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Quark liberation at high temperature: A Monte Carlo study of SU(2) gauge theory

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Quark confinement in a finite-temperature SU(N) gauge theory is formulated as the realization of a global Z_N symmetry. Spontaneous breakdown corresponds to a transition to a nonconfining, plasma phase. The free energy of a single quark is an order parameter which probes the phase structure, and it may be calculated in the Euclidean theory in terms of a "Wilson line" running the length of the system along the (periodic) time axis. We present results of a Monte Carlo calculation in the SU(2) lattice theory which confirm the transition at a critical temperature computed in terms of the zero-temperature string tension; data for the quark-antiquark potential are presented as well. We discuss the implications of the finite-temperature transition for efforts to calculate zero-temperature quantities on finite-size lattices. Finally, we note that restoration of Z_N symmetry as the temperature is lowered may be understood as a condensation of instantons and other topological objects.

I. INTRODUCTION

The study of hot hadronic matter may yield a particularly clear picture of the physics of quark confinement.[1] A wide variety of arguments, based on asymptotically free perturbation theory[2] or on strong-coupling lattice gauge theories,[3,4] suggests a transition from a confining phase to a plasma phase at some critical temperature and density. By investigating the nature of this transition, one may hope to gain an understanding of the physics of the zero-temperature vacuum as well as to predict new phenomena in nucleus-nucleus collisions,[5] in neutron stars,[6] and in the early stages of a big-bang universe.[7]

In this paper, we discuss the confinement-plasma transition in quarkless Yang-Mills theory. We identify an order parameter which is directly related to the free energy of an isolated (static) quark; the transition then corresponds to the appearance of a magnetization in this order parameter. This may be interpreted as the spontaneous breakdown of a global Z_N symmetry of the gauge theory [the center of SU(N)]. In order to probe the realization of this symmetry, it is convenient to impose an ultraviolet cutoff and thus to deal with the Euclidean lattice theory[8]; we present the results of a Monte Carlo investigation of the dynamics of this theory for the case that the gauge group is SU(2).

Recently, Monte Carlo techniques[9] have been employed in the direct evaluation of vacuum expectation values in lattice gauge theories. Numerical "experiments" have been performed on zero-temperature theories in this way,[10,11] and a smooth connection between the strong- and the weak-coupling regimes has been established. Even with lattices of small size, the confining nature of Yang-Mills theories is apparent. A more precise extrapolation of these results to the continuum limit is a goal of current research, but we may employ present knowledge to make statements concerning the behavior of our system when the lattice is removed.

The Monte Carlo calculation verifies the phase transition in the SU(2) gauge theory as described. At low temperatures, the system is disordered and quarks are confined, while at high temperatures the Ising-type Z_2 symmetry is broken and a (screened) Coulomb phase appears.[12]

Our focus on the realization of the global Z_N symmetry yields a picture of the role of finite-temperature instantons[13] in bringing on the phase transition as it is approached from above.[14] In the language of the Ising model, the instantons are domains of flipped Z_2 spins embedded in a magnetized medium. The phase transition is then a condensation of instantons (and of other topological objects) which destroys the magnetization.

The organization of this paper is as follows.

In Sec. II, we briefly review the formalism of finite-temperature Yang-Mills theory[15] and of its realization on a Euclidean lattice, periodic in the time direction.

In Sec. III, we introduce the operator L which measures $e^{-\beta F_q}$, where F_q is the free energy of an isolated, static quark. This operator is an order parameter. In an ordered phase, it has nonzero expectation value corresponding to finite quark free energy. In a disordered phase, $\langle L\rangle = 0$, implying a divergent quark free energy and quark confinement. The two-point correlation function of L yields the free energy of a quark-antiquark pair.

The fact that $\langle L\rangle = 0$ in the disordered phase is a consequence of Z_N symmetry which arises from an invariance of the Yang-Mills theory under certain aperiodic gauge transformations like those considered by 't Hooft.[16] In the absence of spontaneous breakdown, this symmetry is sufficient to show that nonsinglet configurations of static quarks have divergent free energy. The only configurations with finite free energy are states for which the number of quarks minus antiquarks is an integral multiple of N. The ordered phase spontaneously breaks the Z_N symmetry and admits free quarks.

In Sec. IV, we explain briefly how one removes the lattice regulator and renormalizes to obtain finite physical results, expressed in terms of some physical scale parameter such as the zero-temperature string tension.

In Sec. V, we present numerical results of our Monte Carlo computations. We determine the critical temperature of the confinement-plasma transition by studying the critical behavior of our order parameter and extrapolating to the continuum limit. We discuss the quark free energy and the quark-antiquark free energy,[17] and we argue that the transition is second order.

We also discuss the appearance of a phase transition in the space-time-symmetric "zero-temperature" lattice considered by Creutz.[10] We believe this creates difficulties for any determination of zero-temperature string tension via square Wilson loops unless the lattice considered is very large.

In Sec. VI, we present conjectures concerning the underlying dynamics of the transition. We interpret our results as indicating that as the transition is approached from the plasma phase, a condensation of instantons and other objects takes place. An instanton calculation of the expectation value of our order parameter and a comparison with our Monte Carlo data would throw light on the validity of this conjecture.

In the Appendix, we discuss the statistical analysis of our data.[18]

II. FINITE-TEMPERATURE GAUGE THEORY AND THE LATTICE

The starting point for our analysis is the Feynman path-integral representation for the expectation values of quantum operators at finite temperature,[15]

$$\langle \mathcal{O}\rangle \equiv \frac{\mathrm{Tr}\,\mathcal{O}e^{-\beta H}}{\mathrm{Tr}\,e^{-\beta H}} \tag{2.1}$$

$$= \frac{\int \mathcal{D}A^\mu(\vec{x},t)\,\mathcal{O}(A)\exp[-\int_0^\beta dt\int_V d^3x\,\mathcal{L}(A)]}{\int \mathcal{D}A^\mu(\vec{x},t)\exp[-\int_0^\beta dt\int_V d^3x\,\mathcal{L}(A)]}\,. \tag{2.2}$$

$\mathcal{L}(A)$ is the Euclidean Lagrange density, and $\mathcal{O}(A)$ is the operator $\mathcal{O}$ as a functional of the fields A. The inverse temperature is β and the spatial volume is V. The fields are constrained to be periodic in Euclidean time with period β.

The formal integrations over the fields in (2.2) may be performed by placing the fields on the links of a space-time lattice[8] with spacing a, which serves as an ultraviolet cutoff. The number of links in the time direction is N_t and in the space direction N_s. The inverse temperature and volume are thus given by

$$\beta = N_t a\,, \tag{2.3}$$

$$V = (N_s a)^3\,. \tag{2.4}$$

The continuum limit corresponds to the limits N_t, $N_s \to \infty$, and $a \to 0$ with β and V held fixed. The limit $a\to 0$ may be approximated by making a small compared to scale sizes typical of the operator being evaluated and of the quanta of the field theory generated by $\mathcal{L}$.

For renormalizable field theories such as Yang-Mills theories, the limit of zero lattice spacing is singular. The bare coupling constant g_0 must be adjusted simultaneously with the lattice spacing to yield finite limiting values for physically measurable quantities. The asymptotic freedom of Yang-Mills theories implies that g_0 should be adjusted to zero as the lattice is removed. If the zero-temperature theory truly confines, the (zero-temperature) string tension, which is a measure of the strength of the quark-antiquark force, may be held fixed as this adjustment is made.

A conventional transcription of the gauge theory to the lattice begins with the definition of the link variable

$$U^\mu(\vec{x}) \equiv e^{i\vec{\tau}\cdot\vec{A}^\mu(\vec{x})} \tag{2.5}$$

on the link leaving site $\vec{x}$ in the $\hat{\mu}$ direction, where $\vec{\tau}$ are the generators of SU(N) in the fundamental representation. The Lagrange density may be expressed in terms of these link variables as

$$\mathcal{L}(\vec{x}) = \frac{4}{g_0^2a^4}\sum_{\mu,\nu}\frac{1}{2}\,\mathrm{tr}[1 - U^\mu(\vec{x})\,U^\nu(\vec{x}+a\hat{\mu}) \times U^\mu(\vec{x}+a\hat{\nu})^{-1}U^\nu(\vec{x})^{-1}]\,, \tag{2.6}$$

where the traces are of products around the elementary plaquettes of the lattice. This Lagrangian is invariant under the gauge transformation

$$U^\mu(\vec{x}) \to V(\vec{x}+a\hat{\mu})U^\mu(\vec{x})V^{-1}(\vec{x}), \tag{2.7}$$

where $V(\vec{x})$ is a member of the gauge group

$$V(\vec{x}) = e^{i\vec{\tau}\cdot\vec{\Lambda}(\vec{x})}. \tag{2.8}$$

Once we replace $\mathfrak{D}A$ in (2.2) with $\mathfrak{D}U$, the invariant measure on SU(2) on each link, the path integral is properly defined, and the integration over the fields may be attempted. [Note that the lattice spacing in (2.6) will cancel when $\int dt\, d^3\vec{x} \to a^4\sum$.] In Sec. IV we shall report the results of a Monte Carlo integration of (2.2) for various operators of an SU(2) Yang-Mills theory.

[One may choose to employ a lattice with different spacings in the space and time directions,[19] a_s and a_t. In that case the couplings in (2.6) will be different for spacelike and for timelike plaquettes. Taking the limit $a_t \to 0$ first yields the Hamiltonian lattice theory.[20]]

III. QUARK FREE ENERGIES

Quantities which characterize the phases of a gauge theory are the free energies of static configurations of quarks and antiquarks. Let us introduce operators, $\psi_a^\dagger(\vec{r}_i,t)$ and $\psi_a(\vec{r}_i,t)$, which create and annihilate static quarks with color a at position $\vec{r}_i$ and time t, along with their charge conjugates $\psi_a^{\dagger c}$ and ψ_a^c for antiquarks. These static fields satisfy the equal-time anticommutation relations

$$[\psi_a(\vec{r}_i,t), \psi_b^\dagger(\vec{r}_j,t)]_+ = \delta_{ij}\,\delta_{ab}, \tag{3.1}$$

and similarly for ψ^c, with all other equal-time anticommutators vanishing.

The quark fields obey the static time-evolution equation[21]

$$\left(\frac{1}{i}\frac{\partial}{\partial t} - \vec{\tau}\cdot\vec{A}^0(\vec{r}_i,t)\right)\psi(\vec{r}_i,t) = 0. \tag{3.2}$$

This equation may be integrated to yield

$$\psi(\vec{r}_i,t) = T\exp\left(i\int_0^t dt'\,\vec{\tau}\cdot\vec{A}^0(\vec{r}_i,t')\right)\psi(\vec{r}_i,0). \tag{3.3}$$

The symbol T in this equation denotes a time-ordered exponential.

The operators ψ and ψ^c may be employed to obtain an expression for the free energy of a configuration of N_q quarks and $N_{\bar{q}}$ antiquarks. This is given by[22]

$$\exp[-\beta F(\vec{r}_1,\ldots,\vec{r}_{N_q},\vec{r}_1{}',\ldots,\vec{r}_{N_{\bar{q}}}{}')] = \frac{1}{N^{N_q+N_{\bar{q}}}}\sum_{|s\rangle}\langle s|e^{-\beta H}|s\rangle \tag{3.4}$$

with the summation indicated over all states $|s\rangle$ with heavy quarks at $\vec{r}_1,\ldots,\vec{r}_{N_q}$ and antiquarks at $\vec{r}_1{}',\ldots,\vec{r}_{N_{\bar{q}}}{}'$. Introducing the quark fields

$$e^{-\beta F_{N_q N_{\bar{q}}}} = \frac{1}{N^{N_q+N_{\bar{q}}}}\sum_{|s'\rangle}\left\langle s'\left|\sum_{\{a,b\}}\psi_{a_1}(\vec{r}_1,0)\cdots\psi_{a_{N_q}}(\vec{r}_{N_q},0)\psi_{b_1}^c(\vec{r}_1{}',0)\cdots\psi_{b_{N_{\bar{q}}}}^c(\vec{r}_{N_{\bar{q}}}{}',0)\right.\right.$$
$$\left.\left.\times e^{-\beta H}\psi_{a_1}^\dagger(\vec{r}_1,0)\cdots\psi_{a_{N_q}}^\dagger(\vec{r}_{N_q},0)\psi_{b_1}^{\dagger c}(\vec{r}_1{}',0)\cdots\psi_{b_{N_{\bar{q}}}}^{\dagger c}(\vec{r}_{N_{\bar{q}}}{}',0)\right|s'\right\rangle, \tag{3.5}$$

where now the sum is over all states $|s'\rangle$ with *no* heavy quarks. Since $e^{-\beta H}$ generates Euclidean time translations, i.e.,

$$e^{\beta H}O(t)\,e^{-\beta H} = O(t+\beta) \tag{3.6}$$

for any operator $O(t)$, (3.5) becomes

$$e^{-F_{N_q N_{\bar{q}}}} = \frac{1}{N^{N_q+N_{\bar{q}}}}\sum_{|s'\rangle}\left\langle s'\left|\sum_{\{a,b\}}e^{-\beta H}\psi_{a_1}(\vec{r}_1,\beta)\psi_{a_1}^\dagger(\vec{r}_1,0)\cdots\psi_{a_{N_q}}(\vec{r}_{N_q},\beta)\psi_{a_{N_q}}^\dagger(\vec{r}_{N_q},0)\right.\right.$$
$$\left.\left.\times\psi_{b_1}^c(\vec{r}_1{}',\beta)\psi_{b_1}^{\dagger c}(\vec{r}_1{}',0)\cdots\psi_{b_{N_{\bar{q}}}}^c(\vec{r}_{N_{\bar{q}}}{}',\beta)\psi_{b_{N_{\bar{q}}}}^{\dagger c}(\vec{r}_{N_{\bar{q}}}{}',0)\right|s'\right\rangle. \tag{3.7}$$

Using (3.3) and its charge conjugate, along with (3.1), and introducing the definition of the Wilson line[16] as

$$L(\vec{r}) \equiv \frac{1}{N}\,\mathrm{tr}\,T\exp\left[i\int_0^\beta dt\,\vec{\tau}\cdot\vec{A}^0(\vec{r},t)\right], \tag{3.8}$$

(3.7) becomes

$$e^{-\beta F_{N_q N_{\bar{q}}}} = \mathrm{Tr}[e^{-\beta H}L(\vec{r}_1)\cdots L(\vec{r}_{N_q})L^\dagger(\vec{r}_1{}')\cdots L^\dagger(\vec{r}_{N_{\bar{q}}}{}')]. \tag{3.9}$$

The trace is over states of the pure gluon theory. For one quark, this quantity is especially simple:

$$e^{-\beta F_{1,0}} = \mathrm{Tr}\,e^{-\beta H}L(0). \tag{3.10}$$

Dividing (3.9) by the expression for the vacuum (the state with no quarks or antiquarks) yields the difference $\Delta F_{N_q N_{\bar q}} \equiv F_{N_q N_{\bar q}} - F_{00}$ as an expression of the form of (2.1):

$$e^{-\beta \Delta F_{N_q N_{\bar q}}} = \langle L(\vec{r}_1) \cdots L(\vec{r}_{N_q}) L^\dagger(\vec{r}_1') \cdots L^\dagger(\vec{r}_{N_{\bar q}}') \rangle . \tag{3.11}$$

For one quark,

$$e^{-\beta F_q} \equiv e^{-\beta \Delta F_{10}} = \langle L(0) \rangle . \tag{3.12}$$

Since we are dealing with static, infinitely massive quarks, the free energy of a single-quark system is in itself devoid of meaning. We note, however, that the cluster property demands that

$$\langle L(0) L^\dagger(\vec{r}) \rangle \xrightarrow[|\vec{r}| \to \infty]{} \langle L(0) \rangle \langle L^\dagger(\vec{r}) \rangle = |\langle L(0) \rangle|^2 \tag{3.13}$$

so that

$$F_{q\bar q}(\vec{r}) \equiv \Delta F_{11}(\vec{r}) \xrightarrow[|\vec{r}| \to \infty]{} 2F_q . \tag{3.14}$$

Thus if $F_{q\bar q}(\vec{r})$ is finite for some small value of $\vec{r}$ (a quark-mass renormalization may be necessary to ensure this), whether F_q is finite is truly a test of confinement.

Another approach is to isolate the temperature-dependent piece of F_q. If this can be rendered finite at high temperature, a divergence as the temperature is lowered would be a confinement signal.

Note that (3.11) also serves to define the energy of separation of quarks and antiquarks in the gluonic ground state when the limit $\beta \to \infty$ is taken, that is, β much larger than any typical hadronic scale size.

The Wilson line operator $L(\vec{r})$ is invariant under gauge transformations of the form (2.7) that are periodic, i.e.,

$$V(\vec{r}, \beta) = V(\vec{r}, 0) . \tag{3.15}$$

The class of allowable gauge transformations in the theory is not, however, restricted to those which are periodic. The only physically important constraint is that the links $U^\mu(\vec{r}, t)$ or, in the continuum, the fields $A^\mu(\vec{r}, t)$, remain periodic in time when transformed. A condition sufficient for this is[16]

$$V(\vec{r}, \beta) = V(\vec{r}, 0) C , \tag{3.16}$$

where the matrix C is an element of the center of the gauge group,

$$C = e^{2\pi i j/N} I . \tag{3.17}$$

Here I is the unit matrix and j is an integer. The additional symmetry of the theory under choice of C is a *global* Z_N symmetry which remains after all gauge fixing. The gluon Lagrange density is, of course, unchanged by this transformation. The Wilson line is changed, however, as

$$L(\vec{r}) \to e^{2\pi i j/N} L(\vec{r}) . \tag{3.18}$$

The free energy of a system of N_q quarks and $N_{\bar q}$ antiquarks transforms as

$$e^{-\beta \Delta F_{N_q N_{\bar q}}} \to e^{2\pi i j (N_q - N_{\bar q})/N} e^{-\beta \Delta F_{N_q N_{\bar q}}} . \tag{3.19}$$

Unless

$$N_q - N_{\bar q} = nN \tag{3.20}$$

for some integer n, Eq. (3.19) is sufficient to guarantee

$$e^{-\beta \Delta F_{N_q N_{\bar q}}} = 0 , \tag{3.21}$$

corresponding to divergent free energy, as long as the symmetry is not spontaneously broken. The dynamical realization of the symmetry of (3.15)–(3.17) is thus equivalent, if $N=3$ for example, to confinement of configurations of nonzero triality.

This Z_N symmetry need not be dynamically realized, however. If field configurations which are related by a Z_N transformation are not continuously connected through interpolating field configurations of finite energy and measure, we have spontaneous symmetry breaking.

There are N possible broken-symmetry ground states of the pure gauge system. These configurations are labeled by the N possible distinct expectation values of L,

$$\langle L \rangle = e^{2\pi i j/N} L_0 , \quad j = 0, 1, \ldots, N-1 . \tag{3.22}$$

$\langle L \rangle$ is therefore an order parameter quite similar to the magnetization in a Z_N spin system.

In weak-coupling perturbation theory, the quantity L_0 approaches unity since the free energy of a quark relative to the vacuum is zero in the $g_0 \to 0$ limit.

The transcription of (3.8) to the lattice is straightforward via (2.5):

$$L(\vec{r}) = \frac{1}{N} \operatorname{tr} \prod_{t=1}^{N_t} U^0(\vec{r}, t) . \tag{3.23}$$

The expectation value and correlation function of this operator are the main targets of our Monte Carlo calculation.

IV. THE CONTINUUM LIMIT

The lattice spacing a appears in all our considerations as the only dimensional scale: in particular, for our finite-temperature lattice, the physical inverse temperature is $\beta = N_t a$. Renormalization expresses the lattice spacing in terms of the bare coupling and some physical scale such as the zero-temperature string tension K as

$$a^2 = K^{-1} f(g_0)\,, \tag{4.1}$$

where f for the SU(2) gauge theory may be taken from a calculation such as the zero-temperature Monte Carlo work of Creutz.[10,23] One finds that f is described well by the strong- and weak-coupling approximations

$$\begin{aligned} f &= \ln g_0^2\,, \quad \frac{4}{g_0^2} \lesssim 2 \\ f &= \exp\left[-\frac{6\pi^2}{11}\left(\frac{4}{g_0^2} - 2\right)\right], \quad \frac{4}{g_0^2} \gtrsim 2\,. \end{aligned} \tag{4.2}$$

Since changing g_0 changes a, we can vary the temperature according to

$$\beta = N_t a(g_0^2)\,. \tag{4.3}$$

This relationship may be understood qualitatively by the following reasoning. Suppose a quark and an antiquark are separated by a fixed number of lattice sites. As g_0 increases, the force between the quark and the antiquark increases. The corresponding distance between the quark and the antiquark must, according to asymptotic freedom, be increasing. The lattice spacing is thus an increasing function of g_0. Since $T = \beta^{-1} \sim 1/a$, *increasing the bare coupling corresponds to decreasing the temperature.*

As we mentioned in Sec. II, we may trust the lattice theory to give results relevant to the continuum theory only when a is small. Equation (4.3) shows that we can satisfy such a constraint while probing any temperature by choosing N_t large enough. In particular, it is plausible that any critical coupling g_0^* should vary with N_t in such a way that the critical temperature T^* determined via (4.3) approach a finite limit as N_t grows. This is in fact what we observe in the Monte Carlo experiment. (In order to interpret β as the inverse temperature of an infinite system, we demand $N_s \gg N_t$.)

V. MONTE CARLO ANALYSIS OF THE SU(2) THEORY

Monte Carlo simulations, long known in statistical physics,[9] have proven useful for evaluating expectation values of operators in lattice gauge theory as well. We refer the reader to earlier work[10,11] for details of the technique. For the case at hand, each Monte Carlo simulation was carried out for fixed chosen values of N_t, N_s, the bare coupling g_0, and, implicitly, the lattice spacing a, which may be related to g_0 via (4.1), thus fixing the temperature via (4.3).

A. The order parameter

In Fig. 1 we show a set of pass-by-pass data for the average value of L. Each point represents an

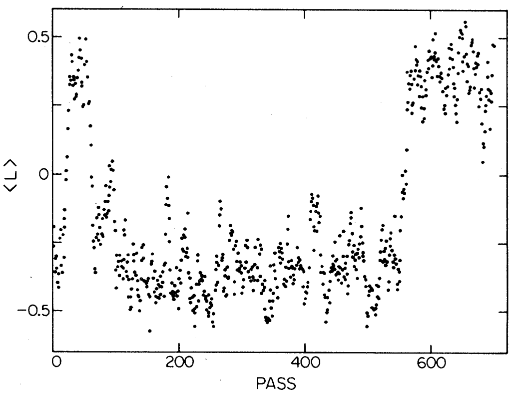

FIG. 1. Pass-by-pass values of the order parameter for $N_t = 1$, $N_s = 5$, and $4/g_0^2 = 0.9$. For a lattice of this size, $4/g_0^{*2} \simeq 0.75$. Starting conditions were $U^\mu = \sigma_x$ everywhere ($L = 0$).

average over the three-volume of the lattice, taken after one pass of touching a heat bath to every link in turn. The SU(2) gauge theory, as probed by our Z_2 order parameter, behaves much similar to an Ising model: after reaching equilibrium, the magnetization fluctuates thermally about some average value, until a large enough fluctuation causes nucleation of a bubble of reversed magnetization which takes over the lattice, reversing the sign of the bulk magnetization. Averages of such data over many passes may be interpreted as the true expectation value. (See the Appendix for a discussion of our statistical analysis.)

Values of $\langle L\rangle$ for $N_t = 3$ as a function of the bare coupling are plotted in Fig. 2. The transition to a low-temperature confining phase is evident. The shape of the curve for $\langle L\rangle$ vs $4/g_0^2$ is only slightly volume dependent: one sees a slight sharpening of

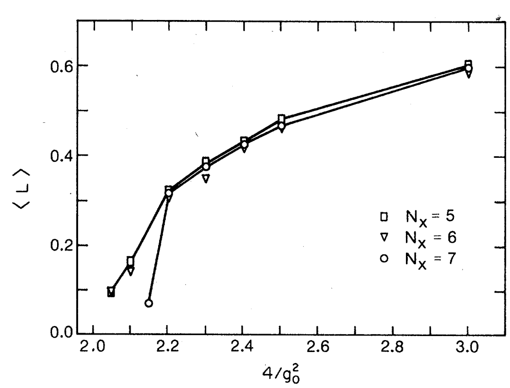

FIG. 2. Magnetization vs inverse coupling for $N_t = 3$ and several values of N_s. Data points for $N_s = 5$ and $N_s = 7$ are joined to guide the eye. Data for $N_s = 8$ (not shown) are close to those for $N_s = 7$. Each lattice shows $\langle L\rangle = 0$ if $4/g_0^2$ is decreased beyond the points shown. Typical error bars are as shown in Fig. 3.

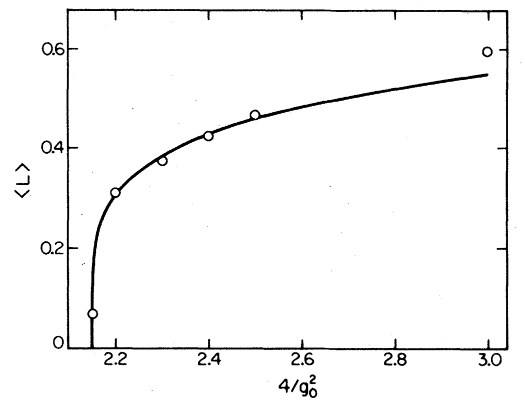

FIG. 3. Power-law fit to the magnetization for $N_t=3$, $N_s=7$. Error bars are the size of the circles.

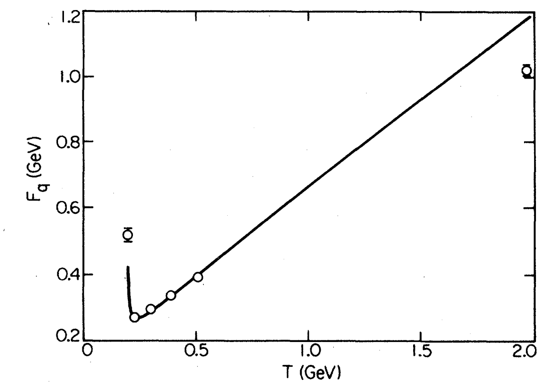

FIG. 4. Quark free energy vs temperature. The points and the curve result from simple transformations of Fig. 3.

the transition as N_s passes $2N_t$.

To obtain a more precise estimate of the critical coupling g_0^* and to learn more about the critical properties, one may fit $\langle L\rangle$ near the transition to the functional form

$$\langle L\rangle = A\left(\frac{4}{g_0^{\,2}} - \frac{4}{g_0^{*2}}\right)^{\beta}. \tag{5.1}$$

The best fit to our data for $N_t=3$, $N_s=7$ is displayed in Fig. 3. We find $4/g_0^{*2}=2.15$ to a high degree of precision, and $\beta=0.207\pm0.008$. (Properly speaking, β cannot be called the critical index, since the fit suffers from the usual experimental difficulty in defining the size of the critical region. For comparison, the critical index for the three-dimensional Ising model is believed[24] to be $\beta_{\text{Ising}} \simeq \frac{5}{16}$.)

We have three reasons for believing the transition to be second order.

(1) The magnetization appears continuous in the coupling, and the curve does not become steeper when N_s is increased past about $2N_t$.

(2) In the neighborhood of g_0^*, many more passes are needed (say 150) to reach apparent thermal equilibrium than far from the transition (typically 30).

(3) Pass-to-pass correlations in $\langle L\rangle$ are strong typically over 3–5 passes far from g_0^*, but extend in range to as many as 15 passes near g_0^*. These correlations are discussed further in the Appendix.

TABLE I. Approximants to T^*. The scale for a and T is set by $\sqrt{K}=400$ MeV.

Lattice size ($N_t\times N_s^3$)	$4/g_0^{*2}$	$a(g_0^{*2})$ (GeV^{-1})	aN_t (GeV^{-1})	T^* (MeV)
1×5^3	0.75	3.2	3.2	310
2×5^3	1.85	2.2	4.4	230
3×7^3	2.15	1.66	5.0	200

B. The critical temperature

Given g_0 and N_t, one may calculate the temperature of the lattice theory via (4.1)–(4.3). We have determined g_0^* for several lattice sizes from plots such as Fig. 3; the values are shown in Table I along with the temperatures to which they correspond. For definiteness we have set $K=(1/2\pi)$ GeV^2, which is approximately the value in the real world derived from Regge trajectories and the string model.[25] Thus, all dimensionful numbers we present would be the ones measured if the real world were described by an SU(2) gauge theory without dynamical quarks.

The critical point of the largest lattice in Table I occurs at a bare coupling corresponding to a length scale $a=\frac{1}{3}$ fm or a mass scale $1/a=600$ MeV. Since one may suppose that the physical scales of the theory are not far from these values, T_{cr} for $N_t=3$ may not by itself be taken for the continuum critical temperature. However, various rough extrapolations yield an estimate in the neighborhood of

$$T_{\text{cr}}(N_t\to\infty)\simeq 170 \text{ MeV}. \tag{5.2}$$

Apart from uncertainty in our extrapolation, Eq. (4.5) is subject to an uncertainty of 30% arising from Creutz's determination of the normalization of (4.2). A change in this normalization would result in a rescaling of (5.2) together with all other dimensional numbers.

C. Quark free energy

Using (3.12) and (4.1)–(4.3), we may transform the axes of Fig. 3 into more physical variables, yielding $F_q(T)$ as shown in Fig. 4. Weak-coupling perturbation theory indicates that F_q should be linear in T up to logarithmic corrections. The almost linear dependence on T evident in Fig. 4 shows that the finite-temperature contributions to this free energy are clearly isolated from the zero-temperature piece. The rapid increase at $T \sim 200$ MeV corresponds to the onset of quark confinement.

D. $q\bar{q}$ potential

The averaging over quark color orientations in (3.4) implies that our $F_{q\bar{q}}(\vec{r})$ is a thermal average over potentials in the possible color channels,

$$e^{-\beta F_{q\bar{q}}} = \tfrac{1}{4}(e^{-\beta V_1} + 3e^{-\beta V_3}) . \tag{5.3}$$

Here $V_1(V_3)$ denotes the $q\bar{q}$ potential in the singlet (triplet) channel for two isospin-$\frac{1}{2}$ particles. The 3 is the degeneracy of the triplet state.

Perturbation theory in the continuum predicts a screened Coulomb potential:

$$V_i(\vec{r}) = c_i g_0^2 \frac{1}{4\pi r} e^{-\mu r} + O(g_0^4) \tag{5.4}$$

with $c_1 = -\frac{3}{4}$, $c_3 = +\frac{1}{4}$, and $\mu^2 = \frac{2}{3} g^2 T^2$. For βg_0^2 small we may expand the exponentials in (5.4), and we find that the $O(g_0^2)$ terms cancel between the singlet and triplet terms. This may be seen immediately[26] from the fact that the diagram in Fig. 5(a) vanishes because $\mathrm{Tr}\,\tau^a = 0$, so that the first contribution to the $q\bar{q}$ potential is the two-gluon diagram in Fig. 5(b).

For our periodic lattice system we make the replacement

$$\frac{1}{4\pi r} \to \frac{1}{a} \phi(\vec{n}) , \tag{5.5}$$

where $\phi(\vec{n})$ is the lattice Green's function satisfying

$$(\nabla^2 + \mu^2 a^2)\phi(\vec{n}) = \delta_{\vec{n},0} \tag{5.6}$$

with periodic boundary conditions. ∇^2 is the lattice Laplacian

$$\nabla^2 \phi(\vec{n}) = \sum_{\hat{\mu}} [\phi(\vec{n} + \hat{\mu}) + \phi(\vec{n} - \hat{\mu})] - 6\phi(\vec{n}) . \tag{5.7}$$

The correlation function $\langle L(0)L(\vec{r})\rangle$ may be extracted from the Monte Carlo calculation in the same way as $\langle L\rangle$; taking the logarithm yields $\beta F_{q\bar{q}}(\vec{r})$ according to (3.11). The high-temperature data presented in Fig. 6 fit the weak-coupling picture discussed above. As we approach the transition we obtain Fig. 7: the potential does not show the $O(g_0^2)$ cancellation of (5.3) and (5.4), and is half as large as the singlet potential alone. This may be understood as the beginning of the onset of confinement physics. The net color flux emanating from the triplet configuration is no longer screened

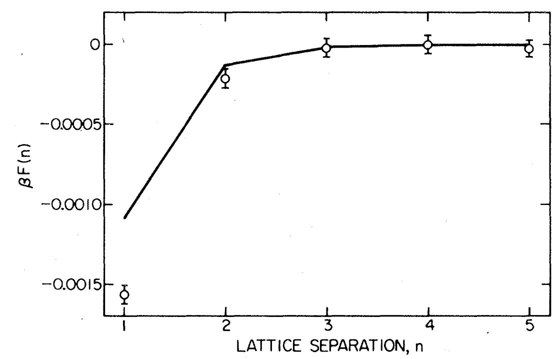

FIG. 6. $q\bar{q}$ potential vs separation (in lattice units) along a lattice axis for $4/g_0^2 = 5$, $N_t = 2$, $N_s = 10$, averaged over directions. The curve is the thermal average of perturbative potentials in the singlet and triplet channels.

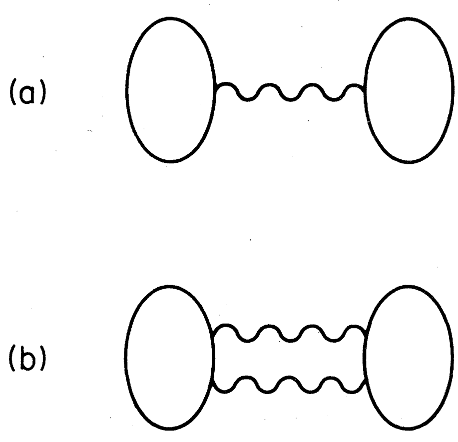

FIG. 5. (a) The one-gluon graph in the correlation function of two Wilson lines, which vanishes. (b) The lowest-order nonvanishing contribution to the color-averaged $q\bar{q}$ potential.

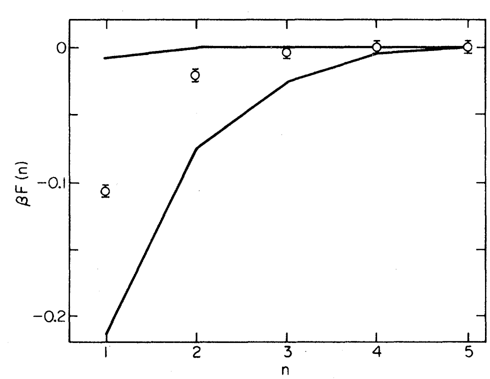

FIG. 7. As in Fig. 6, but for $4/g_0^2 = 2$. For this lattice $4/g_0^{*2} = 1.85$. The lower curve is the screened Coulomb potential in the color-singlet channel; the upper curve is the color-averaged screened Coulomb potential.

TABLE II. Lattice spacing a and physical temperature T for the data in Figs. 6–8 and for the critical point at g_0^*, calculated with (4.1)–(4.3).

	$4/g_0^2$	a (fm)	T (GeV)
Fig. 6	5	1.5×10^{-4}	641
Fig. 7	2	0.38	0.258
g_0^*	1.85	0.43	0.227
Fig. 8	1.8	0.44	0.223

by a thermal bath of glue but rather is confined by pulling a gluon out of the vacuum and binding to it. This costs energy, so that there is an added suppression in the Boltzmann factor for the triplet state. In Table II we indicate the lattice spacings and temperatures for these data.

In Fig. 8 we display the potential at a temperature below the phase transition. Its confining nature is evident; it appears to be a linear potential altered by the periodic boundary conditions. A rough estimate of the slope gives

$$\beta\frac{dF}{dn}\simeq 0.6\,. \tag{5.8}$$

The string tension at this temperature is

$$\sigma=\frac{dF}{dx}=\frac{1}{N_t a^2}\beta\frac{dF}{dn}$$
$$=K\frac{1}{N_t f(g_0)}\beta\frac{dF}{dn} \tag{5.9}$$

yielding for these data, via (4.2), quite a strong string tension, viz.,

$$\sigma=0.4K \quad \text{at } T/T^*=0.98\,. \tag{5.10}$$

E. Zero temperature?

Our prescription for the continuum limit, given in Sec. IV, utilizes the data of Creutz[10] for square Wilson loops in lattices with $N_t=N_s$. His determination of asymptotic freedom scales[23] makes use of estimates of the string tension in the weak-coupling region. However, Fig. 9 shows that for a lattice like his, $\langle L\rangle\neq 0$ when $4/g_0^2\gtrsim 2$ and by our criterion there is *no confinement*.

The root of the confusion is the fact that on a finite lattice there is a competition between two length scales. For very strong coupling, the behavior both of square Wilson loops and of the correlation function $\langle L(0)L(\vec{r})\rangle$ is governed by the dimensionless confinement length $(\sqrt{K}a)^{-1}\ll\beta/a$. For very weak coupling, the string tension is weak and cannot be measured by Wilson loops smaller than the lattice; the decay of the correlation function of L is then governed by the thermal screening length $(\mu a)^{-1}\propto\beta/a$. The delicate area is the one in between, where the crossover of the two scales brings on the finite-temperature transition. Then the fact that $\langle L\rangle\neq 0$ seems to guarantee that, given enough room in the lattice, the correlation function at large separation would show a nonconfining potential. However, the confinement scale may still be small enough that Wilson loops as small as half the lattice may yield an area law.

We remark that in order to have enough room to observe the nonconfining nature of the correlation function near the transition, we would need $N_s\gg N_t$. Going to such a lattice, though, would defer the transition toward weaker coupling (see Fig. 2), leaving us once more in the phase where the correlation function, as well as the square Wilson loop, shows confinement.

The danger in the situation for $N_s=N_t$ lies in determining the scale of the string tension in the transition region. Just as the correlation function of L shows effects of the confinement scale together with the temperature scale, so finite-size ef-

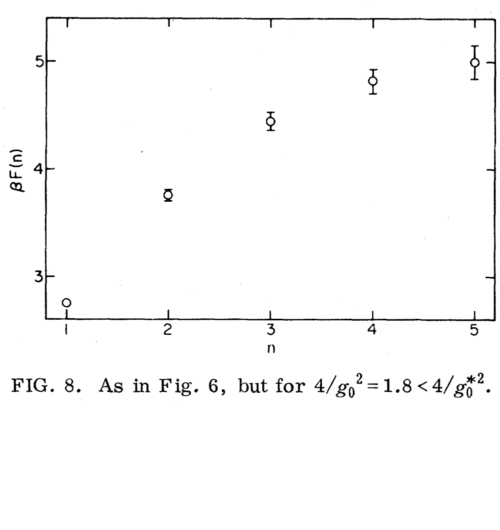

FIG. 8. As in Fig. 6, but for $4/g_0^2=1.8<4/g_0^{*2}$.

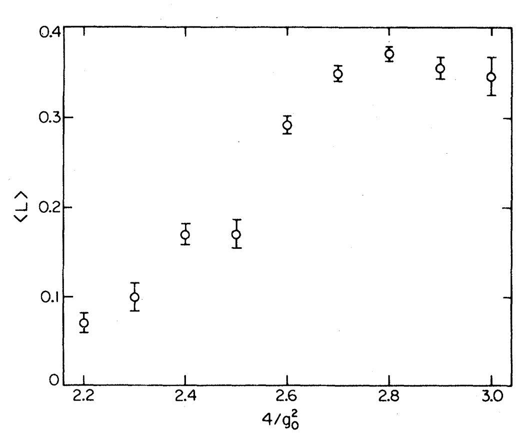

FIG. 9. Rough data for magnetization vs inverse coupling for $N_t=N_s=5$. Error bars do not reflect some uncertainty in when equilibrium is reached.

fects might affect the coefficient of the area law for Wilson loops.

The lesson of this discussion should be the importance of employing a large lattice in measuring the zero-temperature string tension. One may hope that $4/g_0^{*2}$ increases with N_t for $N_s = N_t$ as it does for $N_s \gg N_t$, so that the finite-temperature transition may be pushed in this way past the zero-temperature crossover and well into the weak-coupling region. Then the string tension measured would be more trustworthy. The work of Bhanot and Rebbi[11] on a 16^4 lattice probably satisfies this constraint; it is interesting that their numbers agree well with the earlier work of Creutz.

It is difficult to estimate how fast $4/g_0^{*2}$ increases with N_t for the space-time symmetric lattice. For $N_s \gg N_t$, the considerations of Sec. IV show that the assumption of a finite continuum T^* yields a renormalization relation $g_0^{*2}(N_t)$ via

$$\beta^* = N_t a(g_0^{*2}(N_t)) = \text{constant}. \qquad (5.11)$$

On the other hand, a thermodynamic interpretation is not obvious for $N_t a$ when $N_s = N_t$, and (5.11) may not be applied reliably. A calculation of $\langle L \rangle$ on the lattice of Bhanot and Rebbi would dispel any suspicion of lingering finite-size effects.

A bump in the bulk heat capacity of the gauge theory has been reported by Lautrup and Nauenberg[27] in a 4^4 lattice near $4/g_0^2 = 2$. It is tempting to associate this with the finite-temperature transition rather than with the zero-temperature strong-to-weak-coupling crossover, which folklore would hold to be singularity free. This issue as well would be decided by a calculation on a large lattice where the crossover and the transition should be well separated.

F. Gluon thermodynamics

We have unsuccessfully attempted to isolate the finite-temperature contributions to the free energy and entropy of the gluon gas. This information would be useful for determining the limits of validity of perturbation theory. The free-energy density $\mathfrak{F}$ may be calculated via the expectation value of the action (2.6), as

$$\langle \mathcal{L} \rangle = (g_0^2)^2 \frac{\partial}{\partial (g_0^2)} \mathfrak{F}. \qquad (5.12)$$

However, a zero-temperature (large-N_t) piece must be subtracted from $\mathfrak{F}$ to yield a quantity which is finite in the continuum limit. This subtraction requires better statistics than are available to us.

VI. THE DYNAMICS OF THE TRANSITION

The flip-flop behavior in Fig. 1 suggests a picture of the finite-temperature gauge theory as an ensemble of Z_2 bubbles. At very small couplings corresponding to very high temperatures, the system is one large domain with $\langle L \rangle = \pm 1$, which for definiteness we take to be $\langle L \rangle = +1$. As the temperature is lowered, bubbles with $\langle L \rangle = -1$ form. By a fluctuation, one of these bubbles might grow large and fill the volume causing the flip-flop of Fig. 1.

As the critical temperature is approached, the system literally froths with bubbles of $\langle L \rangle = \pm 1$. The Z_N symmetry is restored and $\lim_{T \to T^*} \langle L \rangle = 0$.

A rough approximation in the lattice theory which focuses on this picture goes as follows.[28] Fix a gauge in which $U = 1$ on all timelike links except those in one spacelike layer, i.e., $U^0(\vec{r}, t) = 1$ for $t \neq 1$. Then the remaining timelike links contain the gauge-invariant quantity L, according to (3.23). In the weak-coupling limit, choose a vacuum with $L = 1$; any such vacuum is clearly gauge equivalent to one in which $U^i = 1$ on all spacelike links as well. Now turn up the coupling, while considering *only* states with $U^i = 1$ and $L = \pm 1$. When restricted to these states, the action (2.6) is that of a three-dimensional Ising model with effective temperature g_0^2, i.e., a nearest-neighbor coupling among degrees of freedom $L(\vec{r}) = \pm 1$.

This Ising model of course exhibits a phase transition such that $\langle L \rangle = 0$ for $g_0^2 > g_0^{*2}$. The approach to the transition from the weak-coupling phase may be studied in a droplet picture, where the partition function is expressed in terms of domains of flipped L embedded in the weak-coupling vacuum $L = +1$. A condensation of such domains at all length scales constitutes the phase transition.[29]

There is a continuum analog to this picture. The finite-temperature instanton[13] is a solution of the Euclidean Yang-Mills equations which interpolates between $L = +1$ at infinity and $L = -1$ at the origin,

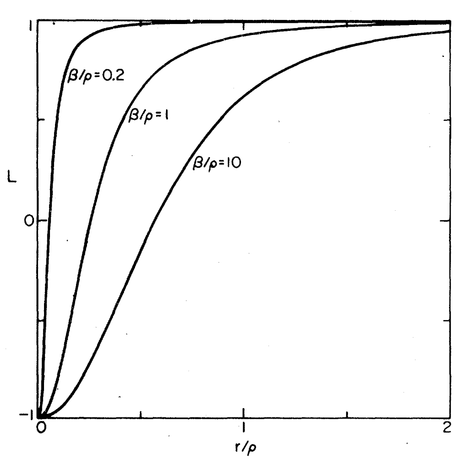

FIG. 10. Magnetization L vs distance r from a finite-temperature instanton of radius ρ, for three values of β.

and thus represents a Z_2 domain configuration which is a stationary-phase point of the path integral (2.1).

An instanton of scale size ρ at the origin is described by the field

$$A^a_\mu = \eta^a_{\mu\nu}\,\Pi(\vec{r},t)\,\partial_\nu \Pi^{-1}(\vec{r},t)\,, \tag{6.1}$$

with

$$\Pi = 1 + \frac{(A/w)\sinh w}{\cosh w - \cos z}\,, \qquad w = \frac{2\pi}{\beta}r,\quad z = \frac{2\pi}{\beta}t,\quad A = \frac{2\pi^2\rho^2}{\beta^2}\,. \tag{6.2}$$

Using $\eta^a_{0\nu} = \delta_{a\nu}$, we may evaluate (3.8) as

$$L(\vec{r}) = L(r) = \cos I(r)\,,$$

where

$$I(r) = \tfrac{1}{2}\int_0^\beta \Pi^{-1}\frac{\partial \Pi}{\partial r}dt = -\pi\left(1 + \frac{N}{D}\right),$$

$$N = \left(\frac{A}{w^2} - 1\right)\sinh w - \frac{A}{w}\cosh w\,,$$

$$D = \left[\left(\frac{A^2}{w^2} + 1\right)\sinh^2 w + \frac{2A}{w}\cosh w \sinh w\right]^{1/2}.$$

$L(r)$ is plotted in Fig. 10.

Thus we are led to picture the phase transition as a condensation of instantons (and possibly of other domainlike objects which may appear as the coupling becomes strong).[30] While a dilute-gas approximation may not be expected to yield significant contributions to quantities such as F_q, a calculation which includes some effects of instanton interactions may result in a curve at least qualitatively similar to Fig. 4.

ACKNOWLEDGMENTS

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APPENDIX: ERROR ANALYSIS

The raw output from the Monte Carlo program, whatever the quantity being measured, consists of pass-by-pass volume averages such as those displayed in Fig. 1. Consider for definiteness the evaluation of $\langle L\rangle$, and denote the volume average after pass i by L_i. The first step is to throw out the data from the beginning of a run which displays the initial drift in L_i from the ordered start. What is left should clearly represent thermal fluctuations about a mean which does *not* drift, along with possible reversals in sign. Cutting out the points associated with the reversals allows us to take absolute values, yielding data $\{x_i\}$ which fluctuate about a constant mean.

It should be obvious that the Monte Carlo algorithm generates sequences $\{x_i\}$ with strong correlations from one pass to the next. We may quantify this effect by calculating the autocorrelation function

$$C(n) = \frac{\langle x_i x_{i+n}\rangle - \langle x\rangle^2}{\langle x^2\rangle - \langle x\rangle^2}\,. \tag{A1}$$

Here the angular brackets represent an averaging over the sequence of data $\{x_i\}$. The accepted 1-σ test for correlation over a "distance" n is to test whether

$$C(n) > \frac{1}{\sqrt{p_n}}\,, \tag{A2}$$

where p_n is the number of pairs with separation n available for averaging in (A1). For small n, $C(n)$ is a decreasing function; the smallest value of n for which (A2) fails is the pass-to-pass correlation length ν referred to at the end of Sec. V A.

Another way of studying correlations is to group the passes into bins of some length l, defining a new variable $m_j^{[l]}$ as the mean of the $\{x_i\}$ in the jth bin. Then we may calculate the nearest-neighbor bin autocorrelation as

$$C'(l) = \frac{\langle m_j^{[l]} m_{j+1}^{[l]}\rangle - \langle m^{[l]}\rangle^2}{\langle m^{[l]2}\rangle - \langle m^{[l]}\rangle^2}\,. \tag{A3}$$

Again we may test whether

$$C'(l) > \frac{1}{(p_l - 1)^{1/2}}\,, \tag{A4}$$

where p_l is the number of bins of length l we are able to construct out of the data. The smallest l for which (A4) fails, that is, for which consecutive bin averages are uncorrelated, gives another estimate for ν, which should be close to that calculated from (A1) and (A2).

If equilibrium has not yet been reached and the mean about which the $\{x_i\}$ fluctuate is still drifting, ν will diverge. Let us assume that enough initial passes have been dropped so that this is not the case.

Then the $\{m_j^{[\nu]}\}$ comprise a set of statistically uncorrelated samples from which $\langle L\rangle = \langle m^{[\nu]}\rangle = \langle x_i\rangle$ may be calculated. The population variance is the usual $\sigma^2 = [p_\nu/(p_\nu - 1)]\langle(m_j^{[\nu]} - \langle L\rangle)^2\rangle$ and the variance in the mean is $s^2 = \sigma^2/p_\nu$. s is the size of the error bars in Figs. 3 and 9.

A complication arises in estimating the uncertainties in the two-point function $G(n) \equiv \langle L(\vec{r})L(\vec{r}+n\hat{\mu})\rangle$ because the sets of data $\{x_i(n)\}$ corresponding to the various $G(n)$ are strongly correlated with each other, a cross correlation which survives when the data are binned into $\{m_j^{[\nu]}(n)\}$ as above. (We drop the $[\nu]$ superscript henceforth.) The covariance matrix is defined as

$$C_{rs} = [\langle m(r)m(s)\rangle - \langle m(r)\rangle\langle m(s)\rangle]\frac{p_\nu}{p_\nu - 1} \qquad \text{(A5)}$$

and the coefficients of correlation as

$$R_{rs} = \frac{C_{rs}}{\sigma_r\sigma_s} \qquad \text{(A6)}$$

with

$$\sigma_n^2 = \frac{p_\nu}{p_\nu - 1}[\langle m(n)^2\rangle - \langle m(n)\rangle^2]\,. \qquad \text{(A7)}$$

Typically, we find $1 - R_{rs} \simeq 10^{-3}$ for all $r \neq s$, showing almost perfect linear correlation among the $\{m_j(n)\}$. At the same time, the matrix C_{rs} possesses a large eigenvalue ϵ_1 corresponding to an eigenvector proportional to $[1,1,1,\ldots]$: clearly, this is relevant to fluctuations where all the $m(n)$ move together rigidly. Discarding this as uninteresting, we take the next largest eigenvalue ϵ_2 of C_{rs} to be the typical variance of uncorrelated fluctuations, and we take $(\epsilon_2/p_\nu)^{1/2}$ for the error bar in all the $G(n)$'s. Taking the logarithm of $G(n)$ and adjusting the error bars accordingly yields Figs. 6–8.

Purists should note that it does not matter whether the above procedure for $G(n)$ is executed before or after taking the logarithm.

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