1) There can be no (classically) stable 4-1 monopoles with

$$E<\frac{8}{9}\frac{4\pi}{g^2}\left(\frac{cg}{b}\right)\delta_{4-1}\,. \quad (5.25)$$

(2) For a classically stable monopole,

$$V_\phi\geq-\tfrac{1}{9}(T_A+T_\phi)\,. \quad (5.26)$$

This effectively sets a bound on the form of the potential energy for which stable monopole solutions exist.

As was the case in Ref. 1, the analysis of stability under dilations represents only a bound on V_ϕ and, if monopoles become unstable under other modes, the equality is not ever achieved for a stable solution. On the other hand, if (near) equality is achieved for a stable solution, it suggests that dilations are directly responsible for the decay of the monopoles. Numerical computations are necessary to obtain more specific information.

VI. NUMERICAL CALCULATIONS

In order to determine the critical value of the parameters of the theory for which the monopoles dissociate in the models of Ref. 1 and Sec. II, trial guesses for the functions $h(r)$ and $K(r)$ were introduced into the expressions for the energy functional, the functional was evaluated numerically, and the trial guesses were varied in such a way as to minimize the energy. For the critical values of the parameters, no (local) minimum for the energy functional could be found.

For the case of the 4-1 monopole, there is an additional function in the ansatz $G(x)$ with a free boundary condition as x approaches zero, and this kind of variational procedure was found to be too difficult. Instead, an attempt was made to solve the classical differential equations for the monopole solution directly, Eqs. (5.16). However, because there are fixed boundary conditions on at least some functions as x approaches zero and when x approaches infinity, solution of the equations is highly nontrivial.

As x approaches infinity there is a fixed boundary condition for all three functions in the ansatz in order for the solution to have finite energy:

$$\left.\begin{array}{l}F(x),\ G(x)\to 1\\ K(x)\to 0\end{array}\right\}\ \text{as } x\to\infty. \quad (6.1)$$

As x approaches zero, for the solutions to be nonsingular they must obey

$$\left.\begin{array}{l}F(x)\to 0\\ K(x)\to 1\end{array}\right\}\ \text{as } x\to 0; \quad (6.2)$$

however, more information is needed before numerical integration can proceed. To solve each one of the required differential equations, the value of the appropriate function [and the first (nonzero) derivative for $F(x)$ and $K(x)$] must be known as x approaches zero, and then one can numerically integrate forward in x until $x\to\infty$. As $x\to 0$, the functions have the form

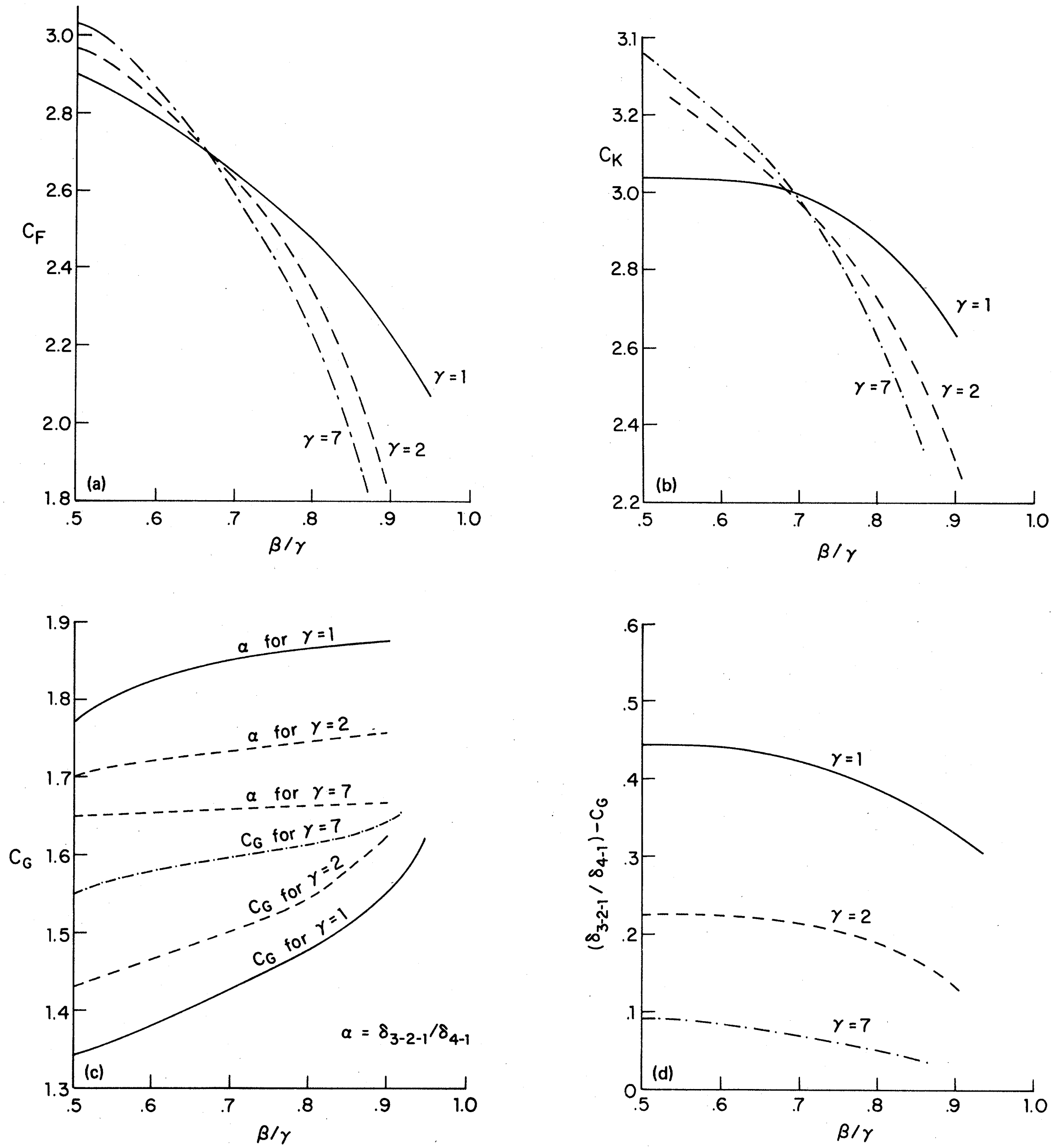

FIG. 6. (a) C_F versus β/γ for several values of γ. (b) C_K versus β/γ for several values of γ. (c) C_G versus β/γ for several values of γ; in addition, $\delta_{3\text{-}2\text{-}1}/\delta_{4\text{-}1}$ versus β/γ is also shown. (d) The difference, $C_G-(\delta_{3\text{-}2\text{-}1}/\delta_{4\text{-}1})$, versus β/γ for several values of γ. In all four figures the value $b/g^2=\frac{1}{3}$ was chosen and the curves are continued only insofar as solutions could be found.

$$K(x)\sim 1 - C_K x^2\,,\qquad F(x)\sim C_F x\,,\qquad G(x)\sim C_G \tag{6.3}$$

(these are consistent with the equations and the constraints). The constants C_F, C_K, and C_G are determined by the condition that the functions satisfy Eq. (6.1) as x approaches infinity. To solve any one of the differential equations the appropriate constant can be varied until the boundary conditions are satisfied at large distances from the core. Therefore, the following procedure was adopted.

(1) As an initiating procedure, a trial function for $F(x)$ was introduced that behaved as

$$F(x) \to cx \text{ as } x \to 0 \text{ for a constant } c,$$

$$F(x) \to 1 \text{ as } x \to \infty .$$

Different trial functions were shown to cause no difference in the final results.

(2) The form for $F(x)$ was used to solve Eq. (5.16a). C_G was varied until the boundary condition $G(x) \to 1$ as $x \to \infty$ was satisfied.

(3) The form for $F(x)$ was used to solve Eq. (5.16c). C_K was varied until the condition $K(x) \to 0$ as $x \to \infty$ was satisfied.

(4) The solutions for steps (2) and (3) were used to solve for $F(x)$ in Eq. (5.16b). C_F was varied until the condition $F(x) \to 1$ as $x \to \infty$ was satisfied. The procedure of varying the constants to meet the necessary boundary conditions in steps (2)–(4) was found to converge within one part in 10^5 after 50 trial guesses for the constants.

(5) The solution for $F(x)$ from step (4) was used to return to step (2) and begin again. The cycle was repeated until the energy of the solution converged to within one part in 10^4.

The solutions for C_F, C_K, and C_G as a function of the parameters β and γ are shown in Figs. 6(a)–6(c) for $b/g^2 = \frac{1}{3}$. The curves for each of the figures have been continued only insofar as a solution could be found. $F(x)$ and $K(x)$ are analogous to the functions $h(r)$ and $K(r)$ for the models of Sec. II and the curves in Figs. 6(a) and 6(b), are consistent with the analysis of Ref. 1. As β/γ increases and the difference in the (free) energy between the 4-1 and 3-2-1 minima becomes larger, C_F and C_K each decrease so that the core region where $F(x) \approx 0$ and $K(x) \approx 1$ is larger. Once the critical values of the parameters is reached, near $\beta/\gamma = 0.93$, subsequent cycles of steps (2)–(5) in the procedure lead to further decreases in C_F and C_K without ever converging—the core grows indefinitely and no time-independent solution exists. The precise form of the curves and the dependence of their form on γ is not presently understood. The curves of C_G versus β/γ in Fig. 6(c) are especially interesting; the fact that C_G, which determines the value of $G(x)$ in the core, is not constrained to equal $\delta_{3-2-1}/\delta_{4-1}$ is what sets this problem apart from that considered in Ref. 1. On the same figure, the value of $\delta_{3-2-1}/\delta_{4-1}$ as a function of β/γ is shown and in Fig. 6(d) the difference, $C_G - \delta_{3-2-1}/\delta_{4-1}$, is shown. As β/γ increases, the difference between C_G and $\delta_{3-2-1}/\delta_{4-1}$ decreases—the monopole core lies closer to the 3-2-1 minimum. The difference does not appear to approach zero at the critical value of β/γ. Once the monopole becomes unstable, the arguments in Ref. 1 imply that the energy from the conversion from metastable phase to stable phase is rapidly stored in the wall of the monopole, further accelerating the growth of the core: this implies that C_G rapidly approaches $\delta_{3-2-1}/\delta_{4-1}$ as the core grows. A typical stable solution for the functions $F(x)$, $G(x)$, and $K(x)$ is shown in Fig. 7.

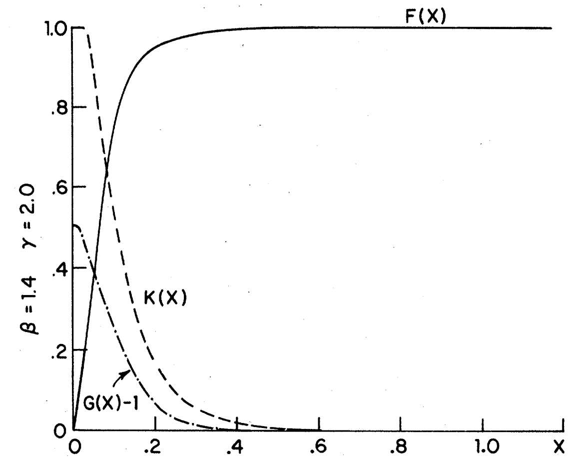

FIG. 7. $G(x)$, $F(x)$, and $K(x)$ for a typical stable solution $\beta = 1.4$ and $\gamma = 2.0$.

In Fig. 8 the dependence of the energy of the monopole solution on β/γ is shown. As in the cases studied in Ref. 1, the energy decreases as the critical value of β/γ is approached, presumably because of the increasing negative energy density of the core. Why the curves cross one another must have to do with the detailed balance between the kinetic energies and potential energies in the solution as a function of β/γ, but no

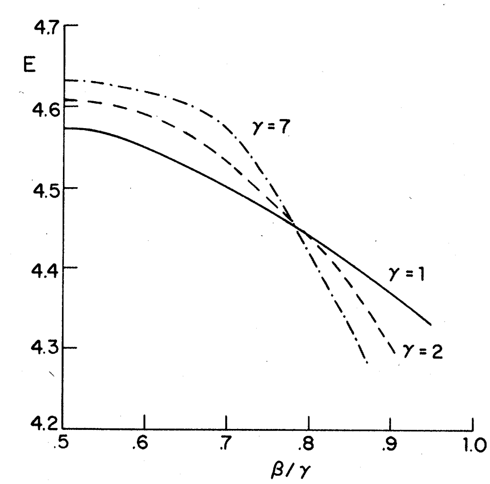

FIG. 8. The energy of the monopole versus β/γ for several values of γ. The coefficient of the integral in Eq. (5.12) has been set to unity.

precise understanding of the phenomena is available at this time. Another interesting parameter to study is the potential energy contribution to the energy V_ϕ, whose bound was computed analytically in Eq. (5.26). V_ϕ versus β/γ is plotted in Fig. 9 and, as expected, V_ϕ decreases sharply as β/γ approaches the critical value; consistent with the bound, V_ϕ never decreases below $-\frac{1}{9}$ for any stable solutions. However, whereas solutions for β/γ less than 0.95 times the critical value converge rapidly and solutions for β/γ greater than 1.05 times the critical value diverge rapidly, for values of β/γ near the critical value convergence or divergence of the solution occurs very slowly; therefore, it was not possible to determine how close to an equality can be achieved in the relation Eq. (5.26).

The most important result is indicated by line 2 in Fig. 3. The line represents the critical values of β and γ for which the monopole dissociates. For values of $\beta(T)$ and γ to the left of line 2, classical solutions can be found; for values to the right, they cannot be found. If the zero-temperature values of β and γ lie to the right of line 2, then as T decreases $\beta(T)$ increases and one proceeds horizontally on the right on Fig. 3 until line 2 is crossed; for that temperature, all 4-1 monopoles dissociate. The computation has not taken account of the possible effects of thermal and quantum fluctuations of the core, so it is probable that the monopoles decay at yet higher temperatures (to the left of line 2). In this sense, line 2 only represents a (rightmost) bound on the monopole dissociation. On the other hand, within the legitimacy of the approximations, b/g^2 has been chosen so that line 3 represents a (leftmost) bound on bubble nucleation. Since line 2 lies to the left of line 3, monopole dissociation occurs at a higher temperature than that required for significant spontaneous bubble nucleation.

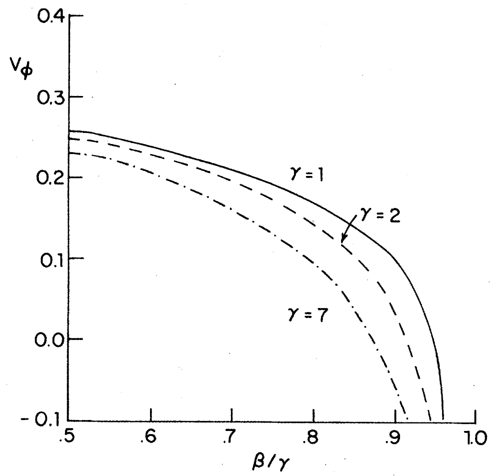

FIG. 9. The potential energy contribution to the total energy versus β/γ for several values of γ. The curves are only continued insofar as solutions could be found, although there is some uncertainty as to where precisely the critical value of β/γ lies. The coefficient of the integral in Eq. (5.22) has been set to unity.

VII. CONCLUSIONS

For the grand unified field theories considered in this paper, dissociation of the monopoles of the SU(4)×U(1) phase has been shown to be a natural phenomenon in the sense that monopole dissociation precedes the spontaneous decay of a metastable SU(4)×U(1) phase.[14] Depending on the region in Fig. 3 in which the zero-temperature values of the parameters β and γ lie and on the number density of 4-1 monopoles, several scenarios seem possible (the uncertainty in the precise positions of lines 2 and 3 will be ignored).

Region A. Even at $T=0$ the symmetry is not broken down to SU(3)×SU(2)×U(1). This region is not physically interesting.

Region B. Even though the 3-2-1 minimum is the global minimum as $T\to 0$, the barrier height between the 4-1 phase and the 3-2-1 phase is too large for there to be sufficient bubble nucleation for the 4-1 → 3-2-1 transition to be completed. Monopoles of the 4-1 phase remain (classically) stable. As $T\to 0$, there is some probability for bubble nucleation (or monopole dissociation from fluctuations of the core) that is too small to keep up with the expansion rate of the Universe, and so the bubbles that are formed do not coalesce and percolate the Universe. The Universe is highly inhomogeneous.

Regions C, D, E. Assuming that the transition from manifest SU(5) to SU(4)×U(1) produces some monopoles, then the first event that occurs after that transition is when T leads to $(\beta(T), \gamma)$ lying on line 2 at which point the monopoles dissociate. (This should occur at $T\sim 10^{13}$ GeV for most β and γ that lie in these regions.) If the number density of monopoles is sufficiently large to compensate for the expansion of the Universe,[1] the monopoles coalesce and complete the 4-1 → 3-2-1 transition. Guth and Weinberg have done a very crude computation that indicates that there should be enough monopoles to complete the transition.[7] When the cores of the 4-1 monopoles coalesce, if these cores are not correlated in SU(3)×SU(2)×U(1), some 3-2-1 monopoles should be produced (see Ref. 1 for a more detailed discussion). If the number density is not sufficient to compensate for the expansion rate of the Universe and (β, γ) lie in region C, the monopoles dissociate but do not coalesce. The cores grow but the space around them continues to supercool in the 4-1 phase. The only possible hope for our Universe would be that it lie inside one of the monopole cores. This possibility is being presently investigated. If the number density of monopoles is not sufficient to compensate for the expansion rate of the Universe, but (β, γ) lie in regions D or E, the monopoles dissociate at some finite T associated with line 2; the cores grow but do not coalesce. As the T associated with line 3 is reached, bubbles form in the space between the monopole cores and all the 3-2-1 regions coalesce to complete the transition. In this last case the transition completes itself at $T \sim \frac{1}{4} T_c$ (except for possibly a very narrow, finely tuned region of parameter space near line 3).

If in fact there is a sufficient density of 4-1 monopoles to compensate for the expansion rate of the Universe, the 4-1 → 3-2-1 transition is completed through monopole dissociation at a higher temperature than if the transition depended on bubble nucleation and there is a larger range of parameter space for which one avoids the kind of supercooling associated with region B which yields an empty and cold universe.[7] Since the transition temperature is higher, there is less latent heat released after the transition and, consequently, less entropy.

It is probably difficult to improve upon the accuracy of the approximations made in this paper, but there are several open questions that would be useful to consider. For example, a better estimate of the number of 4-1 monopoles produced in the SU(5) → SU(4)×U(1) transition would help reduce the number of possible scenarios. More information about bubble growth and monopole formation at finite T is necessary. The environment inside a single decaying monopole at finite T would make an interesting study, especially if the results could consistently describe our own Universe. These issues are being presently studied.

ACKNOWLEDGMENTS

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