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Static spherically symmetric scalar fields in general relativity

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In 1957, Bergmann and Leipnik attempted to find static spherically symmetric solutions of a special form of the field equations of general relativity. They were not able to find explicit expressions for the gravitational potentials, and they did not realize that such expressions could be found by using a different coordinate system. Although Buchdahl developed an elegant procedure for finding, by inspection, the solutions sought by Bergmann and Leipnik, his procedure is severely limited when applied to the spherically symmetric case. Indeed, his procedure fails to identify one whole class of such static solutions. The object of this paper is to show that, under the assumptions of Bergmann and Leipnik, the integration of the field equations is almost trivial, and to identify the missing class of solutions.

I. INTRODUCTION

In 1957, Bergmann and Leipnik[1] considered solutions of the field equations of general relativity

$$R_{ij} - \tfrac{1}{2} g_{ij} R = -\kappa T_{ij} \tag{1}$$

when the energy-momentum tensor has the special form

$$\kappa T_{ij} = \mu (V_i V_j - \tfrac{1}{2} g_{ij} g^{mn} V_m V_n), \tag{2}$$

where V is a scalar, and $V_i = \partial V / \partial x^i$. The authors assume the line element is static, spherically symmetric, and has the form

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 d\sigma^2, \tag{3}$$

where

$$d\sigma^2 = d\theta^2 + \sin^2\theta \, d\phi^2, \tag{4}$$

and both λ and ν are functions of r alone. They were not able to find explicit expressions for λ or ν, and did not realize that such expressions could be found for the gravitational potentials by using a different coordinate system.

Buchdahl[2] developed an elegant procedure for finding, by inspection, solutions of (1) and (2) when a solution of the vacuum equations

$$R_{ij} = 0 \tag{5}$$

is known. His procedure produced the explicit expressions for the solutions sought by Bergmann and Leipnik. Since, by Birkhoff's theorem, the spherically symmetric, static or nonstatic, solution of (5) is unique, Buchdahl's procedure is severely limited when applied to the spherically symmetric case. Indeed, his procedure fails to identify one whole class of such static solutions.

The object of the present paper is to show that, under the assumptions of Bergmann and Leipnik, the integration of the field equations is almost trivial, and to identify the missing class of solutions mentioned above.

II. THE INTEGRATION OF THE FIELD EQUATIONS

It is known[3] that (1) and (2) are equivalent to

$$R_{ij} = -\mu V_i V_j, \tag{6}$$

and that these in turn imply

$$g^{mn} V_{;mn} = V^m{}_{;m} = 0. \tag{7}$$

From the assumption that the line element is static and spherically symmetric, (6) will imply that V_i also has these properties. This does not, however, require that $V = V(r,t)$ be independent of t. For the moment, it will only be assumed that V_i is spherically symmetric and has been placed into the form

$$V_i = (V', 0, 0, \dot{V}), \tag{8}$$

where the prime and dot, respectively, represent partial differentiation with respect to r and t. Using the Takeno[4] formulas to calculate R_{ij} leads to the field equations

$$-\frac{\lambda'}{r} + \frac{1}{2}\left(\nu'' + \frac{(\nu')^2}{2} - \frac{\lambda'\nu'}{2}\right) = -\mu (V')^2, \tag{9}$$

$$\frac{\nu'}{r} + \frac{1}{2}\left(\nu'' + \frac{(\nu')^2}{2} - \frac{\lambda'\nu'}{2}\right) = \mu (\dot{V})^2 e^{\lambda-\nu}, \tag{10}$$

$$\nu' - \lambda' = \frac{2}{r}(e^{\lambda} - 1), \tag{11}$$

$$V'\dot{V} = 0, \tag{12}$$

and (7) becomes

$$\frac{\partial}{\partial r}(r^2 e^{(\nu-\lambda)/2} V') - \frac{\partial}{\partial t}(r^2 e^{(\lambda-\nu)/2} \dot{V}) = 0. \tag{13}$$

From (12), $V' = 0$ or $\dot{V} = 0$. These two cases require different considerations.

Case I. $\dot{V}=0,\ V'\neq 0.$ If $\dot{V}=0$, then the general solution of (10) is

$$r^2\nu' e^{(\nu-\lambda)/2}=h, \tag{14}$$

where h is an arbitrary constant. Since (13) can be integrated to give

$$r^2 V' e^{(\nu-\lambda)/2}=k, \tag{15}$$

where k is an arbitrary constant, (14) and (15) will yield

$$\nu'=\alpha V', \tag{16}$$

where α is an arbitrary constant. Therefore,

$$\nu=\alpha V, \tag{17}$$

where an arbitrary constant of integration has been absorbed by the line element. From (15),

$$e^{\lambda}=r^4(V')^2 e^{\alpha V}/k^2, \tag{18}$$

and the line element (3) has the form

$$ds^2=e^{\alpha V}dt^2-\frac{r^4(V')^2}{k^2}e^{\alpha V}dr^2-r^2d\sigma^2 \tag{19}$$

or

$$ds^2=e^{\alpha V}dt^2-\frac{r^4e^{\alpha V}}{k^2}(dV)^2-r^2d\sigma^2. \tag{20}$$

This suggests the transformation of coordinates $\bar{r}=V(r)$ to obtain

$$ds^2=e^{\alpha\bar{r}}dt^2-\frac{r^4e^{\alpha\bar{r}}}{k^2}(d\bar{r})^2-r^2d\sigma^2, \tag{21}$$

where now $r=r(\bar{r})$, and V_i now has the form

$$V_i=(1,0,0,0). \tag{22}$$

The constant k^2 can be absorbed by linear translations of $\bar{r}$ and t. Dropping the bar, the line element becomes

$$ds^2=e^{\alpha r}dt^2-W^{-4}e^{\alpha r}dr^2-W^{-2}d\sigma^2. \tag{23}$$

$W=W(r)$ is still an unknown function.

Since the gradient V_i has the components (1, 0, 0, 0), the Takeno formulas can be used to give, in this coordinate system, the field equations

$$W''-\alpha W'-\tfrac{1}{2}\mu W=0, \tag{24}$$

$$WW''-(W')^2=-e^{\alpha r}. \tag{25}$$

Equation (24) is a linear differential equation of the second order with constant coefficients, and its general solution is

$$W=Ae^{m_1r}+Be^{m_2r}, \tag{26}$$

where

$$m_1=\tfrac{1}{2}[\alpha+(\alpha^2+2\mu)^{1/2}], \tag{27}$$

$$m_2=\tfrac{1}{2}[\alpha-(\alpha^2+2\mu)^{1/2}], \tag{28}$$

provided $\alpha^2+2\mu\neq 0$. When $\alpha^2+2\mu=0$, the general solution of (24) is

$$W=(A+Br)e^{\alpha r/2}. \tag{29}$$

In both (26) and (29) A and B are arbitrary constants. The roots m_1 and m_2 may, of course, be real or complex numbers.

W as given by (26) will be a solution of (24) if and only if

$$AB(\alpha^2+2\mu)=-1, \tag{30}$$

and (29) will provide a solution of (25) if and only if

$$B=\pm 1. \tag{31}$$

The solutions, therefore, become

$$ds^2=e^{\alpha r}dt^2-(Ae^{m_1r}+Be^{m_2r})^{-4}e^{\alpha r}dr^2 - (Ae^{m_1r}+Be^{m_2r})^{-2}d\sigma^2, \tag{32}$$

where α, A are arbitrary constants, m_1 and m_2 are given by (27) and (28), and B satisfies (30), or

$$ds^2=e^{\alpha r}dt^2-(A\pm r)^{-4}e^{-\alpha r}dr^2 - (A\pm r)^{-2}e^{-\alpha r}d\sigma^2. \tag{33}$$

By obvious transformations, it is possible to put (33) into the isotropic form

$$ds^2=e^{\alpha r}dt^2-r^{-4}e^{-\alpha r}(dr^2+r^2d\sigma^2). \tag{34}$$

Using the transformation $r=1/\bar{r}$, (34) will, after dropping the bar, yield the line element

$$ds^2=e^{\alpha/r}dt^2-e^{-\alpha/r}(dr^2+r^2d\sigma^2), \tag{35}$$

a line element which was found by Yilmaz.[5] It is also contained in Buchdahl's class of solutions as a limiting case.

Returning to the general line element (32), the determination of a class of solutions which are all asymptotically flat will be anticipated by requiring

$$W=Ae^{m_1r}+Be^{m_2r} \tag{36}$$

to be equal to zero at $r=0$, and then transforming $r=0$ to the point at infinity by the inversion $r=1/\bar{r}$. This implies $A=-B$, and coupled with (30) yields

$$A=-B=1/(\alpha^2+2\mu)^{1/2}. \tag{37}$$

Hence,

$$W=e^{\alpha r/2}\sinh(\gamma r)/\gamma, \tag{38}$$

where

$$\gamma=(\alpha^2+2\mu)^{1/2}/2. \tag{39}$$

The inversion $r=1/\bar{r}$ will, after dropping the bar, lead to the line element

$$ds^2 = e^{\alpha r^{-1}}dt^2 - e^{-\alpha r^{-1}}[\gamma r^{-1}/\sinh(\gamma r^{-1})]^4 dr^2 - e^{-\alpha r^{-1}}[\gamma r^{-1}/\sinh(\gamma r^{-1})]^2 r^2 d\sigma^2, \quad (40)$$

a line element which is clearly asymptotically flat at $r=\infty$. When $\gamma=0$, (40) reduces to the Yilmaz line element. When $\alpha=0$, (40) is equivalent to one found by Szekeres.[6] The Buchdahl class of solutions can be obtained by transforming (40) to isotropic form by means of the transformation

$$e^{\gamma r^{-1}} = \left(1+\frac{k}{r}\right)\Big/\left(1-\frac{k}{r}\right), \quad (41)$$

where k is an arbitrary constant. Hence (40) contains all solutions of this class of solutions, and the class can be established by elementary analysis.

Case II. The Missing Class of Solutions: $V'=0$, $\dot{V}\neq 0$. Under these conditions, (13) implies $\dot{V}$ = constant. Without loss of generality, we may take $\dot{V}=1$. The field equations (9)–(11) imply

$$\nu' + \lambda' = \mu r e^{\lambda-\nu}, \quad (42)$$

$$\nu' - \lambda' = 2(e^{\lambda}-1)/r. \quad (43)$$

By differentiating (42) and using (43), ν can be eliminated to give

$$\lambda'' + \frac{3\lambda'}{r}(e^{\lambda}-1) + \frac{2}{r^2}(e^{\lambda}-1)(e^{\lambda}-2) = 0, \quad (44)$$

which can be reduced to a first-order differential of Abelian type. Unfortunately not too much seems to be known about the solutions of this type of differential equation. There is, of course, one trivial solution of (44), $e^{\lambda}=2$. This leads to the line element

$$ds^2 = \mu r^2 dt^2 - 2dr^2 - r^2 d\sigma^2. \quad (45)$$

Although an explicit solution for the gravitational potentials in this coordinate system is unavailable, Eqs. (42) and (43) can be used to find the first few terms of the Taylor expansion of these solutions at any point in space. We illustrate for the point $r=0$. The substitution

$$x = \tfrac{1}{2}\mu r^2 \quad (46)$$

reduces (42) and (43) to

$$\frac{d\nu}{dx} + \frac{d\lambda}{dx} = e^{\lambda-\nu}, \quad (47)$$

$$\frac{d\nu}{dx} - \frac{d\lambda}{dx} = (e^{\lambda}-1)/x. \quad (48)$$

If $\lambda=\lambda(x)$ is to be regular at $x=0$, then $\lambda(0)=0$. Without loss of generality it is possible to require $\nu(0)=0$. Letting a prime represent differentiation with respect to x, (47) and (48) become

$$\nu'(0) + \lambda'(0) = 1, \quad (49)$$

$$\nu'(0) - \lambda'(0) = \lim_{x\to 0}[(e^{\lambda}-1)/x] = \lambda'(0). \quad (50)$$

Hence $\lambda'(0)=\frac{1}{3}$, $\nu'(0)=\frac{2}{3}$. Differentiating (47) and (48) and repeating this procedure yields $\lambda''(0)=-\frac{7}{45}$, $\nu''(0)=-\frac{8}{45}$. This implies

$$\lambda(x) = \tfrac{1}{3}x - \tfrac{7}{90}x^2 + \cdots \quad (51)$$

$$= \tfrac{1}{6}\mu r^2 - \tfrac{7}{360}\mu^2 r^4 + \cdots, \quad (52)$$

$$\nu(x) = \tfrac{2}{3}x - \tfrac{4}{45}x^2 + \cdots \quad (53)$$

$$= \tfrac{1}{3}\mu r^2 - \tfrac{1}{45}\mu^2 r^4 + \cdots. \quad (54)$$

In turn,

$$e^{\lambda} = 1 + \tfrac{1}{6}\mu r^2 - \tfrac{1}{180}\mu^2 r^4 + \cdots, \quad (55)$$

$$e^{\nu} = 1 + \tfrac{1}{3}\mu r^2 + \tfrac{1}{30}\mu^2 r^4 + \cdots. \quad (56)$$

Since the energy-momentum tensor T^i_j satisfies the relationships $T^1_1 = T^2_2 = T^3_3 = -T^4_4$, $T^i_j = 0$, $i\neq j$, this class of solutions must represent a perfect fluid for which this coordinate system is comoving. It is, however, a perfect fluid of a special type because the pressure and density must be equal.

III. CONCLUSION

Up to coordinate transformations, all static spherically symmetric solutions of the field equations $R_{ij} = -\mu V_i V_j$ have now been found. In order to find other spherically symmetric solutions, nonstatic gravitational potentials must be considered. Even in this situation, it is possible to show that a coordinate system always exists for which $V_i = (1,0,0,0)$ or $V_i(0,0,0,1)$. Just as in the static case, two separate classes of solutions will exist depending on whether it is assumed that V_i is a spacelike or timelike vector.

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