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Non-self-dual solutions of SU(3) Yang-Mills theory and a two-dimensional Abelian Higgs model

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We choose a special ansatz for the gauge potentials which corresponds to Witten's ansatz; however, it involves the 5-plet of an SU(2) subalgebra instead of the 3-plet. Among the set of solutions, admitted by our ansatz, only the vacuum with vanishing field strength is self-dual. However, the action and the field equations are, except for one constant factor, equal to Witten's action and field equations and thus describe a two-dimensional Abelian Higgs model.

Recently, the following static ansatz for a solution to SU(3) Yang-Mills-Higgs theory in Minkowski space was given,[1]

$$\phi = -A(r)K\,, \tag{1a}$$

$$W_0 = 0, \quad W_i = i[E, \partial_i E] - iD_2(r)[K, \partial_i E] \tag{1b}$$

$$(i,j,\ldots = 1,2,3)\,,$$

where ϕ is the Higgs field, W_μ $(\mu, \nu, \ldots = 0,1,2,3)$ is the gauge potential, and E and K are the scalars

$$E = \hat{r}_i E_i\,, \tag{2a}$$

$$K = \hat{r}_i K_i = \hat{r}_i \hat{r}_j K_{ij} = \hat{r}_i \hat{r}_j (E_{(i}E_{j)} - \tfrac{2}{3}\delta_{ij} I)\,, \tag{2b}$$

with the SU(2) 3-plet of generators $\vec{E} = (\lambda_7, -\lambda_5, \lambda_2)$ and the 5-plet K_{ij}.

This ansatz has the remarkable feature that, on the one hand, it only admits the trivial solution to the Bogomolny equations which is ruled out by the boundary condition. On the other hand, the equations of motion differ only by a constant factor from the 't Hooft-Polyakov equations of motion[2] and are, furthermore, Euler-Lagrange equations of the energy functional. These properties of the ansatz made it possible to give an existence proof using techniques of Tyupkin *et al.*[3] That is to say, a second Yang-Mills-Higgs monopole for nonvanishing Higgs field self-interaction has been added to the only one so far known, the 't Hooft-Polyakov monopole. Because the existence proof can be extended to the Prasad-Sommerfield limit,[4] the ansatz also leads to a finite-energy solution which does not satisfy the Bogomolny equations.

If one identifies the zero component of the gauge potential with the Higgs field, $W_0 = \phi$, this static Yang-Mills-Higgs solution is also a solution to Euclidean Yang-Mills theory given by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\,\mathrm{tr}\,G_{\mu\nu}G_{\mu\nu} \tag{3}$$

with field strength $G_{\mu\nu}$. However, because the solution is static the action is infinite. If one is interested in solutions with finite action one has to choose a nonstatic ansatz. In the following we are going to choose one which has the same properties as the ansatz used in Ref. 1.

Our ansatz is

$$\begin{aligned} W_0 &= A_0(x_0, r)K\,, \\ W_i &= -[\varphi_2(x_0, r) + 1]\, i[E, \partial_i E] \\ &\quad + \varphi_1(x_0, r)\partial_i K + A_1(x_0, r)\hat{x}_i K\,. \end{aligned} \tag{4}$$

This ansatz is symmetric about the x_0 axis, like Witten's ansatz,[5] which one recovers if one substitutes E's for the K's. It is, of course, also one special form contained in the general ansatz with this symmetry which for SU(3) was given by Yates and by Bais and Weldon[6] and for arbitrary semisimple compact Lie groups by Leznov and Saveliev.[7] These authors, however, exclude our special ansatz in one of the next steps by imposing the self-duality condition.

With the help of the commutation relations

$$\begin{aligned} [E_i, E_j] &= i\epsilon_{ijk}E_k\,, \\ [E_i, K_{jk}] &= i(\epsilon_{ijl}K_{lk} + \epsilon_{ikl}K_{lj})\,, \\ [K_{ij}, K_{kl}] &= (i/4)(\epsilon_{ikm}\delta_{jl} + \epsilon_{ilm}\delta_{jk} \\ &\quad + \epsilon_{jkm}\delta_{il} + \epsilon_{jlm}\delta_{jk})E_m\,, \end{aligned} \tag{5}$$

one easily calculates

$$\begin{aligned} G_{0i} &= -D_0\varphi_2\, i[E, \partial_i E] + D_0\varphi_1 \partial_i K \\ &\quad + F_{01}\hat{x}_i K \end{aligned} \tag{6}$$

and

$$\begin{aligned} \tfrac{1}{2}\epsilon_{ijk}G_{jk} &= D_1\varphi_1 i[K, \partial_i E] + D_1\varphi_2\partial_i E \\ &\quad + (1/r^2)(1-\varphi^2)\hat{x}_i E\,, \end{aligned} \tag{7}$$

where the covariant derivative and the field strength for a two-dimensional Abelian O(2) model was introduced: $D_\alpha \varphi_m = \partial_\alpha \varphi_m + \epsilon_{mn} A_\alpha \varphi_n$, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, $\varphi^2 = \varphi_m \varphi_m$, $(\alpha, \beta, \ldots = 0, 1; m, n, \ldots = 1, 2)$.

Equations (6) and (7) show that in contrast to Witten's ansatz, the self-duality condition $G_{\mu\nu} = \pm \tilde{G}_{\mu\nu}$ is only satisfied for the trivial solution $D_\alpha \varphi_m = 0$, $F_{\alpha\beta} = 0$, and $\varphi^2 = 1$. This comes about because G_{0i} and $\epsilon_{ijk} G_{jk}$ involve different E and K terms. Hence, the vacuum with vanishing G_{0i} and G_{ij} is the only self-dual solution admitted by our ansatz. Because of the O(2) gauge invariance

$$\begin{aligned} \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} &\to \varphi' = e^{-\Lambda\epsilon} \varphi , \\ A_\alpha \to A'_\alpha &= A_\alpha + \partial_\alpha \Lambda , \end{aligned} \tag{8}$$

this solution can always be transformed to $\varphi_1 = 0$, $\varphi_2 = 1$, $A_\alpha = 0$. Further, $\mathrm{tr} G_{\mu\nu} \tilde{G}_{\mu\nu}$ and thus the topological charge is zero for our ansatz (4).

The equations of motion

$$\partial_\nu G_{\mu\nu} = i[G_{\mu\nu}, W_\nu] , \tag{9}$$

however, involve the same terms on both sides and reduce to

$$D_\alpha D_\alpha \varphi_m = (1/r^2) \varphi_m (\varphi^2 - 1) , \tag{10a}$$

$$\partial_\beta (r^2 F_{\alpha\beta}) = 6 \epsilon_{mn} \varphi_m D_\alpha \varphi_n . \tag{10b}$$

These are, on the other hand, the Euler-Lagrange equations of the action

$$A = 32\pi \int_{-\infty}^{\infty} \int_0^\infty dx_0 dr \left(\tfrac{1}{2} (D_\alpha \varphi_m)^2 + \tfrac{1}{24} r^2 F_{\alpha\beta}{}^2 + \frac{1}{4r^2} (1 - \varphi^2)^2 \right), \tag{11}$$

which we get from the full action with our ansatz (4).

Except for a factor 3 in (10b) and (11) our action and equations of motion are identical to Witten's action and equations of motion. To find Yang-Mills solutions, one thus has in either case to find solutions of a two-dimensional Abelian Higgs model in a space of constant negative curvature. Only the relative factors of the kinetic energy term of the gauge fields are different. And, of course, the technique to find solutions to the Abelian Higgs model, which was successful in Witten's case, cannot be applied here because, if solutions exist, they are certainly not self-dual. This is exactly what makes our technique interesting: Using the 5-plet K_{ij} instead of the 3-plet E_i excludes the self-dual solutions but preserves the equations of motion except for constant factors.

Techniques which rely on the equations of motion and not on the self-duality condition may well be successfully transcribed. Glimm and Jaffe's[8] construction of multimerons is an example supporting this statement. Like Glimm and Jaffe we can choose the ansatz

$$A_\alpha = -\partial_\alpha \Lambda, \quad \varphi_1 = \sin\Lambda\, \phi, \quad \varphi_2 = \cos\Lambda \phi \tag{12}$$

with

$$\Lambda = -\sum_p \tan^{-1} \frac{r}{x_0 - a_p} + \sum_p \tan^{-1} \frac{r}{x_0 - b_p} \tag{13}$$

and by an O(2) rotation (8) transform it to $A'_\alpha = 0$, $\varphi'_1 = 0$, and $\varphi'_2 = \phi$. Thus, the equations of motion (10) reduce to

$$r^2 (\partial_0{}^2 + \partial_1{}^2) \phi = \phi^3 - \phi . \tag{14}$$

For this equation Jonsson *et al.*[9] proved existence of solutions. These solutions are not very interesting because they have infinite action[9] and in our case zero topological charge. Nevertheless, this example and the results of Ref. 1 show that using higher SU(2) multiplets for an embedding of an ansatz is an interesting technique although it did not yet provide a non-self-dual solution with finite action and answer the question whether the non-self-dual finite-action CP^N solutions[10] have Yang-Mills analogs.

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