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Finite-size scaling and asymptotic freedom of the SU(2) lattice gauge model

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The dependence of the plaquette energy on lattice size L^4 for the four-dimensional SU(2) lattice gauge model is obtained in the crossover region by Monte Carlo calculations for $L=4$ to 10. Extending finite-size scaling theory to non-Abelian gauge models, a fit to the data verifies the validity of scaling and asymptotic freedom. The value obtained for the coefficient γ_0 in the beta function, corresponding to the one-loop term in perturbation theory, is $\gamma_0=0.041$, in good agreement with the result of SU(2) gauge field theory, $\gamma_0=11/24\pi^2=0.046\ldots$. We estimate an upper bound to the glueball mass $m\cong 1.2\sqrt{\kappa}$ where κ is the string tension.

Monte Carlo calculations for the SU(2) lattice gauge model in four dimensions have shown that the Wilson loop satisfies an area-law dependence[1,2] in both the weak- and the strong-coupling regimes. An approximate boundary dividing these two regimes is determined by the rapid crossover in the variation of the area-law coefficient as a function of β at $\beta\cong 2.2$, where $\beta=4/g^2$ and g^2 is the bare coupling constant. Recently, at this value of β, a sharp maximum was found in the specific heat of the lattice,[3] using the Wilson form of the action. Furthermore, only about half of this maximum was accounted for by the nearest-neighbor plaquette-plaquette correlations indicating that longer-range correlations were important. Near this peak more evidence for a growing correlation length $\xi(\beta)$ was given by a nonvanishing dependence of the specific heat on the size L of the lattice for $L=4$, 5, and 6. In the weak-coupling regime the correspondence between the behavior of ξ and the area-law coefficient κa^2 can be understood by scaling, which implies that $\kappa a^2 \propto \xi^{-2}$, where κ is the string tension and a is the lattice spacing.

In order to establish the relevance of lattice calculations to the problem of quark confinement, it is necessary to show that this weak-coupling regime satisfies the asymptotic-freedom properties of continuum non-Abelian gauge field theory.[4,5] While the present Monte Carlo data for κa^2 are consistent with asymptotic freedom in a narrow range of values of β, it is important to obtain further quantitative confirmation for this hypothesis. In this paper we propose a new test of asymptotic freedom by extending finite-size scaling theory to the case that the correlation length $\xi(\beta)$ increases exponentially with β, and we apply this theory to very accurate values of the plaquette energy $E(\beta,L)$ obtained by Monte Carlo calculations for the SU(2) lattice gauge model as a function of both β and L. Finite-size scaling theory[6] has been applied successfully in the past to determine the critical behavior of spin systems, and more recently to the U(1) lattice gauge model in four dimensions.[7] For this model, the correlation length ξ has a power-law behavior near the critical value β_c, $\xi\cong|\beta-\beta_c|^{-\nu}$, where the critical[7-9] exponent $\nu\cong\frac{1}{3}$. For non-Abelian lattice gauge models we expect that $\xi\sim e^{c\beta}$ where c is approximately a constant.

In the scaling regime, the dependence of the free energy per plaquette $F(\beta,L)$ on the linear size L of the lattice for periodic boundary conditions takes the form[6]

$$\Delta F(\beta,L)=L^{-d}f(L/\xi) \quad , \tag{1}$$

where $\Delta F(\beta,L)\equiv F(\beta,L)-F(\beta,\infty)$, d is the dimension of the lattice, and $f(x)$ is a scaling function. The corresponding contribution to the energy per plaquette $E(\beta,L)\equiv(d/d\beta)\ F(\beta,L)$ is given by

$$\Delta E(\beta,L)=L^{-d}\epsilon(L/\xi)\frac{d\ln\xi}{d\beta}\ , \quad (2)$$

where $\epsilon(x)=-xdf(x)/dx$. If the scaling regime on the lattice corresponds to SU(2) gauge field theory, then the function $d\beta/d\ln\xi$ is the renormalization-group beta function

$$\frac{d\beta}{d\ln\xi}=-y\beta+8\gamma_0+32\gamma_1\frac{1}{\beta}+\cdots\ , \quad (3)$$

where $y=d-4$, $\gamma_0=11/24\pi^2$, and $\gamma_1=34/192\pi^4$. These coefficients are independent of the renormalization cutoff scheme. For $d=4$, the prediction that $\Delta E(\beta,L)L^4$ scales can be tested by applying Eq. (3), which implies $\xi\propto\beta^{-51/121}e^{3\pi^2\beta/11}$, without knowing the form of the scaling function $\epsilon(x)$. Furthermore, the coefficients of the beta function can also be determined by a scaling fit which will be described in detail below.

The scaling function $\epsilon(x)$ is determined by the asymptotic form of the plaquette-plaquette correlation function $G(r,\beta,L)$ averaged over plaquette orientations,

$$G(r,\beta,L)=\langle E_pE_{p'}\rangle-\langle E_p\rangle^2\ , \quad (4)$$

where r is the separation between plaquettes labeled by p and p', $\langle\cdots\rangle$ denotes the thermal average, and $\langle E_p\rangle=E(\beta,L)$. We can obtain $\Delta E(\beta,L)$ by integrating the relation

$$\frac{d}{d\beta}\Delta E(\beta,L)=\sum_r[G(r,\beta,L)-G(r,\beta,\infty)]$$

which depends only on $G(r,\beta,L)$ for large values of r. In the scaling regime the asymptotic form of $G(r,\beta,L)$ is conjectured to be

$$G(r,\beta,L)\cong\frac{g(r/\zeta,r/L)}{r^p}\ , \quad (5)$$

where the exponent p will be determined below. We find that for $d=4$

$$\Delta E(\beta,L)=L^{d-p}\int_{L/\xi}^{\infty}dx\,x^{p-d-1}c(x)\ , \quad (6)$$

where

$$c(x)=\Omega_d\int_0^x dy\frac{1}{y^{p-d+1}}[g(y,y/x)-g(y,0)]$$
$$-\Omega_d\int_x^\infty dy\frac{g(y,0)}{y^{p-d+1}} \quad (7)$$

and Ω_d is proportional to the area of the unit sphere. Setting $p=2d$, we recover[10] the scaling form for $\Delta E(\beta,L)$, Eq. (2), provided we neglect higher-order terms in $1/\beta$ in the asymptotic-freedom expansion, Eq. (3). To incorporate these terms additional contributions to the correlation functions are needed which lead to corrections to scaling which will not be considered here.

We have carried out Monte Carlo calculations for values of $\beta=2.05$ to 2.70 in steps of 0.05 and for $L=4$ to 10, and evaluated $E(\beta,L)$. For this range of β and $L\leq7$ it was found[11] that $\Delta E(\beta,L)\gtrsim10^{-3}$. To obtain the L dependence, we need $E(\beta,L)$ to high accuracy. The statistical error in $E(\beta,L)$ is given by $(1/\beta L^2)[C(\beta)/6N]^{1/2}$, where $C(\beta)$ is the specific heat and N is the number of Monte Carlo interactions. We require $E(\beta,L)$ to an accuracy of $\sim10^{-4}$, which implies $N\gtrsim10^7/6L^4$. The Monte Carlo data for $E(\beta,L)$ are shown in Fig. 1 together with a least-squares fit in accordance with scaling and asymptotic freedom which will be described below. The statistical errors were used in the fit together with an estimated systematic error of 1.5×10^{-4}.

The validity of the scaling equation for $\Delta E(\beta,L)=E(\beta,L)-E(\beta,\infty)$, Eq. (2) together with the asymptotic freedom dependence for the correlation length ξ, Eq. (3), can be tested by plotting $\Delta E(\beta,L)L^4$ as a function of L/ξ. From Fig. 1 we note that $E(\beta,L)$ approaches rapidly its asymptotic value $E(\beta,\infty)$ which can therefore be determined from the data for $L=8$, 9, and 10. Since the scaling relation does not depend on the overall magnitude of $\xi(\beta)$ we have chosen arbitrarily $\xi(2.3)=1.0$. The resultant plot is shown in Fig. 2 for lattice sizes $L=4$, 5, and 6. It is evident that within the errors the data fall on a single curve corresponding to the existence of a unique scaling function $\epsilon(x)$, Eq. (2) Note that as L increases the errors for $\Delta E(\beta,L)$ are greatly magnified in this plot by the factor L^4, and for this reason we have left out the $L=7$ data, although they are also consistent with scaling. All our data for $E(\beta,L)$ can be fitted by least squares, with $E(\beta,\infty)$ as a free parameter, assuming a Padé approximate for $\epsilon(x)$ of the form

$$\epsilon(x)=\frac{a_0+a_1x}{1+a_2x+a_3x^2} \quad (8)$$

suggested by Fig. 2. The resultant values of $E(\beta,\infty)$ are given in Table I, and we obtain $a_0=0.089\pm0.004$, $a_1=0.0019\pm0.0015$, $a_2=0.403\pm0.002$, and $a_3=0.0455\pm0.0005$.

The plaquette energy $E(\beta,\infty)$ for an infinite-size lattice can be accurately fitted in the scaling domain by a polynomial in β. From this fit we obtain the specific heat $C(\beta,\infty)=\beta^2dE(\beta,\infty)/d\beta$ shown in Fig. 3. Also shown in Fig. 3 are the corresponding values of the specific heat $C(\beta,L)$ for $L=4$, 5, and 6 which were obtained by adding the size-dependent scaling contribution $\beta^2d\Delta E(\beta,L)/d\beta$. The resultant values of $C(\beta,4)$ are in excellent agreement with the specific heat obtained previously from a direct evaluation of the energy fluctuations.[3] It is clear that as L increases the specific-heat peak broadens and the maximum value decreases approaching rapidly

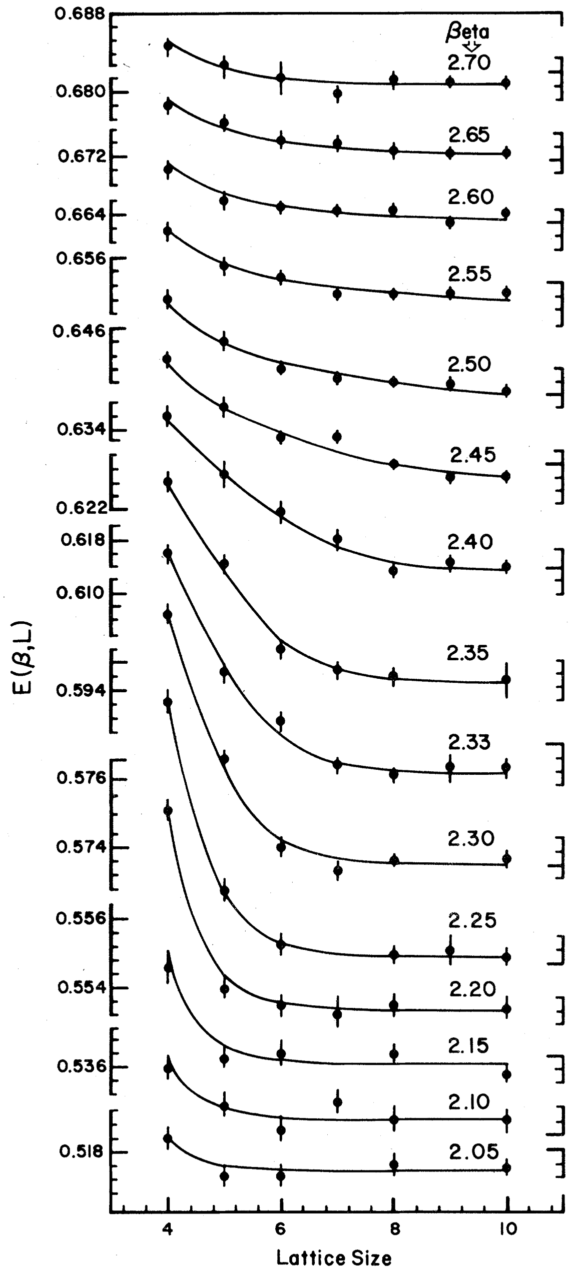

FIG. 1. Monte Carlo values of the plaquette energy $E(\beta,L)$ with statistical errors. The vertical scale has an origin shift for each value of β with a scale factor 0.0004 per division. The curves correspond to a least-squares fit in accordance with scaling and asymptotic freedom described in the text.

the limiting behavior for an infinite lattice. Although the specific-heat maximum persists at $\beta \cong 2.2$, this L dependence implies that the crossover between weak and strong coupling is not associated with an ordinary phase transition.

To test further the validity of the asymptotic-freedom beta function, we assume that the coefficients γ_0 and γ_1 are undetermined, $y=0$, and neglect the β variation in Eq. (3) setting $\beta=\bar{\beta}\sim 2.0$. Then $\xi \propto e^{c\beta}$, where $c=(8\gamma_0+32\gamma_1/\bar{\beta})^{-1}$, and c becomes a parameter in the least-squares fit. The result is $c=3.03\pm 0.11$ which is in good agreement with the predicted value $c=2.5\ldots$. Alternatively, neglecting the higher-order terms in the β function, we obtain $\gamma_0=1/8c=0.041\pm 0.001$ in good agreement with the theoretical prediction $\gamma_0=11/24\pi^2=0.046\ldots$. Similar results are obtained if we allow the dimension d in Eq. (2) to become a free parameter, and we find $d=3.9\pm 0.1$ in strong support of the scaling hypothesis, while $c=3.11\pm 0.13$. Finally, we have allowed also the parameter y in Eq. (3) to become a

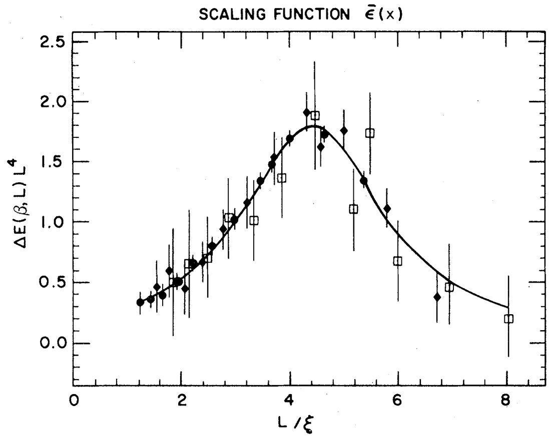

FIG. 2. The scaling plot of $\Delta E(\beta,L)\,L^4$ vs L/ξ with $\xi=e^{[3\pi^2(\beta-2.3)]/11}$: ◆ $(L=4)$, ◆ $(L=5)$, and ⊡ $(L=6)$. The curve is a least-squares fit for a Padé approximate to $\epsilon(x)=(11/3)\pi^2\bar{\epsilon}(x)$, Eq. (2).

TABLE I. Plaquette energy for infinite-size lattice.

β	$E(\beta,\infty)$
2.05	0.5174
2.10	0.5345
2.15	0.5518
2.20	0.5692
2.25	0.5862
2.30	0.6020
2.33	0.6112
2.35	0.6169
2.40	0.6300
2.45	0.6416
2.50	0.6519
2.55	0.6613
2.60	0.6700
2.65	0.6781
2.70	0.6855

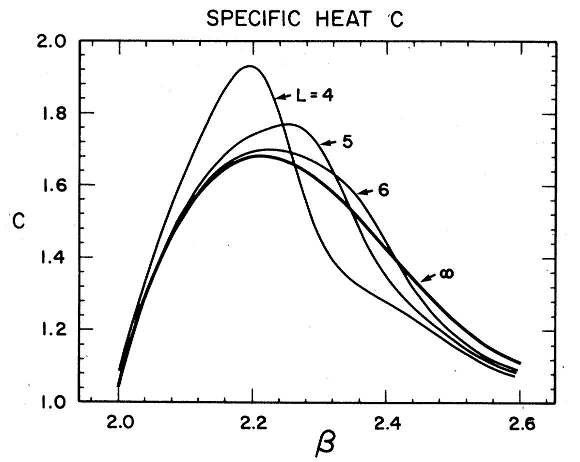

FIG. 3. The specific heat C as a function of β for $L=4$, 5, 6, and ∞ obtained from the scaling fit to the energy.

free parameter which implies that $\xi \propto |(\beta-\beta_c)|^{-1/y}$, where $\beta_c = 1/cy$. We obtain again the same result for c, while y is small, but only an upper bound $|y| \lesssim \frac{1}{45}$ can be determined by our least-squares fit.

The results presented here give direct evidence that the weak-coupling regime on the lattice satisfies scaling properties and the asymptotic-freedom behavior of continuum SU(2) gauge field theory. However, our scaling analysis so far does not give the scale of the correlation length ξ. This quantity is of physical interest because it determines a relation for the mass m of the lowest excitation of the gauge field theory commonly known as the glueball.[12] Since the results of our analysis indicate that the scaling regime extends down to $\beta \sim 2.05$, we can obtain an estimate for an upper bound to m if we make the reasonable assumption that $\xi \gtrsim 1$ for scaling to be valid. We then find that $ma = \xi^{-1} = 184\beta^{51/121}e^{-3\pi^2\beta/11}$. Applying this result to the asymptotic-freedom fit for the area-law coefficient[1,2] gives the relation $m \cong 1.2\sqrt{\kappa}$ or, alternatively, $\kappa a^2 \cong 0.7\xi^{-2}$. This second relation is expected to be valid for a Wilson loop of fixed size on the lattice provided the area of the loop $\gtrsim \xi^2$ and $\xi \gtrsim 1$. This gives the range in β over which the area-law coefficient for fixed-size loops satisfies asymptotic freedom in good agreement with current Monte Carlo calculations.[1,2] However, in order to determine the scale of ξ without special assumptions further theoretical input regarding the form of the scaling function $\epsilon(x)$ is required, and this is currently under investigation.[13]

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[10]In the limit $g^2 \to 0$ perturbation theory gives $p = 8d$.
[11]The lowest-order strong-coupling expansion gives $\Delta E(\beta,L) = 2(\beta/4)^{(4L-1)}$. This expression is an order of magnitude smaller than the observed value of $\Delta E(\beta,L)$ for $2.05 < \beta < 2.20$, and $L=4$, 5, and 6 which indicates that the crossover to the strong-coupling regime occurs for $\beta < 2.05$.
[12]Recent attempts have been made to determine the correlation length ξ by evaluating directly $G(r,\beta,L)$, Eq. (4), by Monte Carlo calculations; see B. Berg, Phys. Lett. 97B, 401 (1980), and Ref. 2. However, we believe that their results are not justified, because the rapid decrease of $G(r,\beta,L)$ with r permits only the determination of G for small r while the scaling hypothesis, Eq. (5), applies to the asymptotic behavior of $G(r,\beta,L)$.
[13]M. Creutz (private communication).