Dynamically broken chiral symmetry with bag confinement

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We lay the groundwork for a model of low-lying mesons in the spirit of the MIT bag model but which also takes into account dynamical chiral-symmetry breaking. We depart from the static cavity approximation in that we dynamically break chiral symmetry prior to confinement and we confine only in the $q\bar{q}$ relative coordinate. We show that single-gluon exchange in the $q\bar{q}$ system is a strong component of the force that drives the symmetry breaking. We verify that our approximations respect the chiral Ward identity prior to confinement. For the pion, we find that our boundary condition on the relative coordinate results in a small chiral breaking in that it raises the pion mass from zero to ~120 MeV.

I. INTRODUCTION

The MIT bag model of quark confinement[1] with perturbative quantum-chromodynamic (QCD) corrections has been quite successful in reproducing the observed hadron spectrum with the notable exception of the pion.[2] This state tends to come out much too massive,[3] but more importantly, there is no evidence in the calculated structure for any (approximate) Goldstone-boson characteristics. The many successes of current algebra[4] [PCAC (partial conservation of axial-vector current)] demand that such a picture be considered unacceptable. Some progress on the mass problem has recently been made by Donoghue and Johnson.[5] They note that for a light state, such as the pion, it is important to remove center-of-momentum fluctuation contributions to the energy of the state calculated using the MIT bag method, as this method does not calculate the system energy in an eigenstate of total momentum. On the other hand, there are compelling reasons to adopt the view described by Pagels and by Pagels and Stokar[6] in which the pion is a consequence of dynamical symmetry breaking in QCD and in which the current-algebra quantity f_π (pion decay constant) can be calculated with confidence.

A major shortcoming in the MIT bag treatment of the pion is the inadequate treatment of the strong $q\bar{q}$ binding in this channel. Their pion owes its existence to the chiral-breaking confining force which gives a large mass to the π-ρ system. The pion mass is subsequently lowered via a spin splitting effect from gluon exchange. But gluon exchange is a chiral-invariant force, it is attractive in the pion channel, and if it is sufficiently attractive to bind a chiral multiplet to negative mass squared,[7] then it can destabilize the vacuum and leave a Goldstone pion via the Nambu-Jona-Lasinio mechanism.[8] Therefore we believe that the important elements are present in the MIT bag model but that the order of importance of the various forces should be reconsidered.

In this paper we follow up on the idea that chiral-invariant forces at medium distances in QCD (intermediate between the asymptotically free region and confining distances) are responsible for the dynamical symmetry breaking. We then subsequently impose a confining boundary condition to the otherwise Goldstone wave function. In neither the original MIT bag,[1] the improved pion bag of Donoghue and Johnson,[5] nor the present model is any attempt made to maintain the axial-vector-current conservation at the surface of the bag. However, this is far less serious a problem in our model as we shall attempt to show.

The MIT bag picture[1] of hadrons may be described as an effective Lagrangian density of the form

$$L = L_K + L^F_{\rm QCD} + L_B + L_{{\rm U(1)}_A}, \qquad (1.1)$$

where L_K consists of quark and gluon kinetic terms, $L^F_{\rm QCD}$ contains the conventional quark-gluon and gluon-gluon interaction terms of QCD which are usually treated in the perturbative fashion of loopwise Feynman-diagram expansions, and L_B describes the confining effect of the bag volume energy density which is believed[9] to arise from path-integral effects which are not obtainable even by summing to all orders of a perturbation theory using $L^F_{\rm QCD}$. $L_{{\rm U(1)}_A}$ describes small instanton effects which break the axial U(1) symmetry. The calculational approach of the MIT group has been to use $L = L_0 + L_1$ where

$$L_0 = L_K + L_B, \quad L_1 = L^F_{\rm QCD} + L_{{\rm U(1)}_A} \qquad (1.2)$$

and L_0 is treated nonperturbatively, while L_1 is treated as a perturbation. ($L_{{\rm U(1)}_A}$ effects were added by Horn and Yankielowicz.[3]) However, since L_B violates chiral symmetry, L_0 is not chiral invariant even for zero quark mass. Spontaneous-symmetry-breaking–induced PCAC never has a chance to appear due to the obliteration of the initial invariance in L_K. Some effort is now being made[10] to find a chiral-invariant form of L_B.

Here we consider an alternative calculational grouping procedure (in the meson sector only),

$$L_0 = L_K + L^F_{\rm QCD} + L_{{\rm U}(1)_A}, \quad L_1 = L_B \tag{1.3}$$

treating the effects of *confinement* as a small effect where appropriate. Now L_0 *is* chiral invariant for vanishing quark masses, and we expect that if L_0 is treated carefully enough to find bound states (e.g. in mean-field perturbation theory), then, for the effective value of the strong coupling constant $\alpha_s > \alpha_s^{\rm crit}$ some critical value to be determined dynamically, spontaneous symmetry breaking (SSB) will occur and we will be able to find a Goldstone pion. Note that there is a consistency check on this procedure: Since α_s decreases as the appropriate length scale ($\equiv \Lambda^{-1}$) decreases; we must find such Goldstone pions with a natural size sufficiently small so that confinement effects can be neglected on that length scale and yet sufficiently large such that $\alpha_s > \alpha_s^{\rm crit}$. For mesons with a small enough spatial extent it is clear that the effects of L_B must be small. This chiral-breaking L_B will also affect the usual PCAC relations; in particular, inferred current quark masses may differ from those obtained in the usual PCAC analyses.

A few comments may be in order on the nature and origin of L_B. Actually, as far as is presently conjectured,[9] the object in full QCD that appears is $L_{\rm inst}$, not L_B, where the former may represent the effects of instantons.[11] However, all of the effects of $L_{\rm inst}$ vanish if only one quark is massless,[11] and we start here from a chiral-SU(2)-invariant L_K in which both the up and down quarks are massless. (Actually we only need these quark masses to be sufficiently small so that the chiral-symmetry-breaking effects that they induce are negligible compared to everything else on the scale of SSB effects.) Furthermore, such instanton effects would be expected to leave the pion massless as they, and so any confinement produced by them, violate chiral U(2) but in a chiral-SU(2)-preserving manner.

Recall, however, that the quark masses are scale-dependent quantities, with variation determined by the renormalization group,[12] and their values grow with increasing distance scale. Such growth itself would have a confinementlike effect (at least, until their own growth freezes the variation with scale of the fermion self-energy graphs in the manner of the Applequist-Carazzone decoupling theorem[13]). This effect can give rise to a nonzero pion mass since it is an explicit chiral-SU(2)-symmetry-breaking effect proportional to the "ultraviolet" (current-algebra) quark masses.[4,12] We will return to this point in Sec. IV.

It should be noted that there may be more than one mechanism that can produce confinement, that are active on different scales. We have in mind the instanton effects[9] and the growing coupling constant analyzed by Ball and Zachariasen.[14] The one with the shortest length scale will dominate the spectrum, shielding any others which could then only show up in large-momentum-transfer scattering. It is very important to maintain $L_{\rm inst}$ effects since, as shown by 't Hooft,[11] they provide the chiral-U$(1)_A$ breaking observed in the π-η mass splitting. Horn and Yankielowicz[3] examined this quantitatively for instantons of scale $\rho \lesssim \mu^{-1}$ and found consistency with the experimental mass values. We neglect the π-η splitting interaction here. Any further attraction of this type in the pion channel must be included along with gluon exchange as part of the driving force for dynamical symmetry breaking.[15]

The outline of the rest of this paper is as follows: In Sec. II, we display the gluon-exchange Bethe-Salpeter description of the $q\bar{q}$ bound states using mean-field perturbation theory. Cutoffs and approximations are introduced to make the problem tractable, and the wave function for the massless pion is obtained in the SSB regime. In Sec. III, we study the definition of the electromagnetic current and the corresponding Ward identity. We then formulate the axial-vector current and examine its anomalous Ward identity. The divergence of the current is used to identify the pion decay constant f_π. In Sec. IV, confining boundary conditions for a relative-coordinate bag are quoted[16] and applied to the wave function from Sec. II. The result is the energy of the pion state as a function of bag size. The bag volume energy is added to this and the total minimized to find the pion mass. Finally, Sec. V briefly summarizes our results and describes our conclusions.

II. SPONTANEOUSLY BROKEN CHIRAL-SYMMETRIC WAVE FUNCTIONS

Our basic idea as described above is to take into account the known short-range $q\bar{q}$ forces to dynamically break chiral symmetry before imposing confinement. In this section we obtain wave functions for $q\bar{q}$ states to replace the free wave functions used in the usual bag model. We propose to calculate the one-particle-irreducible generating functional $\Gamma[\psi,\bar{\psi}]$ in an approximation that has the following properties: (i) it is capable of producing a dynamically broken vacuum, (ii) the strength and range of the $q\bar{q}$ forces can be related to the masses, couplings and wave functions of the pseudoscalar Goldstone bosons and the other mesons that arise naturally (scalars, vectors, and axial vectors), (iii) the approximation does not introduce explicit chiral breakings, i.e., it respects the axial-vector-current Ward identity, and (iv) it is tractable.

We take a phenomenological approach to this, patterned after the Nambu–Jona-Lasinio (NJL) model.[8] The NJL model remains the only really tractable approximation to dynamical symmetry breaking. The simplicity stems from the fact that their interaction term is a four-fermion point coupling. That is, of course, not the case for QCD but we can argue that a separable interaction is a sensible approximation and this gives us the needed simplicity. Approximations of this type have been developed by a number of authors[17-20] including one of us (R.H.).[17] Janus and Cahill[19] and Janus[20] have compared the separable approximation to numerical solutions of the nonlinear Dyson equation for a $\phi^2\sigma$ model and found good agreement. More will be said about these developments in our conclusions, Sec. V.

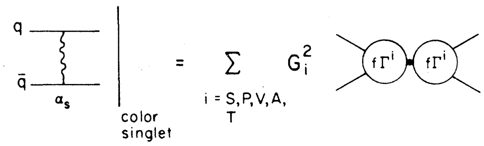

FIG. 1. Approximation of t-channel gluon exchange by a separable form representing the lowest bound states.

To be specific let us take the simplest $q\bar{q}$ force, i.e., the single-gluon exchange, to illustrate the approximation. The action A for gluon exchange can be written

$$A = \int d^4x\,\bar{\psi}(x)(i\gamma\cdot\partial - m_q)\psi(x) - \frac{\pi\alpha_s}{2}\int d^4x\,d^4y\,\bar{\psi}(x)\lambda^r\gamma_\mu\psi(x)\frac{-i}{4\pi^2(x-y)^2}\bar{\psi}(y)\lambda^r\gamma^\mu\psi(y)\,, \tag{2.1}$$

where m_q is the current quark mass, and λ^r are the Gell-Mann matrices. If we were to replace $1/(x-y)^2$ by $\delta^4(x-y)$ (times an appropriate dimensional constant) this would give essentially the NJL model.

Our approach is based on the following physical picture. The gluon-exchange kernel K in a $q\bar{q}$ Bethe-Salpeter equation is attractive in the meson channels (color singlets). It is possible in principle to make an eigenfunction expansion of K and for energies near an eigenvalue, K can be well represented by the outer product of eigenfunctions of the corresponding bound state. Although the ground work has been laid to solve the Bethe-Salpeter equation and the Schwinger-Dyson equation to generate symmetry breaking, we will not be that ambitious here. We use this argument to motivate our basic ansatz as shown in Fig. 1. We approximate the one-gluon exchange by a simple separable form in the meson channels. The specifics of this approximation are developed in this section.

A. Phenomenological action and generating functional

We will replace the action, Eq. (2.1), with the following:

$$\begin{aligned} A = &\int dX\,dx\,\bar{\psi}_2\{i\gamma\cdot\partial_2[\delta^4(x)+g(x)] - m_q\delta^4(x)\}\psi_1 \\ &+\tfrac{4}{9}\int dX\,dx\,dy\Big\{G_P{}^2[\bar{\psi}_2\tau^\alpha f(x)\psi_1][\bar{\psi}_4\tau^\alpha f(y)\psi_3] + G_P{}^2[\bar{\psi}_2\tau^\alpha i\gamma_5 f(x)\psi_1][\bar{\psi}_4\tau^\alpha i\gamma_5 f(y)\psi_3] \\ &\qquad - G_V{}^2[\bar{\psi}_2\tau^\alpha\gamma_\mu f(x)\psi_1][\bar{\psi}_4\tau^\alpha\gamma^\mu f(y)\psi_3] - G_V{}^2[\bar{\psi}_2\tau^\alpha\gamma_\mu\gamma_5 f(x)\psi_1][\bar{\psi}_4\tau^\alpha\gamma^\mu\gamma_5 f(y)\psi_3]\Big\}, \end{aligned} \tag{2.2}$$

where $x_1 = X = -x/2$, $x_2 = X + x/2$, $x_3 = X - y/2$, $x_4 = X + y/2$, and $\bar{\psi}_1 = \bar{\psi}(X - x/2)$ etc. α is to be summed from 0 to 3. In going from Eq. (2.1) to (2.2) there are a number of implicit summations and definitions that are suggested by Fig. 1. In Eq. (2.2), $(\bar{\psi}\tau^\alpha f(x)\psi)$ for example has the quantum numbers of a meson, whereas $(\bar{\psi}\lambda^r\gamma_\mu\psi)$ in Eq. (2.1) has the quantum numbers of a gluon. The function $f(x)$ is introduced in order to give a soft q–q–bound-state-meson vertex. This is taken to be a function with a simple form in momentum space:

$$f(x)=\int \frac{d^4p}{(2\pi)^4}\,e^{-ip\cdot x}\frac{\Lambda^2}{\Lambda^2-p^2}=\frac{i\Lambda^3K_1(\Lambda R)}{4\pi^2 R}\ , \tag{2.3}$$

where $R=(-x^{0^2}+x^2)^{1/2}$, and K_1 is the modified Bessel function. This shows an exponential cutoff in R with range $1/\Lambda$ in the spatial direction. As $\Lambda\to\infty$, $f(x)\to\delta^4(x)$. *The significance of Λ is that it is a measure of the range of the $q\bar{q}$ force that is relevant in determining the properties of the lowest-lying bound state.* (For example, although the Coulomb potential has an infinite range, only a finite range has a significant bearing on the lowest bound state.) The coupling strengths are determined in terms of α_s and Λ from overlap integrals which we describe below. The gluon-exchange force gives equal couplings in the pseudoscalar and scalar channels and in the vector and axial-vector channels and zero in the tensor channel which has been incorporated into Eq. (2.2). The $\frac{4}{9}$ is a convenient factor [$=\frac{16}{9}$ (color-singlet projection) $\times\frac{1}{2}$ (flavor projection) $\times\frac{1}{2}$ (term in the action, not a graph)], which we choose not to include in the definition of the G's.

The action is now in the form in which one can use the NJL techniques to generate dynamical symmetry breaking via a self-consistent field approximation. We do this but we use the language of mean-field perturbation theory[21] or "$1/N$ expansion"[22] which are all equivalent for this problem. The idea is to introduce composite or constraint fields for the mesons in such a way that the action is bilinear in quark fields. One can then do the path integrations over quark fields leaving a nonlocal action in the meson fields that can be expanded in perturbation theory. In this way we obtain the one-particle-irreducible generating functional $\Gamma[\bar{\psi},\psi,$ meson fields$]$. We give only a few intermediate steps (see above references for more details). Add to A in Eq. (2.2) the term

$$\begin{aligned}-\tfrac{4}{9}\int dX\Bigg\{&\left[G_P\int dx\,\bar{\psi}\!\left(X+\frac{x}{2}\right)\tau^\alpha f(x)\psi\!\left(X-\frac{x}{2}\right)+\frac{3\mu}{2\sqrt{2}}\sigma^\alpha(X)\right]^2\\&+\left[G_P\int dx\,\bar{\psi}\!\left(X+\frac{x}{2}\right)\tau^\alpha i\gamma_5 f(x)\psi\!\left(X-\frac{x}{2}\right)+\frac{3\mu}{2\sqrt{2}}\pi^\alpha(X)\right]^2\\&-\left[G_V\int dx\,\bar{\psi}\!\left(X+\frac{x}{2}\right)\tau^\alpha\gamma_\mu f(x)\psi\!\left(X-\frac{x}{2}\right)+\frac{3\mu}{2\sqrt{2}}V_\mu^\alpha(X)\right]^2\\&-\left[G_V\int dx\,\bar{\psi}\!\left(X+\frac{x}{2}\right)\tau^\alpha\gamma_\mu\gamma_5 f(x)\psi\!\left(X-\frac{x}{2}\right)+\frac{3\mu}{2\sqrt{2}}A_\mu^\alpha(X)\right]^2\Bigg\}\ .\end{aligned}$$

Varying the action with respect to the meson fields gives constraints which relates them to bilinears in the quark fields. The mass parameter μ is introduced solely to give the meson fields conventional dimensions. The numerical factors are chosen for convenience. The resulting action is

$$A=\int dX\,dx\,\bar{\psi}\!\left(X+\frac{x}{2}\right)S^{-1}(X,x)\psi\!\left(X-\frac{x}{2}\right)-\frac{\mu^2}{2}\int dX\Big\{[\sigma^\alpha(X)]^2+[\pi^\alpha(X)]^2-[V_\mu^\alpha(X)]^2-[A_\mu^\alpha(X)]^2\Big\} \tag{2.4a}$$

where

$$\begin{aligned}S^{-1}(X,x)=\;&i\gamma\cdot\partial_2[\delta^4(x)+g(x)]-m_q\delta^4(x)\\&-f(x)[g_P\tau^\alpha\sigma^\alpha(X)+g_P\tau^\alpha i\gamma_5\pi^\alpha(X)\\&-g_V\tau^\alpha\gamma^\mu V_\mu^\alpha(X)-g_V\tau^\alpha\gamma^\mu\gamma_5 A_\mu^\alpha(X)]\end{aligned} \tag{2.4b}$$

and

$$g_P\equiv\frac{2\sqrt{2}}{3}G_P\mu\ ,\quad g_V=\frac{2\sqrt{2}}{3}G_V\mu\ , \tag{2.5}$$

and $x_1=X-x/2$, $x_2=X+x/2$ as before.

The one-particle-irreducible generating functional to lowest order in the mean-field approximation is

$$\Gamma[\psi,\bar{\psi},\sigma^\alpha,\pi^\alpha,V_\mu^\alpha,A_\mu^\alpha]=A-i\,\mathrm{Tr}\ln S^{-1}\ . \tag{2.6}$$

A is the "classical" term; the second term is a saddle-point correction. This is the starting point for all further calculations in this section.

From Eq. (2.6) we can get immediately the effective potential $V(\sigma)$:

$$V(\sigma)=-\Gamma(0,0,\sigma^\alpha(X)=\delta_{\alpha 0}\sigma,0,0,0)\ . \tag{2.7}$$

We only allow a nonvanishing minimum in the isoscalar part of the σ^α field. Before writing this out, however, we need to have the propagator S for a constant field σ.

B. Quark propagator

We have introduced two nonlocal functions $f(x)$, $g(x)$ in the propagator, Eq. (2.4). The role of $f(x)$ is to cut off the short-range core of the $q\bar{q}$

interaction. At short distances $f(x)$ is less singular than $\delta^4(x)$. We view this "soft" term as originating from QCD and expect that any calculation of it would also produce a "soft" contribution to the $i\gamma\cdot\partial$ term of S^{-1}. Hence we include a function $g(x)$ which we also take to be less singular than $\delta^4(x)$. We choose the form of $g(x)$ to be simple in momentum space,

$$g(x)=\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot x}\frac{(\lambda-1)\Lambda^2}{\Lambda^2-p^2}$$
$$=(\lambda-1)f(x)\,. \qquad (2.8)$$

This introduces a new parameter λ, but it can be eliminated in terms of the "dynamical mass" M_D which has been fit to e^+e^- data and will be discussed shortly.

We need to find S, fortunately only for a constant σ field. S satisfies

$$\int d^4x_2 S^{-1}(x_1,x_2)S(x_2,x_3)=\delta^4(x_1-x_3)\,. \qquad (2.9)$$

Since for constant σ, S and S^{-1} depend only on the difference coordinate, this is soluble in momentum space:

$$S^{-1}=\gamma\cdot p\frac{\lambda\Lambda^2-p^2}{\Lambda^2-p^2}-\frac{g_P\sigma\Lambda^2}{\Lambda^2-p^2}-m_q \qquad (2.10)$$

and

$$S=\frac{\gamma\cdot p(\lambda\Lambda^2-p^2)(\Lambda^2-p^2)+g_P\sigma\Lambda^2(\Lambda^2-p^2)+m_q(\Lambda^2-p^2)^2}{D(p^2)}\,, \qquad (2.11a)$$

where

$$D(p^2)=p^2(p^2-\lambda\Lambda^2)^2-[g_P\sigma\Lambda^2+m_q(\Lambda^2-p^2)]^2\,. \qquad (2.11b)$$

Consider for a moment the case in which $g(x)$ is absent ($\lambda=1$). The propagator has three poles. The lowest one we identify with the quark mass, the other two represent the cut in an approximate way. Clearly for $\lambda\neq 1$ there are still only three poles and any other form would proliferate this number. This simplicity guided our parametrization of $g(x)$.

At this point we set $m_q=0$ for simplicity since it does not directly play a large role in determining the pion mass in our approach. The added complication is straightforward and will be included in a global fit in another paper.

The minimum position of the effective potential will fix the value of σ:

$$\left.\frac{dV}{d\sigma}\right|_{\sigma=v}=0\,. \qquad (2.12)$$

From here on we substitute v for σ in the quark propagator.

The quark mass M is given as the smallest root of

$$D(p^2)\,|_{p^2=M^2}=0\,. \qquad (2.13)$$

Hence $D(p^2)$ can be written

$$D(p^2)=(p^2-M^2)(p^2-p_+^{\,2})(p^2-p_-^{\,2})\,, \qquad (2.14)$$

where ($m_q=0$)

$$p_\pm^{\,2}=-\frac{M^2}{2}+\lambda\Lambda^2\pm\frac{M}{2}(4\lambda\Lambda^2-3M^2)^{1/2} \qquad (2.15)$$

for fixed M, and these pole positions are determined by the single parameter $(\lambda\Lambda^2)$. Note from Eq. (2.11) that $g_Pv\neq M$, but rather the two are related by the equation

$$M^2(M^2-\lambda\Lambda^2)^2=g_P^{\,2}v^2\Lambda^4\,. \qquad (2.16)$$

As $\Lambda\to\infty$, $M\to g_Pv$.

For later reference, we define A,B ($m_q=0$),

$$S=\frac{\gamma\cdot pA(p^2)+MB(p^2)}{D(p^2)}\,, \qquad (2.17)$$

where $A(p^2)=(\lambda\Lambda^2-p^2)(\Lambda^2-p^2)$, $B(p^2)=(\lambda\Lambda^2-M^2)(\Lambda^2-p^2)$, and $D(p^2)$ is given by Eq. (2.14).

Our quark mass which we denote by M is the pole position in the propagator. The term dynamical mass M_D as used by Pagels and Stokar[6] is a short-distance quantity defined by

$$\frac{X(p^2)}{Y(p^2)}\to\frac{4M_D^{\,3}}{-p^2}\,,\quad p^2\to-\infty\,,$$

where

$$S^{-1}=\gamma\cdot pY(p^2)-X(p^2)\,.$$

M_D parametrizes the approach to the short-distance limit and it has been fit to e^+e^- data by Hagiwara and Sanda.[23] They obtain a value of $M_D\approx 240$ MeV. In terms of our parameter,

$$4M_D^{\,3}=g_Pv\Lambda^2=M(\lambda\Lambda^2-M^2)\,. \qquad (2.18)$$

This could be used to eliminate λ. Whether or not the present determination of M_D is reliable, we can use our value of M_D to compare with other determinations of it.

C. Effective potential; spontaneous symmetry breaking

The effective potential is given by Eqs. (2.6) and (2.7):

$$V(\sigma) = -6\, \mathrm{Tr}_\gamma i \int \frac{d^4p}{(2\pi)^4} \ln S^{-1}(p) + \frac{\mu^2}{2}\sigma^2 . \qquad (2.19)$$

(The factor of 6 comes from three colors times two flavors.) Wick rotating and evaluating the angular integrals we obtain ($m_q = 0$)

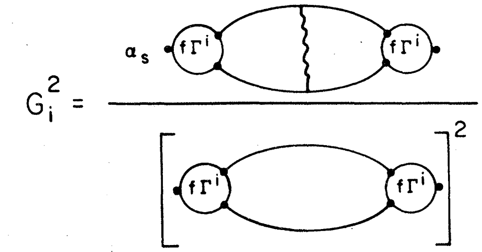

FIG. 2. Normalized overlap integrals for the determination of coupling strengths G_i.

$$V(\sigma) = -\frac{3}{4\pi^2}\int_0^\infty p^2 dp^2 \left\{ \ln[p^2(p^2+\alpha\Lambda^2)^2] + \ln\left(1 + \frac{g_P{}^2\sigma^2\Lambda^2}{p^2(p^2+\lambda\Lambda^2)^2}\right)\right\} + \frac{\mu^2}{2}\sigma^2 . \qquad (2.20)$$

The first logarithm gives an infinite but σ-independent term so it is irrelevant and we drop it. The remaining integral is finite. We then look for a minimum of $V(\sigma)$ for nonzero σ and find v ($=\langle\sigma\rangle$). This gives

$$\mu^2 = 6\frac{g_P}{v}\, \mathrm{Tr}_\gamma i \int \frac{d^4p}{(2\pi)^4}\frac{\Lambda^2}{\Lambda^2 - p^2} S(p) . \qquad (2.21a)$$

Using Eq. (2.16) we can eliminate v in favor of M and obtain

$$\frac{9}{8G_P{}^2} = \frac{\mu^2}{g_P{}^2} = \frac{3\Lambda^4\left[p_+{}^2p_-{}^2\ln\left(\frac{p_+{}^2}{p_-{}^2}\right) + M^2p_+{}^2\ln\left(\frac{M^2}{p_+{}^2}\right) + M^2p_-{}^2\ln\left(\frac{p_-{}^2}{M^2}\right)\right]}{2\pi^2(p_+{}^2 - M^2)(p_-{}^2 - M^2)(p_+{}^2 - p_-{}^2)} . \qquad (2.21b)$$

This gives one relation involving G_P, M, Λ, and λ.

As one decreases the mass M to zero, the interaction strength G_P needed to give that mass goes to a critical value $G_P^{(\mathrm{crit})}$. Taking $M \to 0$ in Eq. (2.21b), we find

$$G_P^{(\mathrm{crit})} = \left(\frac{3\pi^2\lambda}{4\Lambda^2}\right)^{1/2} . \qquad (2.22)$$

This is the value of G_P *at which dynamical symmetry breaking sets in.* G_P must be larger than this in order to obtain a finite constituent quark mass which we call M.

D. Determination of G_i in terms of gluon-exchange strength α_s

We now need to determine $G_{S,P,V,A,T}$ from the gluon-exchange graph which is proportional to α_s. We project the gluon-exchange graph on our separable form as indicated in Fig. 1. The coefficients $G_i{}^2$ are determined by closing this off with wave functions on the right and left as indicated in Fig. 2. Writing this out in momentum space we obtain

$$-\pi\alpha_s \frac{1}{(p-q)^2}(\gamma^\mu)_{13}(\gamma_\mu)_{42}\delta_{13}\delta_{42}$$

and

$$\frac{\Lambda^2}{\Lambda^2 - q^2}\frac{\Lambda^2}{\Lambda^2 - p^2}\tfrac{1}{2}(\tau^\alpha_{12}\tau^\alpha_{43})[G_S{}^2(I)_{12}(I)_{43} + G_P{}^2(i\gamma_5)_{12}(i\gamma_5)_{43} - G_V{}^2(\gamma_\mu)_{12}(\gamma^\mu)_{43}$$
$$- G_A{}^2(\gamma_\mu\gamma_5)_{12}(\gamma^\mu\gamma_5)_{43} + G_T{}^2(\sigma^{\mu\nu})_{12}(\sigma_{\mu\nu})_{43}] ,$$

where $p_1 = P/2 - q$, $p_2 = P/2 + q$, $p_3 = P/2 - p$, $p_4 = P/2 + p$, for scattering $1 + 2 \to 3 + 4$. We projected onto a color singlet, dropping a factor of $\frac{16}{9}$ from both sides, but displayed the flavor tensor.

We now calculate the overlap in the initial and final state with the wave function

$$\phi_j^\alpha(q) = S(q)\frac{\Gamma^j\Lambda^2\tau^\alpha}{\Lambda^2 - q^2}S(q) ,$$

where the Γ^j are the Dirac matrices. We make the projections at $P = 0$ and for chiral-symmetric wave functions, i.e., the strengths G_i are fixed prior to symmetry breaking. Hence $S(q) = \gamma\cdot q(\Lambda^2 - q^2)/(\lambda\Lambda^2 - q^2)$,

$$\pi\alpha_s \mathcal{O}_{j\beta,k\gamma} = \tfrac{1}{2}\sum_{i,\alpha} \delta_i G_i^2 N_{j\beta,i\alpha} N_{i\alpha,k\gamma}, \tag{2.23}$$

where

$$\mathcal{O}_{j\beta,k\gamma} \equiv -\Lambda^4 i\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{\mathrm{Tr}[S(p)\Gamma^j\tau^\beta S(p)\gamma^\mu S(q)\Gamma^k\tau^\gamma S(q)\gamma_\mu]}{(q-p)^2(\Lambda^2-p^2)(\Lambda^2-q^2)},$$

$$N_{j\beta,i\alpha} \equiv \Lambda^4 i\int \frac{d^4p}{(2\pi)^4}\frac{\mathrm{Tr}[S(p)\Gamma^j\tau^\beta S(p)\Gamma^i\tau^\alpha]}{(\Lambda^2-p^2)^2}, \tag{2.24}$$

and

$$\delta_i = 1 \text{ for } S,P,T, \quad \delta_i = -1 \text{ for } V,A.$$

These are diagonal in the space-time indices i,j, and the flavor indices α,β. Clearly the scalar ($\Gamma = I$) and pseudoscalar ($\Gamma = i\gamma_5$) cases are equal as are the vector ($\sigma = \gamma_\mu$) and axial vector ($\Gamma = \gamma_\mu\gamma_5$). This is just a result of chiral symmetry. Further $\mathcal{O}$ is zero for the tensor case ($\Gamma = \sigma_{\mu\nu}$) because the trace is zero before integrating. Therefore $G_S^2 = G_P^2$, $G_A^2 = G_V^2$, and $G_T^2 = 0$. Carrying out the flavor traces, the pseudoscalar case gives

$$\pi\alpha_s\mathcal{O}_P = G_P^2 N_P^2, \tag{2.25}$$

where

$$\mathcal{O}_P \equiv -\Lambda^4 i\int\frac{d^4q}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}\frac{\mathrm{Tr}[S(q)^2\gamma^\mu S(p)^2\gamma_\mu]}{(\Lambda^2-q^2)(\Lambda^2-p^2)(p-q)^2} \tag{2.26a}$$

$$= \frac{\Lambda^2}{16\pi^4\lambda^3}\left[(1-\lambda)^2 + \lambda\frac{\pi^2}{3}\right] \tag{2.26b}$$

and

$$N_P \equiv \Lambda^4 i\int\frac{d^4q}{(2\pi)^4}\frac{\mathrm{Tr}[S(q)^2]}{(\Lambda^2-q^2)^2} \tag{2.26c}$$

$$= \frac{\Lambda^2}{4\pi^2\lambda}. \tag{2.26d}$$

These integrals are evaluated in the Appendix. Finally, the relation between α_s and G_P is

$$\alpha_s = \frac{G_P^2\lambda\Lambda^2}{\pi[(1-\lambda)^2 + \lambda\pi^2/3]}. \tag{2.27}$$

If we take for G_P the smallest value that gives spontaneous symmetry breaking, i.e., the critical value $G_P^{(\mathrm{crit})} = (3\pi^2\lambda/4\Lambda^2)^{1/2}$, we obtain

$$\alpha_s^{(\mathrm{crit})} = \frac{3\pi}{4}\frac{\lambda^2}{(1-\lambda)^2 + \lambda\pi^2/3}. \tag{2.28}$$

We return to this result after establishing a reasonable range of λ.

Similarly we can find G_V,

$$\pi\alpha_s(\mathcal{O}_V g^{\mu\nu}) = -G_V^2(N_V g^{\mu\lambda})(N_V g_\lambda{}^\nu), \tag{2.29}$$

where

$$g_{\mu\nu}\mathcal{O}_V \equiv -\Lambda^4 i\int\frac{d^4q}{(2\pi)^4}\, i\int\frac{d^4p}{(2\pi)^4}\frac{\mathrm{Tr}[S(q)\gamma_\mu S(q)\gamma^\alpha S(p)\gamma_\nu S(p)\gamma_\alpha]}{(\Lambda^2-q^2)(\Lambda^2-p^2)(p-q)^2} \tag{2.30a}$$

$$= -\frac{\Lambda^2 g_{\mu\nu}}{64\pi^4\lambda^3}\left[(\lambda^2+1)\left(4-\frac{\pi^2}{3}\right) + (\pi^2-9)\right] \tag{2.30b}$$

and

$$g^{\mu\nu}N_V = \Lambda^4 i\int\frac{d^4p}{(2\pi)^4}\frac{\mathrm{Tr}\, S(p)\gamma^\mu S(p)\gamma^\nu}{(\Lambda^2-p^2)^2} \tag{2.30c}$$

$$= -\frac{\Lambda^2}{8\pi^2\lambda}g^{\mu\nu}. \tag{2.30d}$$

These integrals are evaluated in the Appendix. Using these results, we find

$$\frac{G_V^2}{G_P^2} = \frac{(\lambda^2+1)(4-\pi^2/3) + \lambda(\pi^2-9)}{(1-\lambda)^2 + \lambda\pi^2/3} = 0.710 - \frac{0.046\lambda}{(1-\lambda)^2 + \lambda\pi^2/3}. \tag{2.31}$$

This ratio varies only between 0.696 and 0.710 for any positive λ. In the NJL model, this ratio is 0.5 and thus this is a departure from a four-fermion interaction.

E. Meson propagators

The meson propagators are obtained from the generating functional Eq. (2.6). The pion propagator is, for example,

$$\left.\frac{\delta^2\Gamma}{\delta\pi^\beta(y)\delta\pi^\alpha(x)}\right|_{\text{sources off}} = -[\mathfrak{D}_P^{-1}(x,y)]_{\alpha\beta} \tag{2.32}$$

"Sources off" means that after taking functional derivatives of Γ, the classical fields are all set to zero with the exception of $\sigma^0 = v$. In momentum space we have

$$\mathfrak{D}_{(P,S)}^{-1}(P^2) = -6g_P{}^2\Lambda^4 i\int\frac{d^4q}{(2\pi)^4}\frac{\mathrm{Tr}\left[(i\gamma_5, I)S\left(q-\frac{P}{2}\right)(i\gamma_5, I)S\left(q-\frac{P}{2}\right)\right]}{(\Lambda^2-q^2)^2}+\mu^2, \tag{2.33}$$

where μ^2 is given by Eq. (2.21). Using Eq. (2.17) for S gives

$$\mathfrak{D}_{(P,S)}^{-1}(P^2) = -24\Lambda^4 g_P{}^2 i\int\frac{d^4q}{(2\pi)^4}\left[\frac{A\left(q-\frac{P}{2}\right)A\left(q+\frac{P}{2}\right)\left(q^2-\frac{P^2}{4}\right)\mp M^2B\left(q-\frac{P}{2}\right)B\left(q+\frac{P}{2}\right)}{(q^2-\Lambda^2)^2D\left(q-\frac{P}{2}\right)D\left(q+\frac{P}{2}\right)}-D(q)^{-1}\right]. \tag{2.34}$$

For the pseudoscalar case, the bracket vanishes for $P^2=0$ as it must since the π is a Goldstone boson and the current quark mass $m_q=0$. This is the first check that the cutoff procedure does not violate chiral symmetry.

The vector and axial-vector propagators can be calculated similarly but we will delay that discussion to another paper. All the propagators can be calculated in closed form using the integrals in the Appendix.

F. Wave functions

The bound-state wave functions are 4×4 matrices in Dirac space transforming like $\psi\bar\psi$. For the pion, the momentum-space wave function is

$$\vec\phi_\pi(q,P) = \vec\tau\phi_\pi(q,P),$$

$$\phi_\pi(q,P) = S\left(q+\frac{P}{2}\right)i\gamma_5\tilde f(q)S\left(q-\frac{P}{2}\right). \tag{2.35}$$

We will go to the center of mass where $P=(E_B,\vec 0)$, where E_B is the bound-state energy, which is zero for the pion. The x-space wave function is given by

$$\phi_\pi(x,P) = \int d^4q\, e^{-iq\cdot x}\phi_\pi(q,P). \tag{2.36}$$

At equal times $q\cdot x = -\vec q\cdot\vec x$, and one can evaluate the integral over q_0 by residues. The remaining integral over $\vec q$ can be evaluated by series as sketched in the Appendix.

In Sec. IV, where we impose confinement, we need to know more than the wave function Eq. (2.36). This wave function falls exponentially for larger r and has an eigenvalue $E_B=0$. Confinement in the relative coordinate imposes a boundary condition at finite R to replace the condition at ∞. In other words we need a general solution of the Bethe-Salpeter equation with an inhomogeneous term which will include growing behavior for large r. This is not difficult since our Bethe-Salpeter kernel is separable. Consider

$$\phi_\pi(k,P) = \phi_\pi^0(k,P)+\frac{6g^2}{\mu^2}S\left(k+\frac{P}{2}\right)i\gamma_5\tilde f(k)S\left(k-\frac{P}{2}\right)\times i\int\frac{d^4q}{(2\pi)^4}\mathrm{Tr}[i\gamma_5\tilde f(q)\phi(q,P)]. \tag{2.37}$$

$\phi_\pi^0(k,P)$ is a free wave which we will discuss further in Sec. IV. The solution is trivial: Taking the appropriate trace and integral gives

$$H = H^0 + H\frac{6g^2}{\mu^2}i\times\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[i\gamma_5\tilde f(q)S\left(q+\frac{P}{2}\right)i\gamma_5\tilde f(q)S\left(q-\frac{P}{2}\right)\right], \tag{2.38a}$$

where

$$H\equiv i\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}[i\gamma_5\tilde f(k)\phi_\pi(k,P)] \tag{2.38b}$$

and similarly for H^0. Therefore

$$\phi_\pi(k,P) = \phi_\pi^0(k,P) + 6g^2\mathfrak{D}_\pi(P^2)H^0S\left(k+\frac{P}{2}\right)i\gamma_5\tilde f(k)S\left(k-\frac{P}{2}\right), \tag{2.39}$$

where we have used the equation for $\mathfrak{D}_\pi^{-1}$, Eq. (2.33). In Sec. IV we will Fourier transform this equation and impose the confining boundary condition. Note that the unconfined eigenvalue condition is recovered in Eq. (2.39) by going to the $P^2=0$ pole.

G. Summary of the parameters of the chiral-invariant model

We have introduced parameters in a way that makes our action, Eqs. (2.4), resemble a σ model. However, there are no kinetic terms for the meson fields to fix their normalization, and hence we have redundant parameters. In this section we remove the redundancy by normalizing the pion propagator.

We also summarize the parameters of the model and indicate how they can be determined.

We wish to show here that all the parameters introduced so far can be determined in terms of M, Λ, and λ. M is the "constituent" quark mass due to spontaneous symmetry breaking, while Λ and λ we introduced in the cutoff functions: $\tilde{f}(p)=\Lambda^2/\Lambda^2-p^2$, $\tilde{g}(p)=(\lambda-1)\tilde{f}(p)$. (We do not include the current quark mass m_q since it breaks chiral symmetry which is conceptually part of the pion-mass calculation described below.)

We introduced the parameter μ^2 so that the composite fields would have conventional dimensions, i.e., $[\mu^2\sigma^2]\sim\text{mass}^4$. The coupling of σ to quarks is via $g_p\bar{\psi}\psi\sigma$. Since σ enters nowhere else in the action, Eqs. (2.4), only the ratio g_p/μ is a meaningful parameter. This ratio enters in the gap equation—Eq. (2.21b):

$$\mu^2=g_p{}^2F_1(\lambda,\Lambda,M)\,. \tag{2.40}$$

Hence the scalar- and pseudoscalar-meson propagators are of the form

$$\mathfrak{D}^{-1}=g_p{}^2F_2(\lambda,\Lambda,M,s)\,, \tag{2.41}$$

where F_1 and F_2 are introduced solely to show parameter dependence. *We now use the redundancy to fix g_P such that the composite-π propagator pole has unit residue:*

$$\frac{d}{ds}\mathfrak{D}_\pi{}^{-1}(s)\bigg|_{s=0}=-1\,. \tag{2.42}$$

Now that g_P is fixed, we can determine v since g_Pv is related to M, Λ, and λ through Eq. (2.15):

$$v=\frac{M(\lambda\Lambda^2-M^2)}{g_P\Lambda^2}\,. \tag{2.43}$$

G_P is determined by Eq. (2.5) which, together with Eq. (2.40), gives

$$G_P{}^2=\frac{9}{8}\frac{g_P{}^2}{\mu^2}=\frac{9}{8}F_1(\lambda,\Lambda,M)^{-1}\,. \tag{2.44}$$

Finally we have defined the dynamical mass M_D,

$$4M_D{}^3=M(\lambda\Lambda^2-M^2)\,,$$

which we see is also expressible in terms of λ, Λ, and M.

How then can we determine, λ, Λ, and M? In principle we could use α_s, M_D, and the decay constant f_π. However, it is not that easy. (i) f_π: We show in the next section that $f_\pi=v$ just as in the σ model because we have an axial-vector-current Ward identity and we have normalized the pion propagator to have unit residue. Experimentally, f_π in this convention is $\approx$93 MeV. (ii) α_s: We related our coupling strength $\Lambda^2G_P{}^2$ to α_s considering single-gluon exchange to be solely responsible for the dynamical symmetry breaking. However, we know it is not. The effect of small instantons gives a large force at these distances as calculated by Caldi.[15] We know this force is needed to break the π-η degeneracy and further Caldi finds it is strong enough to single-handedly cause dynamical symmetry breaking. So we cannot fit to the physical α_s but we can ask if, for reasonable parameters, gluon exchange alone can cause dynamical symmetry breaking. Our answer to this is "very possibly" as we will see shortly. It does seem to be clear that both terms need to be included in the chiral-breaking driving force. (iii) M_D: We have trouble directly fitting our model to the value quoted in Ref. 23. We believe, however, that M_D is only crudely determined.

We present our numerical results in Figs. 3 and 4, in which we plot f_π and $\alpha_s|_{\text{gluon}}$ as functions of Λ. We have chosen the constituent quark mass to be $M=300$ MeV as a representative value. We treat λ and M_D as alternative parameters and plot families of curves for both.

Let us look at some general features of these curves. (i) One expects that as α_s decreases to the critical value, M decreases, and since Λ is the only other mass parameter, M/Λ decreases. Alternatively, for M constant, Λ must increase. The slowly falling curves in Fig. 4 for constant λ exhibit this behavior. The end points marked $\Lambda=\infty$ are the values $\alpha_s^{(\text{crit})}$ given by Eq. (2.28), since then $M/\Lambda=0$. (ii) The shaded areas in Figs. 3 and 4 for $M_D<0.24$ are regions where our pa-

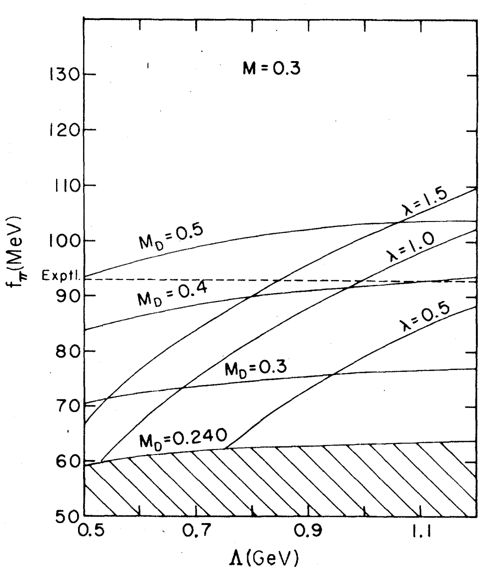

FIG. 3. Pion decay constant f_π vs dynamical-symmetry-breaking cutoff Λ, for fixed quark mass M. The families of curves correspond to fixed dynamical mass M_D or fixed λ (see text). The shaded area is excluded. The values of M and M_D are given in GeV.

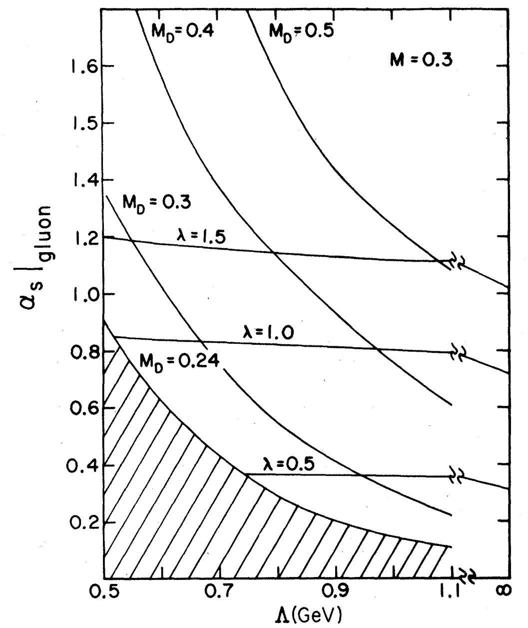

FIG. 4. Values of α_s necessary to obtain $M = 0.3$ GeV from the core of one-gluon exchange, cut off at distance Λ^{-1}. Families of curves are as in Fig. 3. The points at $\Lambda = \infty$ correspond to α_s^{crit}(see text).

rametrization of the propagator breaks down. That is, it has three poles at M, p_-, p_+. For $M_D = M/2^{1/3}$, $M = p_-$, this gives $M_D = 240$ MeV for our choice of $M = 300$ MeV. Our parametrization of the propagator is about the simplest conceivable one and it may seem disconcerting to see that a value for M_D consistent with experiment allows such bizzare infrared behavior. Recall, however, that the $p_\pm$ poles are actually a crude representation of the cut for the dispersive representation of the quark propagator. As is true for the electron propagator in QED, one might expect that the true cut touches the pole: It is interesting to speculate whether a better representation of the cut might lead to a prediction for M_D by satisfying this condition as a constraint. (iii) For the purpose of fixing parameters for what follows, we will choose $M_D = 400$ MeV. Then for f_π equal to its experimental value, $\Lambda \approx 1$ GeV and $\alpha_s|_{gluon} \simeq 0.76$. (iv) Pagels and Stokar obtain[6] $M_D \approx 275$–300 MeV. This would force us to accept values of $f_\pi \approx 70$ MeV. We believe this is encouraging, since the effects of confinement have not yet been included: The systematics of such an inclusion must be to increase the pion wave function at the origin (since normalization is preserved and the large-distance components are suppressed by confinement) and so also the value of f_π. The slow variation of f_π versus Λ for fixed M_D shown in Fig. 4 means that the confinement correction to f_π and the value of M_D must be very well known to infer Λ. We are encouraged in our crude choice in (iii) above by a different argument discussed in Sec. IV which determines a very similar value of Λ. (v) Figure 4 shows that, for fixed λ, α_s decreases as Λ increases; if $\alpha_s^{(crit)}$ were zero, this would parallel the asymptotic freedom property of QCD very closely. Thus there is a much larger range of Λ that is consistent with there being sufficient strength for SSB to occur than might be naively expected.

The experimental value of α_s at the very large scale given by Λ is extremely poorly known. Further, this approach has been carried out to lowest nontrivial order only, so the precise value of α_s is unspecified since an explicit renormalization subtraction scheme has not been determined. However, we have made rather rough approximations and wish only to ask if $\alpha_s^{expt} \sim \alpha_s$ calculated here.

The parameter $\Lambda_{\overline{MS}}$ is used to characterize α_s^{expt} as

$$\alpha_s^{expt}(Q^2) = \frac{12\pi}{25\ln(Q^2/\Lambda_{\overline{MS}}^2)} \tag{2.45}$$

in the Q^2 region above the ψ/J where four quark flavors are active (where $\overline{MS}$ refers to the modified minimal-subtraction scheme). The value of $\Lambda_{\overline{MS}}$ is believed[24] to lie in the range 100–600 MeV. We are interested in much lower values of Q^2, so we have

$$\alpha_s^{\text{``expt''}}(Q^2) = \frac{12\pi}{23\ln(Q^2/m_\psi^2) + 25\ln(m_\psi^2/\Lambda_{\overline{MS}}^2)} \tag{2.46}$$

to account for the fewer flavor degrees of freedom active below charm threshold. This formula is good to $O(\alpha_s)$ down to $Q^2 \sim m_\phi^2$. Thus

$$\alpha_s^{expt}(1\ \text{GeV}^2) \sim 0.8^{+0.8}_{-0.6}, \tag{2.47}$$

where the asymmetric uncertainty reflects the widely accepted belief that $\alpha_s(Q^2)$ does not decrease as $Q^2 \to 0$. (Difficult to prove in the nonperturbative regime; but see Ball and Zachariasen, Ref. 14.)

We have argued above that Λ represents the mean scale of Q of gluon exchanges involved in constructing the meson bound states. Thus Eq. (2.47) is to be compared with $\alpha_s|_{gluon} \simeq 0.76$ obtained for $\Lambda = 1$ GeV. Of course, this does not prove that the picture of meson structure drawn in this section is correct. Nonetheless, it is a reasonable consistency check on our basic assumption: Gluon-exchange forces may very well be strong enough, on a scale *smaller* than the confinement scale, to induce spontaneous symmetry breaking and so significantly affect the contributions of confinement energy to the meson spectrum.

III. ELECTROMAGNETIC AND AXIAL-VECTOR CURRENT AND f_π

In the previous section we represented the bound-state couplings to quarks by a nonlocal interaction involving given functions $f(x)$ and $g(x)$. Since these are not treated as dynamical variables the definition of currents in this model is nontrivial. It is nevertheless possible and we do so in this section. We find the locally conserved electromagnetic current in this model in order to lay the ground work for calculating the electromagnetic form factors. The demonstration in this section that there is a locally conserved axial-vector current is crucial since only then can we claim that our softening of the q-$\bar{q}$-meson vertices is chiral invariant. We show that the axial-vector current Ward identity is satisfied and this leads in turn to a simple expression for f_π.

A. Electromagnetic current

We will identify the electromagnetic current by coupling a photon field to our action, Eq. (2.4), in a gauge-invariant way. We then verify that it satisfies the local current-conservation Ward identity.

The technique to get gauge invariance of nonlocal products involves introducing a line integral of the gauge field. Let us write the action Eq. (2.4) in a more compact form:

$$A = \int dx_1 dx_2 \bar{\psi}(x_2) S^{-1}(x_2, x_1)\psi(x_1) - \frac{\mu^2}{2}\int dX\, F(X), \tag{3.1a}$$

where

$$S^{-1}(x_2, x_1) = i\gamma\cdot\partial_2[\delta^4(x_2 - x_1) + g(x_2 - x_1)] - \mathfrak{M}(x_2 - x_1). \tag{3.1b}$$

We now generalize the action to one that is invariant under the electromagnetic U(1) gauge transformation.

$$A_{\rm EM} = \int d1\, d2\, \bar{\psi}(2) S_{\rm EM}^{-1}(2,1)\psi(1) - \frac{\mu^2}{2}\int dX\, F(X), \tag{3.2a}$$

where

$$S_{\rm EM}^{-1}(2,1) = \exp\left[iQ\int_1^2 \mathfrak{A}_\mu dy^\mu\right] S^{-1}(2,1). \tag{3.2b}$$

A path for the line integral will be specified below. $A_{\rm EM}$ is invariant under the gauge transformation,

$$\psi(x) \to \psi'(x) = e^{-iQ\alpha(x)}\psi(x),$$

$$\bar{\psi}(x) \to \bar{\psi}'(x) = e^{iQ\alpha(x)}\bar{\psi}(x), \tag{3.3}$$

$$\mathfrak{A}_\mu(x) \to \mathfrak{A}'_\mu(x) = \mathfrak{A}_\mu(x) - \partial_\mu\alpha(x).$$

To calculate an electromagnetic effect we simply use the generating functional Γ, Eq. (2.6), but replace A and S^{-1} by $A_{\rm EM}$ and $S_{\rm EM}{}^{-1}$.

$$\Gamma_{\rm EM}(\psi, \ldots, A_\mu) = A_{\rm EM} - i\,{\rm Tr}\ln S_{\rm EM}{}^{-1}. \tag{3.4}$$

The photon couples to any process through the vertex,

$$\Gamma^\mu(x_3, x_2, x_1) = \frac{\delta S_{\rm EM}{}^{-1}(x_2, x_1)}{\delta\mathfrak{A}_\mu(x_3)} = iQ\int_1^2 \delta^4(y - x_3)dy^\mu S^{-1}(2,1). \tag{3.5}$$

Going to momentun space we define the vertex with the momentum routing shown in Fig. 5(a) which gives

$$\Gamma^\mu(p_2, p_1) = Q\frac{\partial}{\partial p_{2\mu}}\int d\xi\{\gamma\cdot(p_2 - \xi p_3)[1 + \tilde{g}(p_2 - \xi p_3)] - gv\tilde{f}(p_2 - \xi p_3)\}. \tag{3.6}$$

The ξ integral results from evaluating the dy^μ integral on a straight-line path between x_1 and x_2. The substitution is

$$y^\mu = \xi(x_2 - x_1)^\mu + x_1^\mu.$$

The tilde means four-dimensional Fourier transform. Note that for the point-interaction case, i.e., $\tilde{g}(p) = 0$, $\tilde{f}(p) = 1$, that Eq. (3.6) becomes $\Gamma^\mu = Q\gamma^\mu$.

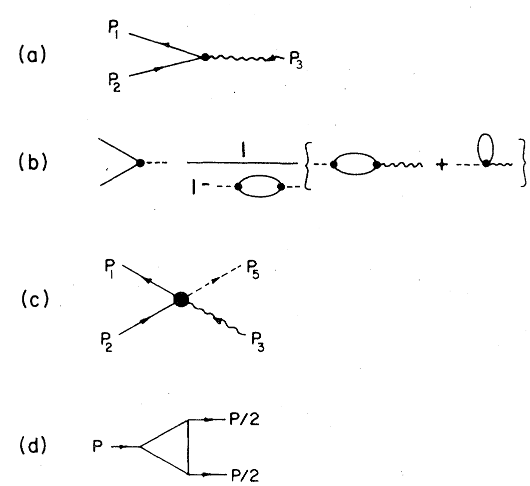

FIG. 5. (a) Momenta in the quark-current vertex. (b) Pion-intermediate-state term in the axial-vector-current vertex. (c) Momenta for quark–pion–axial-vector-current vertex. (d) Momentum sharing necessary to interpret the integral I in the Appendix as a triangle graph.

The Ward identity following from local charge conservation is

$$p_{3\mu}\Gamma^{\mu}(p_2,p_1)=Q[S^{-1}(p_2)-S^{-1}(p_1)]. \quad (3.7)$$

Our vertex satisfies this identity. To see it, use the fact that $\tilde{f}(p)$, and $\tilde{g}(p)$ are functions of p^2, which gives

$$p_3\cdot\frac{\partial}{\partial p_3}\tilde{f}(p_2-\zeta p_3)=-\frac{d}{d\zeta}\tilde{f}(p_2-\zeta p_3) \quad (3.8)$$

and similarly for $\tilde{g}$. This gives

$$p_{3\mu}\Gamma^{\mu}(p_2,p_1)=Qp_3\cdot\gamma -Q\int_0^1 d\zeta\frac{d}{d\zeta}[\gamma\cdot(p_2-\zeta p_3)\tilde{g}(p_2-\zeta p_3) -gv\tilde{f}(p_2-\zeta p_3)] \quad (3.9)$$

which upon integration gives (3.7). If we insert our assumed forms for $\tilde{f}$ and $\tilde{g}$, then $\Gamma^{\mu}(p_2,p_1)$ can be evaluated in closed form.

B. Axial-vector current

We can proceed in a similar way to find the axial-vector current by building a gauge-invariant action and isolating the coefficient of the axial-vector gauge field. (This gauge field is introduced solely as a means of finding the current and is then dropped.) Hence we look for an action that is invariant under the transformation:

$$\begin{aligned}&\psi(x)\to\psi'(x)=e^{i\vec{Q}_5\cdot\vec{\beta}(x)}\psi(x)\,,\\&\bar{\psi}(x)\to\bar{\psi}'(x)=e^{i\vec{Q}_5\cdot\vec{\beta}(x)}\bar{\psi}(x)\,,\\&\vec{\mathfrak{B}}_\mu(x)\to\vec{\mathfrak{B}}'_\mu(x)=\vec{\mathfrak{B}}_\mu(x)+\partial_\mu\vec{\beta}(x)\,,\end{aligned} \quad (3.10)$$

where $\vec{Q}_5$ is the set of three axial-vector charges which generate $SU(2)\times SU(2)$ together with the three vector charges $\vec{Q}$. $\vec{Q}_5$ acting on quark fields is $(\vec{\tau}/2)\gamma_5$. [There is an additional U(1) axial-vector generator but that invariance will eventually be broken.] The meson fields also transform under this gauge transformation. We will focus only on the $\vec{\pi}$ and isoscalar σ in the action Eq. (2.4):

$$(\sigma+i\gamma_5\vec{\tau}\cdot\vec{\pi})\to(\sigma'+i\gamma_5\vec{\tau}\cdot\vec{\pi}')=e^{i\vec{\tau}\cdot\vec{\beta}\gamma_5}(\sigma+i\gamma_5\vec{\tau}\cdot\vec{\pi})e^{i\vec{\tau}\cdot\vec{\beta}\gamma_5}. \quad (3.11)$$

Following the electromagnetic case,

$$A_{\rm ax}=\int d1\,d2\,\bar{\psi}(2)S_{\rm ax}^{-1}(2,1)\psi(1)-\frac{\mu^2}{2}\int dX\,F(X)\,, \quad (3.12)$$

where

$$S_{\rm ax}^{-1}(2,1)=\exp\left(i\gamma_5\int_{(1+2)/2}^{2}\frac{\vec{\tau}}{2}\cdot\vec{\mathfrak{B}}_\mu dy^\mu\right)S^{-1}(2,1)\times\exp\left(-i\gamma_5\int_1^{(1+2)/2}\frac{\vec{\tau}}{2}\cdot\vec{\mathfrak{B}}_\mu dy^\mu\right), \quad (3.13)$$

and where $S^{-1}(2,1)$ is given by Eq. (3.1b). We choose to break up the line integral at the midpoint in order to have a more symmetric expression which has technical advantages later. The axial-vector-current vertex must satisfy the Ward identity

$$p_{3\mu}\vec{\Gamma}_5^{\mu}(p_2,p_1)=-\frac{\vec{\tau}}{2}[\gamma_5S^{-1}(p_1)+S^{-1}(p_2)\gamma_5]. \quad (3.14)$$

The axial-vector gauge couplings can be obtained from the generating functional $\Gamma_{\rm ax}$ given by

$$\Gamma_{\rm ax}[\psi,\ldots,\vec{\mathfrak{B}}_\mu]=A_{\rm ax}-i\,{\rm Tr}\ln S_{\rm ax}^{-1}. \quad (3.15)$$

The "Gaussian term" $\mathrm{Tr}\ln S^{-1}$ is needed in order to get a nonzero v and cannot be neglected in calculating the axial-current vertex for nonzero v. The axial-vector vertex is obtained by coupling the $\vec{\mathfrak{B}}^\mu$ field to quarks, but also to include the $\vec{\pi}$ intermediate state.

$$\vec{\Gamma}_5^{\mu}(3,2,1)=\frac{\delta S_{\rm ax}^{-1}(2,1)}{\delta\vec{\mathfrak{B}}_\mu(3)}+\int\frac{\delta S^{-1}(2,1)}{\delta\pi_i(4)}\left[\frac{\delta^2\Gamma}{\delta\pi_i(4)\delta\pi_j(5)}\right]^{-1}\frac{\delta^2\Gamma_{\rm ax}}{\delta\pi_j(5)\delta\vec{\mathfrak{B}}_\mu(3)}\,d4\,d5\,. \quad (3.16)$$

The second term is represented diagrammatically in Fig. 5b. The axial-vector meson mixes also, but we will neglect it here. Carrying out the functional derivatives gives

$$\vec{\Gamma}_5^{\mu}(3,2,1)=\frac{\delta S_{\rm ax}^{-1}(2,1)}{\delta\vec{\mathfrak{B}}_\mu(3)}-f(2-1)g_p\tau_i i\gamma_5\int\mathfrak{D}_\pi\left(\frac{1+2}{2},5\right)_{ij}\left[i\,{\rm Tr}\frac{\delta S^{-1}}{\delta\pi_j(5)}S\frac{\delta S_{\rm ax}^{-1}}{\delta\vec{\mathfrak{B}}_\mu(3)}S-i\,{\rm Tr}S\frac{\delta^2S_{\rm ax}^{-1}}{\delta\pi_j(5)\delta\vec{\mathfrak{B}}_\mu(3)}\right]d5\,. \quad (3.17)$$

Referring to $S_{\rm ax}^{-1}$, Eq. (3.13), note that the $\delta^2S_{\rm ax}^{-1}$ term would vanish if the interaction was local, i.e., if $g(x)\to 0$ and $f(x)=\delta^4(x)$.

The first term in Eq. (3.17) is the direct coupling of the axial-vector current to quarks:

$$\frac{\delta S_{\rm ax}^{-1}(2,1)}{\delta\vec{\mathfrak{B}}_\mu(3)}=i\frac{\vec{\tau}}{2}\gamma_5\int_{(1+2)/2}^{2}\delta^4(y-x_3)dy^\mu S^{-1}(2,1)-S^{-1}(2,1)i\frac{\vec{\tau}}{2}\gamma_5\int_1^{(1+2)/2}\delta^4(y-x_3)dy^\mu\,. \quad (3.18)$$

Going to momentum space, using the same definition as the electromagnetic case [see Eq. (3.16)] gives

$$\frac{\delta S_{ax}^{-1}}{\delta \vec{\mathcal{B}}_\mu}(p_2,p_1)=\frac{\vec{\tau}}{2}\gamma_5\frac{\partial}{\partial p_2^\mu}\int_0^1 d\zeta\{\gamma\cdot p[1+\tilde{g}(p)]-gv\tilde{f}(p)\}-\frac{\partial}{\partial p_2^\mu}\int_{-1}^0 d\zeta\{\gamma\cdot p[1+\tilde{g}(p)]-gv\tilde{f}(p)\}\frac{\vec{\tau}}{2}\gamma_5\,, \tag{3.19}$$

where p is defined as $(p_2+p_1-\zeta p_3)/2$. Similarly we define the $q\bar{q}$-π-$\mathcal{B}_\mu$ coupling in momentum space with the momenta shown in Fig. 5(c). This gives

$$\Gamma_{ij}^\mu(p_3,p_2,p_1)=-\frac{\tau_i\tau_j}{2}g_p\gamma_5\frac{\partial}{\partial p_{2\mu}}\left[\int_0^1 d\zeta\tilde{f}(p)-\int_{-1}^0 d\zeta\tilde{f}(p)\right], \tag{3.20}$$

where $p=(p_2+p_1-\zeta p_3)/2$. If we multiply this by p_3 we get

$$p_{3\mu}\Gamma_{ij}^\mu(p_3,p_2,p_1)=\tau_i\frac{\tau_j}{2}\gamma_5 g_p\left[\tilde{f}\left(p_1-\frac{p_5}{2}\right)+\tilde{f}\left(p_2+\frac{p_5}{2}\right)-2\tilde{f}\left(\frac{p_1+p_2}{2}\right)\right]. \tag{3.21}$$

We could give the expression for the full axial-vector-current vertex in momentum space but it is not very illuminating. However, it is interesting to look at the divergence

$$p_{3\mu}\vec{\Gamma}_5^\mu(p_2,p_1)=\frac{\vec{\tau}}{2}E(p_2,p_1)+\frac{\tilde{f}\left(\frac{p_1+p_2}{2}\right)g_p\vec{\tau}\gamma_5}{\mathfrak{D}_\pi^{-1}(p_3)}G(p_3)\,, \tag{3.22}$$

where

$$E(p_2,p_1)\equiv-\gamma_5 S^{-1}(p_1)-S^{-1}(p_2)\gamma_5-2\gamma_5 g_p v\tilde{f}\left(\frac{p_1+p_2}{2}\right), \tag{3.23}$$

$$G(P)\equiv-3g_p i\int\frac{d^4q}{(2\pi)^4}\mathrm{Tr}\left\{\tilde{f}(p)\gamma_5 S\left(q-\frac{P}{2}\right)E\left(q-\frac{P}{2},q+\frac{P}{2}\right)S\left(q+\frac{P}{2}\right)-S(q)\gamma_5\left[\tilde{f}\left(q-\frac{P}{2}\right)+\tilde{f}\left(q+\frac{P}{2}\right)-2\tilde{f}(q)\right]\right\}. \tag{3.24}$$

The right-hand side of Eq. (3.22) collapses to give the Ward identity, Eq. (3.14). Note that we have used Eq. (2.33) to establish that

$$G(P)=v\mathfrak{D}_\pi^{-1}(P)\,. \tag{3.25}$$

We can now calculate f_π using this result. First note that $G(P)$ is the divergence of something—which we now define to be $G^\mu(P)$, $G^\mu(P)P_\mu \equiv G(P)$. By invariance $G^\mu(P)$ is proportional to P^μ. Call the proportionality constant $f_\pi(P^2)$. The amplitude for a π_i to couple to the axial-vector current $V_{\mu j}^5$ is

$$\delta_{ij}G^\mu(P)=\delta_{ij}f_\pi(P^2)P^\mu\,. \tag{3.26}$$

Taking the divergence and using Eq. (3.25) in the limit as $p\to 0$ gives

$$f_\pi(0)=v\,, \tag{3.27}$$

since $\mathfrak{D}_\pi(P^2)$ has a unit residue at the pion pole. The convention in Eq. (3.26) is chosen to agree with the σ model in which $f_\pi=v$ also. The decay constant defined this way has an experimental value $f_\pi=93$ MeV.

IV. MASS SHIFT OF THE PION DUE TO CONFINEMENT

We now impose confinement by relaxing the boundary condition on the wave function at infinity in the relative $q\bar{q}$ coordinate x, and imposing an appropriate condition at finite separation. This new boundary condition is developed in another paper.[16] We will lift a result from that paper and use it here. We refer the reader to Ref. 16 for details.

In the MIT-bag cavity approximation, mesons are built up by putting a q and $\bar{q}$ in a static cavity. In our picture we have a "cavity" only in the $q\bar{q}$ relative coordinate. The $q\bar{q}$ system can be in an eigenstate of total momentum P. [In practice we will choose $P=(E,\vec{0})$.] Physically we picture this as a model of QCD confinement which restricts the separation of a $q\bar{q}$ pair. Our implementation of this closely parallels the MIT-bag development. We find a linear boundary condition on the $q\bar{q}$ wave function which corresponds to zero particle flux across the surface. We then supply a volume energy and a quadratic boundary condition which guarantees classical stability of the surface, i.e., zero flux of energy and momentum across the surface.

We will impose the following linear boundary condition on the meson wave functions:

$$\hat{r}\cdot\vec{\gamma}\,\phi\gamma^0-\gamma^0\phi\hat{r}\cdot\vec{\gamma}+i(\phi-\gamma^0\hat{r}\cdot\vec{\gamma}\phi\hat{r}\cdot\vec{\gamma}\gamma^0)\Big|_{\text{surface}}=0\,, \tag{4.1}$$

where ϕ is a meson wave function in the center-of-mass frame as described in Sec. II F. The unit vector $\hat{r}$ is directed in the $\vec{x}=\vec{x}_2-\vec{x}_1$ direction. This condition implies $\hat{r}\cdot\vec{J}=0$, where $\vec{J}$ is the relative-coordinate current density[25]

$$\vec{J} \equiv \mathrm{Tr}(\bar{\phi}\,\vec{\gamma}\,\phi\gamma^0 - \bar{\phi}\,\gamma^0\phi\vec{\gamma})\,. \tag{4.2}$$

This is analogous to the MIT-bag condition for single quarks:

$$(i\hat{r}\cdot\vec{\gamma} - 1)\psi = 0, \quad \hat{r}\cdot(\bar{\psi}\vec{\gamma}\psi) = 0\,.$$

The condition Eq. (4.1) is not strictly applicable to the problem in this paper and we need to justify its use here. The assumptions in deriving Eq. (4.1) were: (i) $q, \bar{q}$ have equal mass, (ii) $q\bar{q}$ are in the center-of-momentum frame, (iii) the confining region is spherical in the three-space relative coordinate, and (iv) the particles are noninteracting. Points (i) and (ii) pose no problem. Point (iii) restricts the states we can look at. Point (iv) is not true for our case. The pion starts out as a zero-mass bound state before confinement.

We nevertheless feel that Eq. (4.1) gives a reasonable starting point. The problem of finding a boundary condition for interacting particles is that particle current can flow in the Wick-rotated relative time direction and it is no longer sufficient to just look at the three-space part. This is a relativistic effect coming from retardation. One then needs to consider a four-dimensional relative coordinate cavity.[26] But such a cavity will not be a four-sphere at finite energy, and the shape must be determined by stability. These problems are beyond the scope of the present paper. We will ignore the possibility of our bag leaking in the fourth direction. We will set the relative time equal to zero and look at the mass shift caused by squeezing the wave function into a three-space bag.

We calculate the mass shift for the pion only. All other states are near the $2M$ threshold and so confinement is expected to have large effects. Indeed, these other states may be better calculated in the MIT fashion as confinement first, then short-distance gluon-exchange effects as a perturbation. The pion, however, is well below threshold; its structure has been determined above by shorter-distance interactions leading to spontaneous symmetry breaking: We expect only small effects due to confinement as it acts only on the long-distance tail of the wave function for this state.

The solution of the inhomogeneous Bethe-Salpeter equation for the pion is given by Eq. (2.39). Fourier transformation of the full wave function takes the form

$$\phi(x,P) = \phi^0(x,P) + \phi^1(x,P)\,, \tag{4.3}$$

where $\phi^0(x,P)$ is a free wave.

$$\phi^1(x,P) = \frac{6g^2H^0}{\mathfrak{D}_\pi^{-1}(P^2)} \int d^4k\, e^{-ik\cdot x} S\left(k+\frac{P}{2}\right) i\gamma_5 \tilde{f}(k) S\left(k-\frac{P}{2}\right). \tag{4.4}$$

H^0 is given by

$$H^0 = i\int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}[i\gamma_5 \tilde{f}(k)\phi_\pi{}^0(k,P)]\,. \tag{4.5}$$

We can reduce Eq. (4.4) to a single integral and do it numerically. However, there is a simple effect going on when we impose confinement which gets masked by the technical details. Only the tail of the wave function is relevant in determining the mass shift for a moderate size bag. Since it is insensitive to the inner structure we will let $\Lambda\to\infty$ in this calculation. That greatly simplifies Eq. (4.4). Of course $\mathfrak{D}_\pi^{-1}(P^2)$ blows up. To handle that we use a new cutoff Ω on the dispersion representation of $\mathfrak{D}_\pi^{-1}$. Ω is a chiral-invariant cutoff and in fact was used by NJL. We emphasize this is only a convenience. We can pick Ω to get f_π correctly. [If we transform the appropriate value of Ω into an inverse length scale, we find $(\Omega^2 - 4M^2)^{1/2} \simeq 0.7$ GeV which corresponds nicely with the value of Λ used in Sec. II.] Finally $S(k) = (\gamma\cdot k - M)^{-1}$, $\tilde{f}(k) = 1$. The question of the value of the dynamical mass M_D is moot since M is a constant.

With these simplifications, let us evaluate $\phi(x, P)$. We set the relative time zero, $k\cdot x = -\vec{k}\cdot\vec{k}$. We can get ϕ^0 from Ref. 16.

$$\phi^0(x,P) = \begin{pmatrix} \dfrac{\vec{F}\cdot\vec{\sigma}}{2M} & \dfrac{E+2M}{4M}F \\[2ex] \dfrac{E-2M}{4M}F & \dfrac{\vec{F}\cdot\vec{\sigma}}{2M} \end{pmatrix}, \tag{4.6}$$

$$\phi^1(x,P) = -\frac{3Eg^2}{M\pi^2\mathfrak{D}_\pi^{-1}} \int_0^\infty \frac{k^2dk}{(k^2+M^2)^{1/2}} \frac{\begin{pmatrix} E\vec{G}\cdot\vec{\sigma} & [2(k^2+M^2)+EM]G \\ [2(k^2+M^2)-EM]G & E\vec{G}\cdot\vec{\sigma} \end{pmatrix}}{s-4(k^2+M^2)}, \tag{4.7}$$

where

$$F = j_0(qr)\,, \qquad G = j_0(kr)\,,$$
$$\vec{F} = i\hat{r}qj_i(qr)\,, \quad \vec{G} = i\hat{r}kj_i(kr)\,, \tag{4.8}$$

and $s = E^2 = P^2 = 4(q^2 + M^2)$. We have used the result $H^0 = -E/(2\pi)^4M$. The dispersion integral for $\mathfrak{D}_\pi^{-1}$ is

$$\mathfrak{D}_\pi^{-1}(s) = -\frac{3sg^2}{3\pi^2}\int_{4M^2}^{\Omega^2}\frac{ds'}{s'-s}\left(\frac{s'-4M^2}{s'}\right)^{1/2}. \quad (4.9)$$

We choose g^2 to give $\mathfrak{D}_\pi$ a unit residue, as described in Sec. II:

$$\frac{1}{g^2} = \frac{3}{4\pi^2}\int_{4M^2}^{\Omega^2}\frac{ds'}{s'}\left(\frac{s'-4M^2}{s'}\right)^{1/2}. \quad (4.10)$$

Then $f_\pi = v$,

$$f_\pi^2 = v^2 = \frac{M^2}{g^2} = \frac{3M^2}{4\pi^2}\int_{4M^2}^{\Omega^2}\frac{ds'}{s'}\left(\frac{s'-4M^2}{s'}\right)^{1/2}. \quad (4.11)$$

We now impose the boundary condition. Writing $\phi = \binom{AB}{CD}$, we then find Eq. (4.1) in terms of 2×2 matrices:

$$\hat{r}\cdot\vec{\sigma}C + B\hat{r}\cdot\vec{\sigma} + iA + i\hat{r}\cdot\vec{\sigma}D\hat{r}\cdot\vec{\sigma} = 0. \quad (4.12)$$

(This occurs in one of the four 2×2 blocks, the other three blocks give the same condition, up to multiples of $\vec{\sigma}\cdot\hat{r}$.) Thus our final eigenvalue condition reads

$$\left(\frac{E}{2M}j_0(qR) - \frac{q}{M}j_1(qR)\right)\left(\frac{\pi^2\mathfrak{D}_\pi^{-1}}{g^2}\right) + \frac{3E}{4M}\left[\frac{2}{R}\int_{-\infty}^{\infty}\frac{k\,dk(k^2+M^2)^{1/2}\sin kR}{(k^2-q^2)} + E\frac{d}{dR}\frac{1}{R}\int_{-\infty}^{\infty}\frac{k\,dk\sin kR}{(k^2+M^2)^{1/2}(k^2-q^2)}\right] = 0, \quad (4.13)$$

where R is the bag diameter. These integrals can be evaluated by series as described in the Appendix. This condition must be real since there is no longer a continuum of states and one can check that the imaginary part of the integrals cancels against the imaginary part of $\mathfrak{D}_\pi^{-1}$. In the absence of interactions the condition reads

$$\frac{E}{2M}j_0(qR) - \frac{q}{M}j_1(qR) = 0, \quad (4.14)$$

which is very similar to the MIT condition with the exception of the reinterpretation of R and E and the factor of 2 associated with the center-of-mass reduction. To get to the bound-state region we must continue to positive imaginary values. As one would expect, for the noninteracting case there are no solutions here since both terms in Eq. (4.14) become positive.

Figure 6 shows the lowest solution (denoted $E_\pi|_{BC}$) to this equation for two values of M. As indicated, the curves are well fitted by Bessel-function behaviors of large argument:

$$E_\pi|_{BC}(R) \simeq AR^{-1/2}e^{-2MR}, \quad (4.15)$$

which might be expected from expansion of Eq. (4.13) in the large-R region.[27]

Calculation of the pion mass now proceeds as in the MIT bag calculation: The bag volume energy is added to the quark energy and the whole is minimized at $R_{\min} \equiv R_\pi$,

$$M_\pi(R_\pi) = \frac{Ae^{-2MR_\pi}}{\sqrt{R_\pi}} + \tfrac{4}{3}\pi B\left(\frac{R_\pi}{2}\right)^3. \quad (4.16)$$

(This is properly M as c.m. motion has already been removed.)

$$0 = \left(-2M - \frac{1}{2R_\pi}\right)\frac{Ae^{-2MR_\pi}}{\sqrt{R_\pi}} + \pi B\left(\frac{R_\pi}{2}\right)^2. \quad (4.17)$$

The only difference from the MIT result is the change to the exponential form of R dependence from the R^{-1} dependence of confined states found by the MIT group. We emphasize that this difference is due to the gluon-induced quark binding, and in fact, due to that binding being so strong as to induce SSB thus generating a mass scale for the wave function even for quarks with zero "current-algebra" (ultraviolet) mass.

The values of A obtained from numerical fitting to the solutions are shown in Fig. 6 (A is slightly M-dependent). M was fixed in Sec. II. This leaves B to be determined. In our bag, the quark and antiquark may separate to a distance R; this is only $R/2$ from the fixed center. In the MIT-bag calculations, the c.m. motion is ignored and this

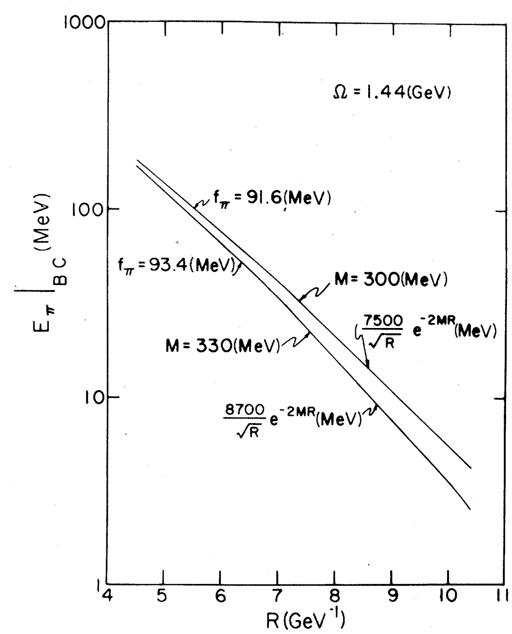

FIG. 6. $E_\pi|_{BC}$ vs bag diameter R. $R/2$ is the radius of the relative bag. $E_\pi|_{BC}$ is the contribution to the pion mass arising from relaxing the large-r boundary condition on the pion wave function and replacing it with the bag boundary condition. Volume-energy effects are not included. An approximate fit for this range of R is indicated on each curve.

gives a good fit to the heavier states. Since there the maximum distance of a quark from the fixed center is R_{MIT} we must have $R/2 = R_{\text{MIT}}$. For our bag to have the same volume energy as MIT's bag, therefore, we can use the same value[2] of B,

$$B = B_{\text{MIT}} = (0.14\ \text{GeV})^4, \tag{4.18}$$

since we have used $R/2$ in this volume term in Eq. (4.16).

It is now trivial to solve (4.17) numerically, and substitute the result in (4.16). We find

$$\begin{aligned} R_\pi &= 7.1\ \text{GeV}^{-1}\quad (M = 300\ \text{MeV}) \\ &= 6.7\ \text{GeV}^{-1}\quad (M = 330\ \text{MeV}) \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} M_\pi &= 120\ \text{MeV}\quad (M = 300\ \text{MeV}) \\ &= 110\ \text{MeV}\quad (M = 330\ \text{MeV}). \end{aligned} \tag{4.20}$$

These values have uncertainties of $\pm 20\%$, not including any systematics of the initial approximations. In both cases, more than half (65%) of the mass comes from the volume-energy term.

These results are encouraging. Recalling that the radius of this state is $R/2$, we see that our pion is very similar in size to that calculated by the MIT group and to other hadrons.[2] However, we have already projected out c.m. motion, and can see why the state is smaller: the gluon interaction has pulled the wave function sharply inwards from the form it would take for massless quarks.

There remain two possible problems:

(1) Our confinement condition is still not chiral-SU(2) invariant. As discussed in the Introduction, this poses the problem of understanding its origin, since the conjectured mechanisms of coupling-constant growth and/or large instanton effects seem to preserve chiral-SU(2) invariance. However, it should be noted that quark mass terms in the Lagrangian ("current-algebra" or "ultraviolet" masses) are scale-dependent quantities: The current-algebra values[4] of order 5–10 MeV are valid on distance scales $\lesssim 1\ \text{GeV}^{-1}$; i.e., in the region of dynamical SSB. As the distance scale increases, these values grow and this also produces a confining effect on quarks. Since this effect is proportional to the current-algebra mass, it is consistent (with current-algebra analyses) for this effect to be the origin of a finite pion mass. None of this precludes other sources of confinement from acting on larger distance scales. If the active mechanism of confinement in the region of 1 GeV^{-1} preserves chiral SU(2), and this can be formulated in bag terms, the result of this section would again be $M_\pi = 0$. We would then have to go back to the Bethe–Salpeter analysis in Sec. II with nonzero quark current-algebra masses to see what value they generate for M_π.

(2) We have used B_{MIT} but not the zero-point fluctuation energy Z_0/R_{MIT} discussed by the MIT group. We have done so because we do not attach the same physical reality to the cavity approximation as they do: The MIT calculation leaves only color magnetic interactions inside the bag whereas we have included gluon "Coulombic" effects. We believe that this attractive interaction reflects the same physics that the negative Z_0 term represents in the MIT calculation.

Milton[28] has argued recently that if a specific renormalization scheme is employed, $Z_0 > 0$ (vs $Z_0 = -1.84$ in Ref. 2). We believe that the renormalization freedom allows the Z_0 term not associated with gluon exchange to be removed entirely: again, the use of this term in Ref. 2 must merely mock up the attractive "Coulombic" effects of gluons that are otherwise omitted. For reference, we note that Milton's Z_0 term causes a small increase in R_π but a large ($\times 2$) increase in M_π; if $Z_0 = -1.84$, R_π is reduced to about 4 GeV^{-1}, and this produces a negative value for M_π. This is probably due to effectively double counting the Coulombic part of the gluon exchange. Finally we note that Milton has described a small negative, fermion zero-point fluctuation term[29]: If this alone is included ($Z_0 = -3/16\pi$) we obtain $R_\pi = 6.7$ GeV^{-1} and $M_\pi = 100$ MeV, which is negligible change within the uncertainties.

V. DISCUSSION AND CONCLUSION

In this paper, we have studied dynamical chiral-symmetry breaking due to strong $q\bar{q}$ forces on a scale smaller than the confining radius. In particular, this leads to a zero-mass Goldstone pion for physical strong-interaction parameter values close to those inferred from experiment. A confinement scheme, similar to the MIT bag system, but expressed in terms of relative quark coordinates was then imposed on the interacting wave function. Since the Goldstone pion is a deep bound state with binding energy $2M$, the wave function is exponentially damped $\sim e^{-Mr}$. The effect of the boundary condition is to replace this large-r behavior with a condition at the surface. We find the boundary condition can be satisfied by adding a small admixture of exponentially growing free wave $\sim e^{Mr}$. This leads to a small mass shift of the pion and a small change to the otherwise Goldstone pion. It is because of the radically different spatial dependence of that wave function from that of free quarks in a cavity that confinement produced only a weak effect on the pion mass. We found that the pion acquired a small mass, reasonably close to its experimental value. This is as opposed to the MIT calculation, which initially produces a large value for the pion mass, followed by perturbative corrections which lower it. More significant than the failure of that approach to obtain a reasonable mass value is its failure to explain the Goldstone character of the state.

As was argued in the Introduction and in Sec. IV, the chiral-symmetry violation of the baglike confinement condition need not be in conflict with current-algebra results if it "turns off", i.e., $B \to 0$ in the limit of vanishing ultraviolet quark masses, $m_q^{\rm UV}$. This will be the case if B represents the effect of the increase of these masses with increasing distance scale. If that is so, then $M_\pi \to 0$ also in the limit $m_q^{\rm UV} \to 0$. This does not preclude the existence of other, chiral-symmetry-preserving confinement mechanisms effective at even larger distance scales. (However, these could not be described simply by a nonvanishing value of B.)

It is difficult to answer the question of whether QCD has sufficient strength, at distances less than the confining radius for light mesons, to cause spontaneous symmetry breaking. We view our calculation of the strength of the short-range core of gluon exchange as evidence that it is a large component in driving dynamical symmetry breaking. The π-η mass splitting is evidence that $U(1)_A$ breaking forces are also significant since they must help drive the pion to negative mass squared but keep the η mass squared positive. We believe that our approach of extracting a separable interaction from the deepest bound state and using it to approximate the nonlinear Schwinger-Dyson (NLSD) equations is a sound one. We cite as evidence Cahill and Janus[19] who solved the NLSD equations numerically and found good agreement of the solution with this approximation in a $\phi^2\sigma$ model. Also, Janus[20] found that when fully corrected propagators (solutions to the NLSD equations) were used, this increased the effective attraction over the bare ladder approximation. If this result is true for QCD, then there is more attraction in gluon exchange than our naive calculation indicates.

The combination of these elements provides a clear outline for an understanding of a $(q\bar{q})$ composite almost-Goldstone pion in confined QCD.

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APPENDIX: INTEGRALS

1. Overlap integrals

In Sec. IID we encountered the following integral:

$$\mathcal{O}_P = -\Lambda^4 i \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}[S(q)^2\gamma^\mu S(p)^2\gamma_\mu]}{(\Lambda^2-q^2)(\Lambda^2-p^2)(p-q)^2}, \tag{2.26a}$$

where $S(q) = \gamma \cdot q(\Lambda^2 - q^2)/(\lambda\Lambda^2 - q^2)$. Taking the trace and performing a Wick rotation gives

$$\mathcal{O}_P = 16\Lambda^4 \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \frac{(\Lambda^2+q^2)(\Lambda^2+p^2)}{q^2(\lambda\Lambda^2+q^2)^2(p-q)^2(\lambda\Lambda^2+p^2)^2 p^2}.$$

The angular integration gives

$$\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2+q^2-2pq\cos\theta} = \frac{1}{32\pi^2 q^2} \int_0^\infty dp^2 (p^2+q^2-|p^2-q^2|).$$

Define $p^2 = \lambda\Lambda^2 x$, $q^2 = \lambda\Lambda^2 y$, then we obtain

$$\mathcal{O}_P = \frac{\Lambda^2}{8\pi^4\lambda^3} \int_0^\infty dy \frac{(1+\lambda y)}{y(1+y)^2} \int_0^y dx \frac{(1+\lambda x)}{(1+x)^2}. \tag{A1}$$

Also in Sec. IID we encounter $\mathcal{O}_V$ which, upon using similar techniques, reduces to

$$\mathcal{O}_V = -\frac{\Lambda^2}{64\pi^4\lambda^3} \int_0^\infty dy \frac{(1+\lambda y)}{y^2(y+1)^2} \int_0^y dx \frac{(x+y)(1+x)}{(x+1)^2}. \tag{A2}$$

Define

$$\omega_{nm} \equiv \int_0^\infty dy \frac{1}{y^n(1+y)^2} \int_0^y dx \frac{x^m}{(1+x)^2} . \tag{A3}$$

Θ_P and Θ_V can be evaluated in terms of the following ω's:

$$\omega_{00} = \omega_{01} = \omega_{10} = \tfrac{1}{2} ,$$
$$\omega_{11} = -\frac{3}{2} + \frac{\pi^2}{6}, \quad \omega_{22} = -\frac{13}{2} + \frac{2\pi^2}{3} , \tag{A4}$$
$$\omega_{21} = \omega_{12} = \frac{7}{2} - \frac{\pi^2}{3}.$$

2. Meson propagators

The π propagator integral can be expressed in terms of

$$\begin{pmatrix} I \\ J \end{pmatrix} (z_1{}^2, z_2{}^2, \Lambda^2; P^2) = i \int \frac{d^4q}{(2\pi)^4} \begin{pmatrix} 1/(q^2-\Lambda^2) \\ 1/(q^2-\Lambda^2)^2 \end{pmatrix} \times \frac{1}{[(q-P/2)^2 - z_1{}^2][(q+P/2)^2 - z_2{}^2]} , \tag{A5}$$

where z_i take on the values, M, P_+, P_-. I is a triangle graph and J is a box graph. Because of the special kinematics they can be expressed in terms of the two-point bubble graph. For example, I looks like the graph in Fig. 5(d).

If we disperse this in the $s(=P^2)$ channel there are three normal thresholds and we get

$$I = -\frac{1}{8\pi^2} \int_{(z_1+z_2)^2}^\infty \frac{ds'\Delta(s', z_1, z_2)}{(s'-s)2s'\left(-\frac{s'}{4} + \frac{z_1{}^2 + z_2{}^2}{2} - \Lambda^2\right)} + \frac{1}{16\pi^2} \int_{4(z_2+\Lambda)^2}^\infty \frac{ds'\Delta(s', 2z_2, 2\Lambda)}{(s'-s)2s'\left(-\frac{s'}{4} + \frac{z_1{}^2 + z_2{}^2}{2} - \Lambda^2\right)}$$
$$+ \frac{1}{16\pi^2} \int_{4(z_1+\Lambda)^2}^\infty \frac{ds'\Delta(s', 2z_1, 2\Lambda)}{(s'-s)2s'\left(-\frac{s'}{4} + \frac{z_1{}^2 + z_2{}^2}{2} - \Lambda^2\right)} , \tag{A6}$$

where

$$\Delta(s, x, y) \equiv ([s - (x-y)^2][s - (x+y)^2])^{1/2} . \tag{A7}$$

These are subtracted bubble graphs that can be evaluated in closed form.

3. Fourier transforms

The integrals in Sec. IV can be cast into series. The basic step is to use the following identity:

$$\int_{-\infty}^\infty \frac{q\,dq \sin qR}{(q^2-a^2)(q^2+M^2)^{1/2}} = 2e^{iaR} \int_0^R dt K_0(Mt) \cos at + 2i \sin aR \int_R^\infty dt K_0(Mt) e^{iat} . \tag{A8}$$

A similar formula holds for the other integral in Eq. (4.13). The second integral in Eq. (A8) can be converted to a $\int_0^R$ integral, and the $\int_0^\infty$ integral can be evaluated in closed form. Then a small argument expansion of K_0 gives a simple series expansion.

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$$\frac{\partial}{\partial T}(\rho_1\rho_2) + \vec{\nabla}_X \cdot \left(\frac{\rho_2\vec{J}_1 + \rho_1\vec{J}_2}{2}\right) + \vec{\nabla}_x \cdot (\rho_1\vec{J}_2 - \rho_2\vec{J}_1) = 0,$$

in terms of total and relative coordinates. The second term is zero in the center-of-mass system and the third term involves our current.

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