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Realistic calculations of solar-neutrino oscillations

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We reexamine the possible effects of oscillations on the apparent solar-neutrino flux, integrated over the theoretical solar spectrum weighted by the ^{37}Cl or ^{71}Ga neutrino capture cross section. Spectral, thermal, and distance averaging do not reduce all oscillations to their mean values. The averaged $\nu_e \to \nu_e$ transition probability can fall as low as 0.1. Annual variations are evaluated.

The discrepancy between the observed[1] solar-neutrino capture rate 2.1 ± 0.3 SNU (solar-neutrino units) and the calculated[2] rate 7.5 ± 1.5 SNU has been widely attributed to neutrino oscillations.[3,4] If all oscillation wavelengths are much shorter than the mean radius $\bar{R} = 1.5 \times 10^{11}$ m of the Earth's orbit, the formulas are simple and mixing of three neutrinos can suppress the averaged transition probability $\langle P(\nu_e \to \nu_e)\rangle$ by factors down to $\frac{1}{3}$. If any oscillation wavelengths are comparable to $\bar{R}$, however, the picture becomes both more interesting and more complicated. The averaged transition probability can fall below $\frac{1}{3}$ (still with three neutrinos)[4] and there can be annual variations related to the eccentricity of the Earth's orbit.[3-5] In this circumstance calculations must include[4] integration over the solar emission spectrum weighted by the detector capture cross section.

We present updated and rather complete calculations of this kind for existing ^{37}Cl and future ^{71}Ga detectors. The spectrum-averaged value $\langle P(\nu_e \to \nu_e)\rangle_s$ for the ^{37}Cl experiment can be as low as 0.1 for two neutrinos[4,6] or 0.05 for three neutrinos, for certain mass differences and symmetrical mixing. Annual variations related to orbital eccentricity come mainly from a narrow ^7Be line; we evaluate the maximal possible variations.

We suppose that the electron neutrino ν_e is a linear superposition of nondegenerate mass eigenstates ν_i with masses m_i $(i = 1, 2, \ldots, n)$:

$$|\nu_e\rangle = \sum_i U_{ei} |\nu_i\rangle , \tag{1}$$

where U is a unitary mixing matrix. Given an initial relativistic ν_e of energy E, the probability of finding ν_e after flight path L is

$$P(\nu_e \to \nu_e) = 1 - \sum_{i<j} 4 |U_{ei}|^2 |U_{ej}|^2 \sin^2(\tfrac{1}{2}\Delta_{ij}) , \tag{2}$$

where $\Delta_{ij} = \frac{1}{2}\delta m_{ij}^2 L/E$ and $\delta m_{ij}^2 = m_i^2 - m_j^2$. If all wavelengths $4\pi E/\delta m_{ij}^2$ are much less than the size of the emitting solar core $\Delta L \sim 10^8$ m or the annual variation of Earth-Sun distance $\Delta L \sim 5 \times 10^9$ m (assuming a long time average), the oscillations in Eq. (2) are not resolved and the averaged transition probability is then

$$\langle P(\nu_e \to \nu_e)\rangle = 1 - \sum_{i<j} 2 |U_{ei}|^2 |U_{ej}|^2 . \tag{3}$$

This simple formula has lower bound $\langle P(\nu_e \to \nu_e)\rangle \geq 1/n$ for n-neutrino mixing. If the oscillation wavelengths are not all $\ll \Delta L$, we must use Eq. (2) to make an energy average over the source and detector.

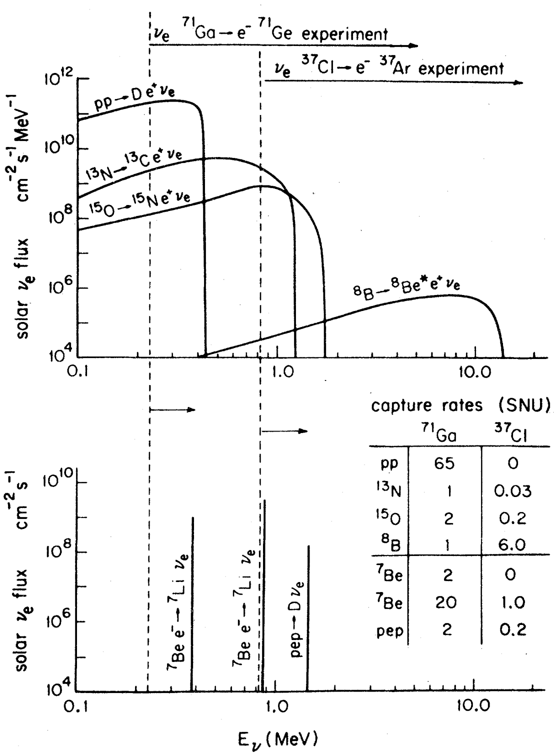
The solar emission spectrum for the standard model[2] is shown in Fig. 1, with the thresholds and rates for the $\nu_e + {}^{37}Cl \to e^- + {}^{37}Ar$ and $\nu_e + {}^{71}Ga \to e^- + {}^{71}Ge$ neutrino capture reactions. We examine what happens to the factors $\sin^2(\frac{1}{2}\Delta_{ij})$ in Eq. (2) for $L = \bar{R} = 1.5 \times 10^{11}$ m when averaged over the spectrum weighted by the capture cross section. Figure 2 shows the averaged[7] value of $\sin^2(\frac{1}{2}\Delta) = \sin^2(\frac{1}{4}\delta m^2 L/E)$ at $L = \bar{R}$ vs δm^2, for ^{37}Cl and ^{71}Ga detectors. Changing L would simply rescale δm^2. For an ideal narrow line spectrum the result would be a sinusoid oscillating between 0 and 1; the figure shows results for the real solar spectrum. Consider for example the ^{37}Cl case.

(i) For small enough δm^2 there is no effect: $\sin^2(\frac{1}{2}\Delta)$ is small throughout the range of E.

(ii) As δm^2 increases above 10^{-13} eV^2, the first contributions to $\sin^2(\frac{1}{2}\Delta)$ arise from the region near 1 MeV (dominated by the ^7Be line at $E = 0.862$ MeV); they are approximately sinusoidal at first, bounded by 0.2.

(iii) As δm^2 increases above 10^{-11} eV^2, the ^8B continuum around 8 MeV gives an additional sinusoid that rises initially to about 0.8.

(iv) As δm^2 increases beyond 10^{-10} eV^2, the broad spectral contributions damp toward constants, and the narrow lines from ^7Be electron capture and pep fusion give superimposed oscillations that are finally damped by thermal broad-



	^{71}Ga	^{37}Cl
pp	65	0
^{13}N	1	0.03
^{15}O	2	0.2
^{8}B	1	6.0
^{7}Be	2	0
^{7}Be	20	1.0
pep	2	0.2

FIG. 1. Continuous bands and discrete lines of the solar emission spectrum in the standard model, based on Ref. 2. Thresholds for ^{37}Cl and ^{71}Ga detectors are indicated.

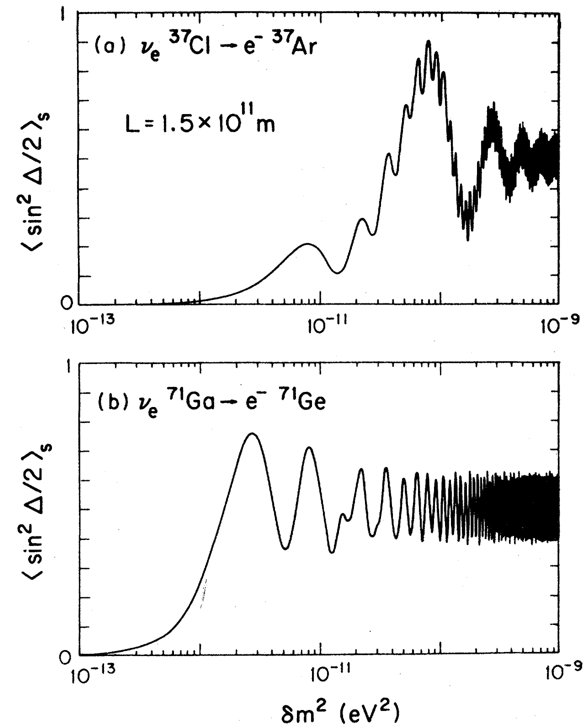


FIG. 2. Spectrum-averaged values of $\sin^2(\frac{1}{2}\Delta)$ vs δm^2 at $L = 1.5\times 10^{11}$ m for (a) ^{37}Cl detectors and (b) ^{71}Ga detectors.

ening ($\Delta E \sim 1$ keV) for $\delta m^2 \gtrsim 10^{-8}$ eV^2. In the ^{71}Ga results of Fig. 2(b), the dominant *pp* flux enters first and the persistent oscillation comes mainly from a narrow ^{7}Be line.

The spectrum-averaged transition probability $\langle P(\nu_e \to \nu_e)\rangle_s$ can be read directly from Eq. (2) and Fig. 2 for any neutrino mixing, at any fixed L. We use the subscript s to emphasize spectrum averaging as opposed to complete oscillation averaging for which $\langle \sin^2(\frac{1}{2}\Delta)\rangle = \frac{1}{2}$ and Eq. (3) applies. It is instructive to consider some particular examples.

Two-neutrino mixing. Assuming maximal mixing $|U_{ei}|^2 = \frac{1}{2}$, the complete oscillation average of Eq. (3) achieves its lower bound $P = \frac{1}{2}$. With spectrum averaging alone, however, we have

$$\langle P(\nu_e \to \nu_e)\rangle_s = 1 - \langle \sin^2(\tfrac{1}{2}\Delta_{12})\rangle_s \qquad (4)$$

that can fall to 0.1 for δm_{12}^2 near 0.8×10^{-10} eV^2, for a ^{37}Cl detector.

Three-neutrino mixing. With symmetrical mixing $|U_{ei}|^2 = \frac{1}{3}$ we have

$$P(\nu_e \to \nu_e) = 1 - \tfrac{4}{9}[\sin^2(\tfrac{1}{2}\Delta_{12}) + \sin^2(\tfrac{1}{2}\Delta_{23}) + \sin^2(\tfrac{1}{2}\Delta_{31})]. \qquad (5)$$

The complete oscillation average then reaches its lower bound $\langle P\rangle = \frac{1}{3}$. The spectrum average, however, depends on two independent δm_{ij} parameters, and we distinguish several cases (with the convention $m_1 < m_2 < m_3$).

(i) All $|\delta m_{ij}^2| \gg 4\pi E/L$; L averaging sets all $\sin^2(\frac{1}{2}\Delta_{ij})$ to $\frac{1}{2}$ as discussed earlier, the complete oscillation average.

(ii) Only $\delta m_{32}^2, \delta m_{31}^2 \gg 4\pi E/L$; the Δ_{23} and Δ_{31} oscillations average out completely, while Δ_{12} is subject only to spectral averaging, giving

$$\langle P(\nu_e \to \nu_e)\rangle_s = \tfrac{5}{9} - \tfrac{4}{9}\langle \sin^2(\tfrac{1}{2}\Delta_{12})\rangle_s . \qquad (6)$$

Hence $\langle P\rangle_s$ can fall to 0.15 for a ^{37}Cl detector for $\delta m_{12}^2 = 0.8\times 10^{-10}$ eV^2. In this mass regime symmetrical three-neutrino mixing does not allow $\langle P\rangle_s$ to fall quite as low as two-neutrino mixing does.

(iii) $\delta m_{31}^2 \approx \delta m_{32}^2 \approx 4\pi E/L \gg \delta m_{21}^2$. Here $\sin^2\frac{1}{2}\Delta_{21}$ is essentially zero and $\langle P\rangle_s = 1 - \frac{8}{9}\langle \sin^2(\frac{1}{2}\Delta_{31})\rangle_s$ can fall to 0.12 for a ^{37}Cl detector. A similar result holds if $\delta m_{31}^2 \approx \delta m_{21}^2 \gg \delta m_{32}^2$.

(iv) All oscillation lengths $4\pi E/\delta m_{ij}^2 \gtrsim L$. The three oscillatory terms on the right-hand side of Eq. (5) are subject to spectrum averaging only, and we now have two independent mass scales δm_{21}^2 and δm_{31}^2 to consider. The results for $\langle P\rangle_s$ can in principle be deduced from Fig. 2, but it is more transparent to make a contour plot in the $\delta m_{21}^2, \delta m_{31}^2$ plane, as we show in Fig. 3.

In these contour plots $\delta m_{31}^2 > \delta m_{21}^2 > 0$ by convention (i.e., $m_3 > m_2 > m_1$). There is a symmetry about the diagonal $\delta m_{21}^2 = \frac{1}{2}\delta m_{31}^2$, in the sense that $\langle P \rangle_s$ is unchanged by the substitution $\delta m_{21}^2 \to \delta m_{31}^2 - \delta m_{21}^2$ in the symmetric mixing model. Figure 3 shows that long-wavelength oscillation effects can take $\langle P \rangle_s$ below 0.1 for suitable δm_{ij}^2 for the ^{37}Cl detector (in fact values as low as 0.05 are possible) and close to 0.1 for the ^{71}Ga detector.

We see that $\langle P \rangle_s$ values can differ quite widely between ^{37}Cl and ^{71}Ga detectors for long-wavelength regimes. If such a difference were established experimentally (modulo uncertainties in the solar model) it would be evidence for a long-wavelength oscillation, and contour plots such as in Fig. 3 could be used to restrict the allowed ranges of δm_{ij}^2.

The Earth-Sun distance L varies annually between perihelion, $L = R_P$, and aphelion, $L = R_A$, $\Delta L = R_A - R_P = \bar{R}/30$. To use Fig. 2 that was calculated for the mean $L = \bar{R} = 1.5 \times 10^{11}$ m, we must multiply the scale of δm^2 by the varying factor $\bar{R}/L$. In principle we can then read the annual variation in $\langle \sin^2(\frac{1}{2}\Delta)\rangle_s$ directly from Fig. 2. The steeper the local slope $d\langle \sin^2(\frac{1}{2}\Delta)\rangle_s / d(\ln \delta m^2)$, the faster the time variation of the spectrum average of $\sin^2(\frac{1}{2}\Delta)$. The time variation is dominated by the ^{7}Be spectral line at $E = 0.862$ MeV, through its contribution of $0.14 \sin^2(\frac{1}{4}\delta m^2 L/E)$.

The most dramatic annual variation results for $(\Delta L \delta m^2)/(2E) \gtrsim \pi$ corresponding to a half-cycle or more of the ^{7}Be oscillation. The lowest δm^2 that achieves this is about 2×10^{-10} eV^2, in which case the annual variation could be 1 SNU for the ^{37}Cl detector and 20 SNU for the ^{71}Ga detector. Only a few such cycles could be discriminated experimentally. A case for an annual variation in the solar-neutrino data has been presented by Ehrlich.[5]

The contribution of a narrow line to annual variations is simply expressed through Eq. (2), taking E as the line energy and multiplying by its overall weight factor. The contribution of the continuum is usually ignored, because the line widths $\Delta E/E$ here are much larger than the eccentricity $\Delta L/L$. In calculations of annual changes, we find the continuum variations are less than 5% of their total contribution, and can be neglected in a first approximation.

For the symmetric solar minimizing solution with three neutrinos, the capture rate s (in SNU) varies approximately with L as

$$s = \alpha/3 + \beta - \beta V, \tag{7}$$
$$V = \tfrac{4}{9}[\sin^2(\tfrac{1}{2}\Delta_{12}) + \sin^2(\tfrac{1}{2}\Delta_{23}) + \sin^2(\tfrac{1}{2}\Delta_{31})] .$$

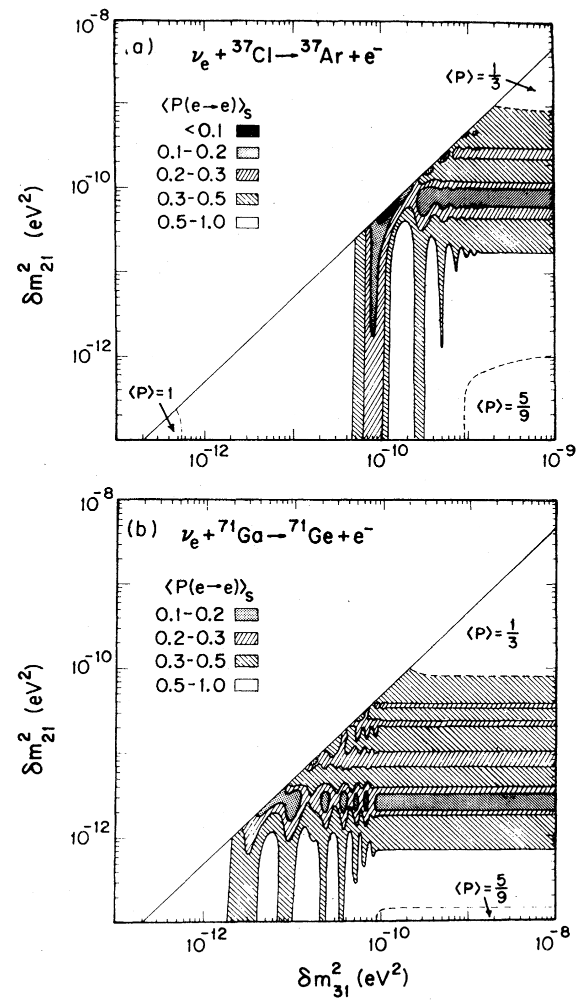


FIG. 3. Contour plot of the spectrum average $\langle P(\nu_e \to \nu_e)\rangle_s$ vs δm_{21}^2 and δm_{31}^2 for symmetrical three-neutrino mixing at $L = 1.5 \times 10^{11}$ m for the (a) ^{37}Cl detector and (b) ^{71}Ga detector. The sector $\delta m_{31}^2 > 2\delta m_{21}^2$ is shown; other sectors are defined by symmetry. In the ^{37}Cl results, the rapid oscillations due to the ^{7}Be and pep lines are included as oscillation averages.

Here α sums the continuum, ^{7}Be (0.384 MeV) and pep contributions, while β is due to the ^{7}Be (0.862 MeV) line. For the standard solar model,[2] $\alpha = 6.5$, $\beta = 1$ SNU for ^{37}Cl and $\alpha = 73$, $\beta = 20$ SNU for ^{71}Ga. The maximum variation of V is 1 (for an optimum choice of the δm_{ij}^2), and the corresponding variation is $\Delta s = \beta$. The effect $\Delta s/\bar{s} = 3\beta/(\alpha + \beta)$ could be as large as 40% for ^{37}Cl and 65% for ^{71}Ga. The maximum variation $\Delta s = 1$ SNU for ^{37}Cl is difficult to test with present experimental uncertainties ± 0.5 SNU per quarter year.

The following δm^2 categories give large variations in V and hence in s:

(i) $\delta m_{21}^2 \simeq \delta m_{31}^2 \gtrsim 2\times10^{-10}$ eV^2, $0 \leqslant V \leqslant 1$

(ii) $\delta m_{21}^2 \ll \delta m_{31}^2 \gtrsim 2\times10^{-10}$ eV^2, $0 \leqslant V \leqslant \frac{8}{9}$

(iii) $\delta m_{31}^2 \gg \delta m_{21}^2 \gtrsim 2\times10^{-10}$ eV^2, $\frac{4}{9} \leqslant V \leqslant \frac{8}{9}$.

If all δm^2 are much larger or much smaller than 2×10^{-10} eV^2, the variation of s with orbital distance is negligible.

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[6]G. J. Stephenson (private communication) drew our attention to this result.

[7]For these averages, we used standard solar model spectra and capture cross sections given in Ref. 2.