Rapid Communications

The Rapid Communications section is intended for the accelerated publication of important new results. Manuscripts submitted to this section are given priority in handling in the editorial office and in production. A Rapid Communication may be no longer than $3\frac{1}{2}$ printed pages and must be accompanied by an abstract. Page proofs are sent to authors, but, because of the rapid publication schedule, publication is not delayed for receipt of corrections unless requested by the author.

Confinement potential and exact solution of the Nambu equation

J. P. Hsu
Physics Department, Southeastern Massachusetts University, North Dartmouth, Massachusetts 02747
(Received 8 May 1981)

By solving Nambu's equation for a simple bound state, we demonstrate that a confinement potential of the form re^{-br} between two particles can be understood as the potential due to the transmission of a bound-state-like field. Such a potential satisfies physical boundary conditions at infinity and can tightly confine particles if b is sufficiently small. Some experimental implications are discussed.

The consequences of quantum chromodynamics (QCD) are very difficult to compare with the enormous amounts of data of strong interactions, especially at low energies. Even the qualitative features of the confinement of quarks have not yet been understood satisfactorily on the basis of the QCD theory. The motivation for studying the Nambu equation[1] for a simple bound state is as follows: It appears that one cannot work out quantitative consequences for QCD without having a clear qualitative picture of field theory beyond perturbation expansions. To get such a picture, one may study the qualitative behavior of Yang-Mills theory[2] in a simplified situation such as that in $2+1$ dimensions.[3] However, it is not clear how to generalize such a qualitative picture in $2+1$ dimensions to the physical $3+1$ dimensions. In view of this, it is worthwhile to study a different simplified situation, namely, a bound state in which the couplings of fields are as simple as possible and the dimensions of physical space remain intact. To study the amplitude of a bound state within the field-theoretic framework, one must consider an approximation which is better than the power-series expansion. Bound states are in general described by the Bethe-Salpeter equation.[4] For our purpose, we only consider a very simple bound-state equation which was first proposed by Nambu in 1950.[1] We are able to obtain a potential of the form re^{-br} by solving the Nambu equation in a particular time-independent limit. This potential between two particles can be related to a force transmitted by a field described by the Nambu equation, just as the Yukawa potential between two nucleons is due to the exchange of a meson described by the Yukawa equation $(\nabla^2-a^2)\phi=0$. This type of potential with small b resembles that of a string model and can provide a very tight confinement of particles.[5]

Let us consider a simple Lagrangian involving the fermion ψ, two massive scalar particles Φ_a and Φ_b, and a massless scalar particle U:

$$\mathfrak{L}=-\bar{\psi}(\gamma_\mu\partial_\mu+m)\psi-\sum_{i=a,b}(\partial_\mu\Phi_i^\dagger\partial_\mu\Phi_i+m_i^2\Phi_i^\dagger\Phi_i)-\tfrac{1}{2}(\partial_\mu U)^2+\sum_{i=a,b}G_i\bar{\psi}\psi(\Phi_i+\Phi_i^\dagger)+g_a\Phi_a^\dagger\Phi_a U+g_b\Phi_b^\dagger\Phi_b U\ . \tag{1}$$

The ladder-approximation scattering amplitude between Φ_a and Φ_b is given by $\langle a'b'|S|ab\rangle$, i.e.,[4]

$$(-i)^2 g_a g_b\int d(12)A_a'^\dagger(1)A_b'^\dagger(2)D(x_1-x_2)B_a(1)B_b(2)$$
$$+(-i)^4(g_ag_b)^2\int d(1234)A_a'^\dagger(1)A_b'^\dagger(2)D(x_1-x_2)\Delta_a(x_1-x_3)\Delta_b(x_2-x_4)D(x_3-x_4)B_a(3)B_b(4)+\cdots\ , \tag{2}$$

where $d(12\ldots)=d^4x_1d^4x_2\ldots$, $A_a'^\dagger(1)=\langle a'|\Phi_a^\dagger(x_1)|0\rangle$, $A_b'^\dagger(2)=\langle b'|\Phi_b^\dagger(x_2)|0\rangle$, $B_a(1)=\langle 0|\Phi_a(x_1)|a\rangle$, $B_b(2)=\langle 0|\Phi_b(x_2)|b\rangle$, etc.; $D(y)$ and $\Delta_{a,b}(y)$ are, respectively, the propagators of U and $\Phi_{a,b}$.

The Feynman amplitude is defined by

$$\mathfrak{F}(12,ab)=B_a(1)B_b(2)-g_ag_b\int d(34)\Delta_a(x_1-x_3)\Delta_b(x_2-x_4)D(x_3-x_4)B_a(3)B_b(4)+\cdots\ . \tag{3}$$

It describes the bound state and satisfies the Nambu equation

$$[(\partial_{1\mu}^2 - m_a{}^2)(\partial_{2\mu}^2 - m_b{}^2) - g_a g_b D(x_1 - x_2)]\, \mathfrak{F}(12,ab) = 0 \ , \qquad \partial_{1\mu} = \partial/\partial x_{1\mu}, \quad \partial_{2\mu} = \partial/\partial x_{2\mu} \ , \tag{4}$$

where the inhomogeneous term $B_a(1)B_b(2)$ is dropped.[4] We note that Eq. (4) is also valid for scattering states. In terms of the center-of-mass coordinate X_μ and the relative coordinate x_μ, one can write $\mathfrak{F}(12,ab)$ in the form

$$\mathfrak{F}(12,ab) = \exp(iP\cdot X)B(x) \ , \tag{5}$$

$$X_\mu = (m_a x_1 + m_b x_2)_\mu/(m_a + m_b), \quad x_\mu = (x_1 - x_2)_\mu \ , \tag{6}$$

where $P_\mu{}^2 = (p_1 + p_2)_\mu{}^2 = -m^2$ is the mass squared of the bound state. It follows from (5) and (6) that the Nambu equation can be written as

$$\left\{\left[\left(\frac{iP_\mu m_a}{m_a + m_b} + \frac{\partial}{\partial x_\mu}\right)^2 - m_a{}^2\right]\left[\left(\frac{iP_\nu m_b}{m_a + m_b} - \frac{\partial}{\partial x_\nu}\right)^2 - m_b{}^2\right] - g_a g_b D(x)\right\} B(x) = 0 \ . \tag{7}$$

We are interested in a static solution of (7). Let us consider a special time-independent limit in which the Nambu equation (7) takes the form[6]

$$\{[\nabla^2 - m_a{}^2(1-\eta^2)][\nabla^2 - m_b{}^2(1-\eta^2)] - g_a g_b D(\vec{r})\} B(\vec{r}) = 0, \quad r > 0 \ , \tag{8}$$

where we have chosen a particular frame of reference in which $\vec{P} = 0$ and $P_4 = i(m_a + m_b)\eta$, $0 \leq \eta \leq 1$. In general we have $P_\mu{}^2 = -(m_a + m_b)^2\eta^2$, so that small values of η correspond to strong binding between Φ_a and Φ_b, and $\eta \approx 1$ implies weak binding. In solving a field equation, one usual assumes a particular form for the solution because there is a rich array of solutions for field equations in general.[7] Based on physical boundary conditions at infinity, we expect the amplitude $B(\vec{r})$ for a bound state to damp like e^{-br}, $b > 0$, for large r. Therefore, we assume a spherically symmetric solution with the form

$$B(\vec{r}) = Ar^n e^{-br}, \quad \vec{r} = (x,y,z) \ . \tag{9}$$

We have a solution if and only if $n = 1$. In this case, Eqs. (8) and (9) lead to the result

$$\begin{aligned} &r^{-1}[12b^2 - 2(1-\eta^2)(m_a{}^2 + m_b{}^2) - g_a g_b/4\pi] \\ &+ 4b[2b^2 - (m_a{}^2 + m_b{}^2)(1-\eta^2)] \\ &+ r[b^2 - m_a{}^2(1-\eta^2)][b^2 - m_b{}^2(1-\eta^2)] = 0, \quad r > 0 \ , \end{aligned} \tag{10}$$

$$D(\vec{r}) = 1/(4\pi^2 r^2) \ .$$

When $m_a = m_b$ we have a solution

$$B(\vec{r}) = Are^{-br} \ , \tag{11}$$

where

$$b^2 = m_a{}^2(1-\eta^2) = m_b{}^2(1-\eta^2) = g_a g_b/(32\pi^2) \ .$$

If $m_a \neq m_b$ one does not have a solution of the form (9).

Since both ends of the Φ_a (Φ_b) line can be attached to the ψ ($\bar{\psi}$) line in Feynman diagrams, the static solution be interpreted as the potential between two fermions. The reason is as follows: The time-independent wave function $B(\vec{r})$ of the bound-state-like field is described by Eq. (8), just like the time-independent wave function of the meson field is described by the Yukawa equation $(\nabla^2 - a^2)\phi(r) = 0$. When one described the interaction between two particles in terms of a potential, one naturally requires that when the potential is inserted into a Schrödinger equation to compute the scattering amplitude, it should reproduce the field-theoretic S matrix in the nonrelativistic range. In this way, the solution $\phi(r) \propto e^{-ar}/r$ of the equation $(\nabla^2 - a^2)\phi = 0$ can be understood as the potential transmitted by a meson field. Similarly, the solution Are^{-br} can be interpreted as the potential transmitted by a bound-state-like field described by Eq. (8). A major difference between these two potentials is that the Yukawa potential can be obtained in perturbation theory, while the confinement potential (11) cannot be obtained in the usual perturbation theory. In general, a potential obtained in perturbation theory will be singular at $r = 0$.

The Lagrangian (1) is evidently too simple to be realistic and too naive to be taken seriously. Nevertheless, we stress that the essential equation for the confinement potential in our discussion is the time-indepenent field equation (8) rather than the Lagrangian (1). We note that any field described by the time-independent Nambu equation (8) can lead to the static confinement potential (11).

To illustrate an essential difference in the qualitative picture between field theories in two spatial dimensions and those in three spatial dimensions, let us consider the Nambu equation in two spatial dimensions. We have

$$\left\{\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - m_a{}^2(1-\eta^2)\right]\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - m_b{}^2(1-\eta^2)\right] - g_a g_b D(\vec{r})\right\} B(\vec{r}) = 0 \ , \tag{12}$$

where $\vec{r} = (x,y)$. If one substitutes the solution of the form (9) with $\vec{r}=(x,y)$, we have

$$n^2(n-2)^2r^{n-4}-\{(n-1)[b(2n+1)(n-1)-bn^2]+bn^2(n-2)\}r^{n-3}+\cdots \quad (13)$$

apart from other terms. These two terms cannot be canceled by other terms and they cannot vanish at the same time for positive b and real n. Thus one does not have a solution of the form (9) for the case of two spatial dimensions, in sharp contrast to the case of three spatial dimensions.

Suppose that the potential between quarks in the charmonium can be approximately described by the potential (11). With the help of the experimental results for the mass spectrum of the charmonium, we can estimate the constants A and b in the potential (11). We find that

$$B(\vec{r})=(0.212\ \mathrm{GeV}^2)re^{-br},\quad b=1.55\ \mathrm{MeV}\ . \quad (14)$$

by inserting (11) into the Schrödinger equation with charmed quark mass $m_c=1.152$ GeV. With these values for A and b, we can fit the charmonium mass spectrum within 5% for S, P, and D states. For example, the numerical values of the total mass (i.e., $2m_c+E$) for 1S, 2S, 3S, and 4S states are 3095, 3684, 4164, and 4598 MeV, respectively. This is to be compared with the experimental results 3097, 3685, 4030, and 4415 MeV.[8] We see that the potential (14) is a rather good approximation. It should be stressed that the parameter b in (14) is small and, therefore, bound-state energies of low-lying charmonium states should be unaffected by the presence of the exponential. Also, in order to improve the fit of the 3S and 4S states one must include the most important correction that due to the fact that they are above the threshold for bare-charm production. Of course, there are other corrections such as the relativistic correction, etc. The results of computer calculations show that the charmonium mass spectrum is sensitive to the parameter A but insensitive to the values of the parameter b, as long as b is smaller than 1.55 MeV.

We note that the physical boundary condition for fields at infinity requires $b>0$. It is very likely that this is true in general in a field-theoretic framework and, therefore, the confinement of particles cannot be absolute in principle. Of course, if b is sufficiently small, we may not be able to detect isolated particles (quarks). Based on (14) one may roughly estimate the energy E_{sq} needed to separate quarks by calculating the maximum potential energy. The maximum of the potential $B(r)$, determined by $dB(r)/dr=0$, occurs at $r_0=1/b$. This gives

$$E_{\mathrm{sq}}\sim B(r_0)\gtrsim 50\ \mathrm{GeV},\quad b\lesssim 1.5\ \mathrm{MeV}\ . \quad (15)$$

Since the value of b cannot be determined accurately, we are unable to obtain a reliable answer for the value of E_{sq} by fitting the charmonium mass spectrum.

This paper describes a small step beyond perturbation theory and gives the salient features of a potential transmitted by a bound-state-like field which satisfies Eq. (8). The significant result is that the time-independent field equation (8) leads to a potential which can tightly confine particles. Such a tight (but not absolute) confinement potential appears to be in harmony with rare events of detecting fractional charges in the experiments of Millikan and of Fairbank and collaborators.[9] According to the present investigation, the essential equation for the confinement potential is the Nambu equation (8). Presumably, a realistic model for quark confinement should have this type of equation in static limit. It is hoped that such discussion will shed light on our ultimate goal of understanding confinement in QCD. We have considered only simple scalar fields in the paper, a detailed study of Yang-Mills fields and fermions with more complicated gauge-invariant couplings is desirable.

Notes added in proof. (a) Quark potential with $V(r)\to 0$ for large r has been considered as a screening effect of the original potential $\propto r$ by the creation of a quark-antiquark pair within the context of infrared confinement. See J. Kogut and L. Susskind, Phys. Rev. Lett. 34, 767 (1975). (b) The considerations in the paper can also be applied to the straton model; see Hung-yuan Tzu, in *Proceedings of the 1980 Guangzhou Conference on Theoretical Particle Physics* (Science Press, Beijing, 1980), Vol. 1, p. 4. For an interesting discussion of the equal-time Bethe-Salpeter equation [related to Eqs. (7) and (8)], see Ruan Tu-nan, Zhu Hsi-quen, Ho Tso-xiu, Qing Cheng-rui, and Chao Wei-qin, *ibid.*, Vol. 2, p. 1390. (c) The author wishes to thank Professor K. C. Chou, Professor H. M. Fried, Professor T. C. Hsien, and colleagues at Brown University for useful discussions.

The author should like to thank the help of M. D. Xin and T. M. Yan. This work was supported in part by NRC and in part by Southeastern Massachusetts University.

[1]Y. Nambu, Prog. Theor. Phys. (Kyoto) 5, 614 (1950).

[2]C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).

[3]R. P. Feynman, Caltech report, 1981 (unpublished).

[4]For a comprehensive discussion of bound states, see K. Nishijima, *Fields and Particles* (Benjamin, New York, 1969), pp. 250–300.

[5]Y. Nambu, Sci. Am. 235 (No. 5), 48 (1976). In this article on the confinement of quarks various mechanisms and models are excellently discussed.

[6]In this limit, the propagator $D(x)$ takes the form $D(\vec{r}) = D(\vec{r}, 0)$. Note that this is consistent with the approach of Faustov and others, in which the relative time $x_0 = x_4/i$ is defined to be zero and one single time is used to describe the two-particle system in the center-of-mass frame. See, for example, R. N. Faustov, Ann. Phys. (N.Y.) 78, 176 (1973); J. P. Hsu and T. N. Sherry, Found. Phys. 10, 57 (1980). The latter gives a more general discussion of one single common time for describing many-particle systems within the four-dimensional symmetry framework. We may remark that the time-independent limit discussed in this paper is different from the nonrelativistic limit of instantaneous interaction for the U field. (This limit will give an effective potential $\propto 1/r$ between Φ_a and Φ_b.)

[7]See, for example, J. P. Hsu, Phys. Rev. Lett. 36, 1515 (1976); C. N. Yang and T. T. Wu, Phys. Rev. D 13, 3233 (1976).

[8]Particle Data Group, Rev. Mod. Phys. 52, S1 (1980). For comparison, our numerical results of the total mass for $1P$, $2P$, $1D$, and $2D$ states are, respectively, 3440, 3951, 3738, and 4201 MeV.

[9]Historically, fractional charges of $\sim 2e/3$ and $\sim e/3$ have been reported by Ehrenhaft (in 1941) and Millikan (in 1910), respectively. For an interesting discussion of these events, see P. A. M. Dirac, in *The Physicist's Conception of Nature,* edited by J. Mehra (Reidel, Boston, 1973), pp. 12–14. For recent detections of fractional charge of $e/3$, see G. S. LaRue, J. D. Phillips, and W. M. Fairbank, Phys. Rev. Lett. 46, 967 (1981), and references therein.