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Static potential in string models

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A class of string models is studied to leading order in a $1/d$ expansion, where d is the dimensionality of space-time. The static potential is found to be given by $V(R)=M^2R(1-R_c^{\;2}/R^2)^{1/2}$, where R_c is a critical distance. At $R=R_c$, the system undergoes a phase transition to a state where a quasistatic string picture is no longer applicable.

I. INTRODUCTION

One of the outstanding problems in non-Abelian gauge theories is the calculation of the static potential between two heavy sources. In continuum gauge theories one can systematically calculate the short-distance Coulomb limit of this potential, but its long-range behavior has thus far been unobtainable. In lattice gauge theories it is straightforward to deduce that its long-range limit is a linear potential, but the departures away from linearity are difficult to compute. It would be desirable to have a model where a single calculational scheme could give the potential over a range of distances.

In this paper we shall study a class of *phenomenological* models for the potential based on ideas of Nambu[1] and Eguchi.[2] These models represent the Wilson loop[3] as statistical averages over fluctuating surfaces. The word phenomenological should be understood in the same sense that the nonlinear σ model is a phenomenological description of quantum chromodynamics (QCD). The current theoretical prejudice for these models is based on the strong-coupling expansion for the Wilson loop in a lattice gauge theory: the contributions to the loop expectation value are obtained from the various surfaces spanning the loop.

The functional-integral quantization of these models was studied by Lüscher, Symanzik, and Weisz.[4] In this important paper, these authors successfully calculated the leading correction to a general Wilson-loop expectation value due to the quantum fluctuations of the string. Their result reduced, for the case of the static potential, to

$$V(R)=M^2R-\frac{\pi(d-2)}{24R}+\cdots, \qquad (1.1)$$

where M^2 is the string tension, and d is the dimensionality of space-time. The $1/R$ piece is a long-distance effect. There are corrections to the above that go as R^{-3}, etc. Equation (1.1) is an expansion around $R=\infty$, and the $1/R$ piece should not be confused with the short-distance Coulomb exchange contribution. Lüscher,[5] and Stack and Stone[6] have pointed out that the $\pi(d-2)/24$ is a universal coefficient that one expects in any model based on a string description.

Lüscher, Symanzik, and Weisz suggested that a large-d expansion might be interesting for developing a nonperturbative approach to the static potential. In this paper we show that one can define a $1/d$ expansion, and we are able to solve the models to leading order in $1/d$. The potential in these models is given by the simple formula

$$V(R)=M^2R(1-R_c^2/R^2)^{1/2}; \qquad (1.2)$$

the parameter R_c will be computed later in the paper. The above expression is linear for large R and makes an abrupt transition to a square-root singularity at $R=R_c$. The expression is nonsense for $R<R_c$. We have not been able to calculate V for $R<R_c$.

The nonanalyticity in Eq. (1.2) for $V(R)$ shall be interpreted as the existence of a phase transition at $d=\infty$ in these models. If one seriously assumes that the string models can be derived from QCD then Eq. (1.2) indicates that there is rapid change away from linearity in the static potential.

The rest of this paper is organized as follows. In Sec. II, the Nambu and the generalized Eguchi models are reviewed, and the formalism for the $1/d$ analysis is presented. The Nambu model is solved to leading order in $1/d$ in Sec. III. The equivalence to leading order of the generalized Eguchi models to the Nambu model is demonstrated in Sec. IV. Section V is a discussion of the results of Secs. III and IV. Appendices A and B justify some simplifying assumptions which were used in Sec. III.

II. BACKGROUND

We shall study a class of models for the vacuum expectation value of the Wilson loop in four-dimensional space-time. The first model is based on the Nambu ansatz for the action of a string.[1]

The other models are generalizations of the model proposed by Eguchi.[2]

Let C be a contour in d-dimensional Euclidean space R^d, and let S be any two-dimensional surface with boundary C, i.e., $\partial S = C$. Introduce coordinates (z^1, z^2) on this surface so that S is parametrized as a mapping $x^\mu(z)$ from a bounded region $\mathcal{R}$ in R^2 into R^d. Let g_{ab} be the induced metric on this surface:

$$g_{ab} = \frac{\partial x^\mu}{\partial z^a}\frac{\partial x^\mu}{\partial z^b}, \tag{2.1}$$

where $a, b = 0, 1$.

The Nambu model for the Wilson loop is defined by

$$W_N[C] = \int_{\{S|\partial S = C\}} [\mathfrak{D}x] \exp\left[-M^2 \int d^2z (\det g_{ab})^{1/2}\right], \tag{2.2}$$

where the functional integral is taken over all surfaces S with boundary C. The Nambu action is the area of the surface. The area is invariant with respect to the "gauge" group of coordinate transformations on S. Gauge fixing must be introduced to make the path integral well defined. Since the invariance group of the action is the full coordinate group, the Wilson-loop expectation value can only depend on the contour C.

To define the generalized Eguchi models, consider again the region $\mathcal{R}$ mapped into the surface S by $x^\mu(z)$. The boundary of R is mapped into C. The area of $\mathcal{R}$ will be denoted by an upper case A. The generalized Eguchi model is defined by

$$W_E[C] = \int_0^\infty dA \, W_E[C, A] \exp(-\tfrac{1}{2}A), \tag{2.3}$$

where

$$W_E[C, A] = \int_{\{S|\partial S = C\}} [\mathfrak{D}x] \exp\left[-\left(\frac{2\nu - 1}{\nu}\right)^{2\nu - 1} \frac{M^{4\nu}}{2\nu} \times \int_{\mathcal{R}} d^2z (\det g)^\nu\right]. \tag{2.4}$$

The factors in the action have been chosen such that M^2 is the string tension. The models are defined only for $\nu > \frac{1}{2}$. The action with $\nu = 1$ was first proposed by Schild[7] as an alternative to the Nambu action. For $\nu > \frac{1}{2}$, these actions have a smaller invariance group. They are only invariant with respect to symplectic coordinate transformations, i.e., those for which $|\partial z'/\partial z| = 1$. A theorem due to Moser[8] guarantees that any two simply connected regions with the same area can be mapped into each other via a symplectic coordinate transformation. As a consequence, W_E can only depend on C and A, and not on any other property of $\mathcal{R}$. The evaluation of (2.3) also requires gauge fixing although the gauge group is much smaller.

To compute the static potential we choose the loop C to be a rectangle in the (01) plane of length T and width R with $T \gg R$. According to Wilson,[3] the loop expectation value is expected to behave as

$$W[C] \sim \exp[-TV(R)], \tag{2.5}$$

where $V(R)$ is identified as the static potential between infinitely massive sources separated by a distance R. We shall study the potential in an expansion similar to the $1/N$ expansion of many-component field theories. In this approximation[9] there are two parameters, g^2 and N. The $1/N$ expansion is defined by studying the behavior of the theory as $N \to \infty$ with $\lambda = g^2 N$ held fixed. In this paper, the role of N will be played by $D = d - 2$, the number of dimensions of space transverse to the string. Lüscher, Symanzik, and Weisz showed that in these models the dimensionless combination $(MR)^{-2}$ plays the role of g^2. As we will see later, a convenient choice for λ is

$$\lambda = \frac{\pi D}{24 M^2 R^2}. \tag{2.6}$$

Functional integrals (2.2) and (2.4) are evaluated by expanding around the stationary point of the action. Since the stationary points of these models are minimum-area surfaces,[7] we are required to expand about the flat rectangle with boundary C. Let S be any surface with the rectangle as the boundary. Choose coordinates on S by mapping into it a rectangular region $\mathcal{R}$ of length a_0 and width a_1 (Fig. 1). This leads to a convenient parametrization of S given by

$$x^0(z) = (Tz^0/a_0) + \eta^0(z), \tag{2.7a}$$

$$x^1(z) = (Rz^1/a_1) + \eta^1(z), \tag{2.7b}$$

$$x^\nu(z) = \phi^\nu(z), \quad \nu = 2, 3, \ldots, d-1. \tag{2.7c}$$

The η's are the longitudinal deformations of the surface, and the ϕ's are the transverse deformations. The η's and ϕ's must vanish on the boundary.

The zero modes associated with coordinate transformations manifest themselves through the η's. In the $\nu = 1$ case, the zero modes are eliminated by using the Faddeev-Popov procedure as demonstrated by Lüscher, Symanzik, and Weisz.[4] Since there are D transverse modes and only two longitudinal modes, the effects of the η's will not appear in the leading order in $1/D$. We will neglect the η's from now on.

Using (2.7), we find that the induced metric is given by

$$g_{ab}=\begin{bmatrix}\left(\frac{T}{a_0}\right)^2+\left(\frac{\partial\vec\phi}{\partial z^0}\right)^2 & \frac{\partial\vec\phi}{\partial z^0}\cdot\frac{\partial\vec\phi}{\partial z^1}\\ \frac{\partial\vec\phi}{\partial z^0}\cdot\frac{\partial\vec\phi}{\partial z^1} & \left(\frac{R}{a_1}\right)^2+\left(\frac{\partial\vec\phi}{\partial z_1}\right)^2\end{bmatrix}. \tag{2.8}$$

As a warm-up exercise, we shall evaluate the contribution to V due to the quadratic transverse fluctuations in the Nambu model. To order $\vec\phi^{\,2}$, the action is given by

$$M^2\int d^2z(\det g)^{1/2}$$
$$=M^2RT+\tfrac{1}{2}M^2\int_0^T dt\int_0^R dr\left[\left(\frac{\partial\vec\phi}{\partial t}\right)^2+\left(\frac{\partial\vec\phi}{\partial r}\right)^2\right]. \tag{2.9}$$

Since we are interested in the limit $T\to\infty$, the time integral in (2.9) may be replaced by an integral from $-\infty$ to $+\infty$. Inserting the above into (2.2) we find that

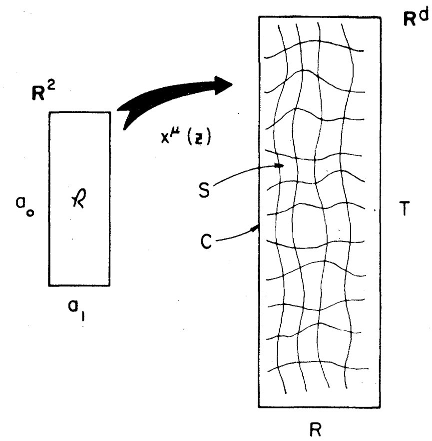

FIG. 1. Choosing convenient coordinates (2.7) to parametrize the surfaces spanning the rectangular $T\times R$ Wilson loop.

$$W_N[C]=\exp(-M^2RT)\int[\mathfrak{D}\phi]\exp\left[-\tfrac{1}{2}M^2\int_{-\infty}^{\infty}dt\int_0^R dr\left(\frac{\partial\vec\phi}{\partial t}\right)^2+\left(\frac{\partial\vec\phi}{\partial r}\right)^2\right] \tag{2.10}$$

$$=\exp[-M^2RT-\tfrac{1}{2}D\,\mathrm{Tr}\ln(-\partial_t^{\,2}-\partial_r^{\,2})]. \tag{2.11}$$

The functional trace may be evaluated by Fourier transformation. The ultraviolet divergences are tamed by using analytic regularization.[10] In analytic regularization, the logarithm is defined by

$$\ln x=-\frac{\partial}{\partial\beta}x^{-\beta}\bigg|_{\beta=0}. \tag{2.12}$$

The trace in momentum space is given by

$$\mathrm{Tr}\ln(-\partial_t^{\,2}-\partial_r^{\,2})$$
$$=-\frac{\partial}{\partial\beta}T\sum_{n=1}^{\infty}\int\frac{d\omega}{2\pi}\frac{1}{[\omega^2+(n\pi/R)^2]^\beta}\bigg|_{\beta=0} \tag{2.13}$$
$$=-\frac{T}{(4\pi)^{1/2}}\frac{\partial}{\partial\beta}\frac{\Gamma(\beta-\frac{1}{2})}{\Gamma(\beta)}\sum_{n=1}^{\infty}\left(\frac{R^2}{\pi^2n^2}\right)^{\beta-1/2}\bigg|_{\beta=0} \tag{2.14}$$
$$=\pi T\zeta(-1)/R \tag{2.15}$$
$$=-\pi T/(12R). \tag{2.16}$$

We conclude that the potential is given by

$$V(R)=M^2R-\pi D/(24R)+\cdots. \tag{2.17}$$

The above result is the analog for the Nambu model of the Lüscher, Symanzik, and Weisz calculation. The universality of the $1/R$ coefficient is discussed in Refs. 5 and 6.

The $1/D$ expansion is best discussed in "physical" coordinates defined by

$$t=Tz^0/a_0, \tag{2.18a}$$
$$r=Rz^1/a_1. \tag{2.18b}$$

In these coordinates, the Nambu action and the Schild action are, respectively, given by

$$I_N=M^2\int_{-\infty}^{\infty}dt\int_0^R dr[\det(\delta_{ab}+\partial_a\vec\phi\cdot\partial_b\vec\phi)]^{1/2}, \tag{2.19}$$

$$I_E=\frac{1}{2\nu}\left(\frac{2\nu-1}{\nu}\right)^{2\nu-1}\left(\frac{M^2TR}{A}\right)^{2\nu}\frac{A}{TR}\times\int_{-\infty}^{+\infty}dt\int_0^R dr[\det(\delta_{ab}+\partial_a\vec\phi\cdot\partial_b\vec\phi)]^\nu, \tag{2.20}$$

where $A=a_0a_1$, and the indices $a,b=0,1$ now refer to (t,r). The $1/D$ expansion is obtained by a standard set of manipulations: We introduce composite fields σ_{ab} for $\partial_a\vec\phi\cdot\partial_b\vec\phi$, and constrain $\sigma_{ab}=\partial_a\vec\phi\cdot\partial_b\vec\phi$ by introducing Lagrange multipliers α^{ab}. For the Nambu model the manipulations produce

$$W_N=\int[\mathfrak{D}\phi][\mathfrak{D}\sigma]\,\delta(\sigma_{ab}-\partial_a\vec\phi\cdot\partial_b\vec\phi)\times\exp\left\{-M^2\int dt\,dr[\det(\delta_{ab}+\sigma_{ab})]^{1/2}\right\} \tag{2.21}$$

$$=\int[\mathfrak{D}\phi][\mathfrak{D}\sigma][\mathfrak{D}\alpha]\exp(-\tilde S_N), \tag{2.22}$$

where

$$\tilde{S}_N = M^2 \int dt\, dr [\det(\delta_{ab} + \sigma_{ab})]^{1/2} + \tfrac{1}{2} M^2 \int dt\, dr\, \alpha^{ab} (\partial_a \vec{\phi} \cdot \partial_b \vec{\phi} - \sigma_{ab}) . \qquad (2.23)$$

In going from (2.21) to (2.22) we used the exponential parametrization of the δ function with the understanding that the α functional integrals run from $-i\infty$ to $+i\infty$ in the complex α-plane. The action $\tilde{S}_N$ is quadratic in ϕ, and the functional integral may be evaluated with the result

$$W_N = \int [\mathfrak{D}\alpha][\mathfrak{D}\sigma] \exp(-S_N) , \qquad (2.24)$$

where

$$S_N = M^2 \int_{-\infty}^{+\infty} dt \int_0^R dr \{[\det(\delta_{ab} + \sigma_{ab})]^{1/2} - \tfrac{1}{2} \alpha^{ab} \sigma_{ab}\} + \tfrac{1}{2} D \operatorname{Tr} \ln(-\partial_a \alpha^{ab} \partial_b) . \qquad (2.25)$$

One can perform the same manipulations in the Eguchi models with the result

$$\tilde{W}_E = \int [\mathfrak{D}\alpha][\mathfrak{D}\sigma] \exp(-S_E) , \qquad (2.26)$$

where

$$S_E = \frac{1}{2\nu}\left(\frac{2\nu-1}{\nu}\right)^{2\nu-1}\left(\frac{M^2TR}{A}\right)^{2\nu} \frac{A}{TR} \int_{-\infty}^{\infty} dt \int_0^R dr [\det(\delta_{ab} + \sigma_{ab})]^{\nu} - \tfrac{1}{2} M^2 \int dt\, dr\, \alpha^{ab} \sigma_{ab} + \tfrac{1}{2} D \operatorname{Tr} \ln(-\partial_a \alpha^{ab} \partial_b) . \qquad (2.27)$$

Equations (2.25) and (2.27) are the effective actions which will be used to generate the $1/D$ expansion by steepest-descent methods.

III. THE NAMBU MODEL

In this section the leading term in the $1/D$ expansion of the static potential in the Nambu model is obtained. The $1/D$ expansion is systematically generated by expanding (2.25) around its stationary points. There are several observations that can be made about the stationary solution.

(1) The system becomes time-translational invariant as $T \to \infty$, and the stationary solution $\sigma = \bar{\sigma}(r)$, $\alpha = \bar{\alpha}(r)$ should be time independent. Throughout the rest of the paper, $\bar{\sigma}$ and $\bar{\alpha}$ will always denote the stationary solution.

(2) Since R is finite we expect $\bar{\sigma}$ and $\bar{\alpha}$ to depend on r.

(3) The equations $\delta S_N / \delta\sigma(r)$ are algebraic.

(4) The equations $\delta S_N / \delta\alpha(r)$ are functional differential equations.

In Appendix A we show that the situation is much simpler than what is indicated above. The conclusions of Appendix A are (1) $\bar{\sigma}$ and $\bar{\alpha}$ may be taken to be t and r independent and (2) $\bar{\sigma}$ and $\bar{\alpha}$ are diagonal matrices. Under these assumptions, the functional trace in Eq. (2.25) is easily evaluated using the method explained in Sec. II. The effective action (2.25) becomes

$$S_N = M^2 RT [(1+\sigma_0)^{1/2}(1+\sigma_1)^{1/2} - \tfrac{1}{2}(\alpha_0\sigma_0 + \alpha_1\sigma_1)] - \pi D T (\alpha_1/\alpha_0)^{1/2}/(24R) , \qquad (3.1)$$

where $\alpha_0 = \alpha_{00}$ and $\alpha_1 = \alpha_{11}$. If one defines a parameter λ by Eq. (2.6) then Eq. (3.1) may be written as

$$S_N = \left(\frac{\pi T}{24R}\right) \frac{D}{\lambda} [(1+\sigma_0)^{1/2}(1+\sigma_1)^{1/2} - \tfrac{1}{2}\, \alpha \cdot \sigma - \lambda(\alpha_1/\alpha_0)^{1/2}] . \qquad (3.2)$$

The expression above is reminiscent of $1/N$ expansions. The leading term is a function of λ with a prefactor of D. The next term in the expansion would be a function of λ multiplied by D^0.

The variational equations for the stationary point of (3.3) are

$$\bar{\alpha}_0 = (1+\bar{\sigma}_0)^{-1/2}(1+\bar{\sigma}_1)^{1/2} , \qquad (3.3a)$$

$$\bar{\alpha}_1 = (1+\bar{\sigma}_0)^{1/2}(1+\bar{\sigma}_1)^{-1/2} , \qquad (3.3b)$$

$$\bar{\sigma}_0 = \lambda(\bar{\alpha}_1/\bar{\alpha}_0^{\,3})^{1/2} , \qquad (3.3c)$$

$$\bar{\sigma}_1 = -\lambda(\bar{\alpha}_0\bar{\alpha}_1)^{-1/2} . \qquad (3.3d)$$

This system of equations is easily solved with the result

$$\bar{\alpha}_0 = (1-2\lambda)^{1/2} , \qquad (3.4a)$$

$$\bar{\alpha}_1 = (1-2\lambda)^{-1/2} , \qquad (3.4b)$$

$$\bar{\sigma}_0 = \lambda(1-2\lambda)^{-1} , \qquad (3.4c)$$

$$\bar{\sigma}_1 = -\lambda . \qquad (3.4d)$$

Inserting the above into (3.2) yields the static potential to leading order in $1/D$:

$$V(R) = M^2 R (1-2\lambda)^{1/2} . \qquad (3.5)$$

The matrix of second derivatives of S_N has two positive eigenvalues and two negative eigenvalues at the stationary point. This is required since the

σ integrals run east-west in the complex plane, but the α integrals run north-south. Solution (3.4) is stable with respect to small fluctuations.

The potential may be written in the form

$$V(R)=M^2R(1-R_c^{\,2}/R^2)^{1/2}, \tag{3.6}$$

where

$$R_c^{\,2}=\pi D/(12M^2). \tag{3.7}$$

A graph of V is given in Fig. 2. It appears as if the solution given by Eq. (3.6) breaks down at a critical distance R_c corresponding to $\lambda=\frac{1}{2}$. We shall interpret this difficulty as a signal of a phase transition in the string models when $D=\infty$. We shall accumulate more evidence of what occurs at this point and postpone our discussion of it to Sec. V.

There are two correlation functions which are by-products of the static potential calculation. The average fluctuation F will be defined by

$$F=(DR)^{-1}\int_0^R dr\,\langle\vec{\phi}(t,r)\cdot\vec{\phi}(t,r)\rangle. \tag{3.8}$$

A short calculation shows that

$$F=(2\pi M^2)^{-1}(\bar{\alpha}_0\bar{\alpha}_1)^{-1/2}\ln(R/R^*) \tag{3.9}$$

$$=(2\pi M^2)^{-1}\ln(R/R^*), \tag{3.10}$$

where R^* is some constant. Equation (2.23) is responsible for the appearance of $\bar{\alpha}$ in (3.9). At the stationary point $\bar{\alpha}_0\bar{\alpha}_1=1$ and their effect disappears in F.

The second correlation function we consider is the function $G(t)$ which measures correlations at two different times:

$$G(t)=(DR)^{-1}\int_0^R dr\,\langle\vec{\phi}(t,r)\cdot\vec{\phi}(0,r)\rangle \tag{3.11}$$

$$=-(2\pi M^2)^{-1}\ln(1-e^{-t/\xi}), \tag{3.12}$$

where

$$\xi=(R/\pi)(\bar{\alpha}_0/\bar{\alpha}_1)^{1/2} \tag{3.13}$$

$$=(R/\pi)(1-2\lambda)^{1/2}. \tag{3.14}$$

The effect of the $\bar{\alpha}$'s makes its appearance through ξ. For large t, the function $G(t)$ has exponential decay with correlation time ξ. The correlation time vanishes as $\lambda\to\frac{1}{2}$.

Another interesting quantity is the ratio of the area of the dominant surfaces at the stationary point to the area of the flat rectangle. This ratio is given by

$$(\det g)^{1/2}=(1-\lambda)(1-2\lambda)^{-1/2}, \tag{3.15}$$

a divergent quantity as $\lambda\to\frac{1}{2}$.

We shall postpone the discussion of these observations to Sec. V.

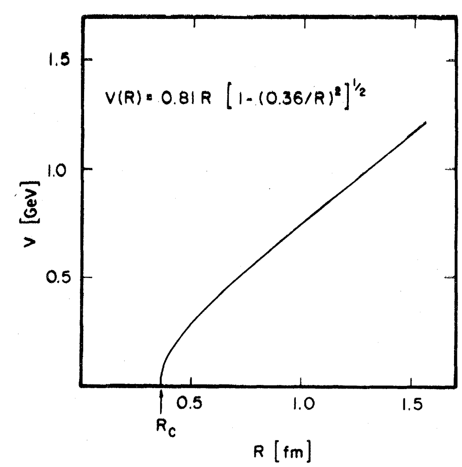

FIG. 2. A plot of the potential with $D=2$, $M^2=(2\pi\alpha')^{-1}$ where $\alpha'=1\ \mathrm{GeV}^{-2}$ is the Regge slope. V is measured in GeV and R in fm. $R_c=0.36$ fm.

IV. THE GENERALIZED EGUCHI MODELS

The $1/D$ expansion of the generalized Eguchi models is more involved than the corresponding one of the Nambu model due to the additional area integration [see Eq. (2.3)]. We shall use our experience with the Nambu model to assume that $\bar{\alpha}$ and $\bar{\sigma}$ are (t,r) independent. We have not succeeded in proving this statement, but we believe it is true. This ansatz is at least self-consistent. The arguments in Appendix A can be used to show that $\bar{\sigma}$ and $\bar{\alpha}$ are diagonal matrices.

Under these assumptions, we will show that to leading order in $1/D$, the generalized Eguchi models are equivalent to the Nambu model. Equation (2.27) simplifies to the form

$$S_E=\frac{A}{2\nu}\left(\frac{2\nu-1}{\nu}\right)^{2\nu-1}\left(\frac{M^2TR}{A}\right)^{2\nu}-\tfrac{1}{2}M^2TR(\alpha_0\sigma_0+\alpha_1\sigma_1)$$
$$-T\pi D(\alpha_1/\alpha_0)^{1/2}(24R)^{-1}. \tag{4.1}$$

The Wilson loop is obtained by evaluating an area integral [see Eq. (2.3)] in addition to the α and σ integrals:

$$W_E[C]\sim\int_0^\infty dA\int_{-\infty}^\infty d\alpha\,d\sigma\exp(-\tfrac{1}{2}A-S_E). \tag{4.2}$$

The easiest way to demonstrate the equivalence of the models is to evaluate the A integral first. This integral is evaluated by steepest-descent methods. The stationary point $\bar{A}$ is found by solving the equation

$$\frac{\partial}{\partial A}\left(\tfrac{1}{2}A+S_E\right)_{\bar{A}}=0\,. \tag{4.3}$$

The solution to this equation is given by

$$\bar{A}=\left(\frac{2\nu-1}{\nu}\right)M^2TR(1+\sigma_0)^{1/2}(1+\sigma_1)^{1/2}\,. \tag{4.4}$$

Substituting the expression for $\bar{A}$ into (4.2) leads to the result

$$W_E\sim\int d\alpha\, d\sigma \exp\left(-\{M^2TR[(1+\sigma_0)^{1/2}(1+\sigma_1)^{1/2}-\tfrac{1}{2}\alpha\cdot\sigma] - T\pi D(\alpha_1/\alpha_0)^{1/2}(24R)^{-1}\}\right)\,. \tag{4.5}$$

The argument of the exponential is precisely Eq. (3.1).

V. DISCUSSION

The analysis of Secs. III and IV indicates that in the models considered there is a phase transition at $D=\infty$. The static potential is not analytic at $\lambda=\frac{1}{2}$, i.e., there is a singularity in the free energy of the system. The area of the dominant surfaces diverged as $(1-2\lambda)^{-1/2}$. The amplitude of the transverse fluctuations approached a constant, but the fluctuations also became completely uncorrelated in time. All these observations are summarized by saying that ϕ^2 is not becoming large but $(\partial\phi)^2$ is diverging.

The analysis of Secs. III and IV was based on the assumption that the surfaces which dominate the functional integral differ from the flat rectangle by a "small amount." This statement is embodied in Eqs. (2.7). At $\lambda=0$, the dominant surface is the flat rectangle. For $0<\lambda<\frac{1}{2}$, the dominant surfaces are small perturbations about the rectangle. As $\lambda\to\frac{1}{2}$, the dominant surfaces are becoming very hilly and no longer resemble the flat rectangle.

For $\lambda>\frac{1}{2}$, one can extrapolate and guess that the path integral is no longer dominated by small fluctuations about the flat rectangle. There is no longer a well-defined class of surfaces which governs the functional integral, i.e., Eqs. (2.7) are no longer starting points for a perturbation expansion. A quasistatic string description of the potential is no longer adequate. In some sense, the string has spread over space.

There are $1/D$ corrections to the results of Secs. III and IV. We started with a nonrenormalizable field theory, and discovered that to leading order in $1/D$ there were no renormalizations required if one used analytic regularization. It is not obvious that this will persist to higher orders in $1/D$. The higher-order corrections might require the introduction of additional nonrenormalizable interactions. Dimensional analysis shows that these interactions cannot change the coefficient of the long-distance $1/R$ piece of the potential, but they can affect all the other higher powers in $1/R$. We shall not worry about these questions. As a working hypothesis we shall assume that the effect of *all* the $1/D$ corrections is to smooth out the sharp transition in the theory at finite D. It is an open question whether the Nambu model and the generalized Eguchi models are equivalent to higher orders in $1/D$.

Since we do not know how to do perturbation theory for $\lambda>\frac{1}{2}$, we can only *speculate* on the short-distance behavior. The limit $\lambda\to\infty$ may be obtained by letting $R\to0$ or $M^2\to0$. At $M=0$ there is no scale in the problem, and one might hope that the potential assumes the scale-invariant form $V\propto1/R$. Peskin[11] has argued that, in gauge theories with a second-order phase transition, this scale-invariant form follows even if one includes renormalization effects. His argument, however, applies to any theory in which the order parameter is a renormalized loop operator; it should apply to the string as well.

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APPENDIX A

This appendix is devoted to the justification of the assumptions used in Sec. III. Many of the remarks made in this section are also applicable to the generalized Eguchi models.

Scaling arguments can be used to extract information about the behavior of effective action (2.25) at the stationary point. Within the framework of analytic regularization

$$\mathrm{Tr}\ln(-\partial_a\alpha^{ab}\partial_b)=\mathrm{Tr}\ln(-\partial_a\rho\alpha^{ab}\partial_b) \tag{A1}$$

for any positive constant ρ. If $\bar{\sigma}$ and $\bar{\alpha}$ are the stationary points of the action, then S_N has to be stationary with respect to an arbitrary variation about $\bar{\sigma}$ and $\bar{\alpha}$. In particular, if $\delta\sigma=0$, $\delta\alpha=\epsilon\bar{\alpha}$ then $\delta S_N[\bar{\alpha},\bar{\sigma}]=0$. The transformation on α is an infinitesimal scale transformation. The functional trace of the logarithm is automatically invariant under such a transformation; thus we conclude that the only other term containing α must be stationary under such a transformation:

$$\int dt\,dr\,\bar{\alpha}^{ab}(t,r)\bar{\sigma}_{ab}(t,r)=0\,. \tag{A2}$$

The $\alpha\cdot\sigma$ term never contributes to the potential in leading order. This may be verified explicitly from the expressions given in Sec. III.

To obtain more information, we have to study the action as $T\to\infty$. The stationary solutions are expected to be t independent because of the time-translational invariance. Under this most reasonable assumption, we may rewrite action (2.25) as

$$S_N/T=M^2\int_0^R dr\{[\det(\delta_{ab}+\sigma_{ab}(r))]^{1/2}-\tfrac{1}{2}\alpha^{ab}(r)\sigma_{ab}(r)\}$$

$$+\tfrac{1}{2}D\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_0^R dr\langle r|\ln[\omega^2\alpha^{00}(r)+i\,\omega\alpha^{01}(r)\partial_r+i\omega\partial_r\,\alpha^{01}(r)-\partial_r\,\alpha^{11}(r)\,\partial_r]|r\rangle\,. \tag{A3}$$

Time-reversal invariance ($\omega\to-\omega$) guarantees that the logarithm term is an even functional of $\alpha^{01}(r)$.

By performing the σ-variational derivatives one obtains the algebraic equations

$$\bar{\alpha}_{00}=[1+\bar{\sigma}_{11}(r)][\det(1+\bar{\sigma})]^{-1/2}, \tag{A4a}$$

$$\bar{\alpha}_{11}=[1+\bar{\sigma}_{00}(r)][\det(1+\bar{\sigma})]^{-1/2}, \tag{A4b}$$

$$\bar{\alpha}_{01}=-\bar{\sigma}_{01}(r)[\det(1+\bar{\sigma})]^{-1/2}. \tag{A4c}$$

From the above it follows that $\det\bar{\alpha}=1$, and if $\bar{\sigma}_{01}=0$ then $\bar{\alpha}_{01}=0$.

We shall now argue that $\bar{\sigma}_{01}=0$ and $\bar{\alpha}_{01}=0$. As $R\to\infty$, the system becomes invariant with respect to the group of Euclidean transformations on the plane. In particular, one expects $\bar{\alpha}$ and $\bar{\sigma}$ to be (t,r) independent by translational invariance, and $\bar{\alpha}^{ab}=\bar{\alpha}\delta^{ab}$ by rotational invariance. At $R=\infty$ the stationary point is given by $\bar{\alpha}=1$, $\bar{\sigma}=0$. Remembering that $R=\infty$ corresponds to $\lambda=0$ we conclude that the power series in λ for $\bar{\alpha}$ begins at order λ^0, and the corresponding one for $\bar{\sigma}$ begins at order λ^1. Let L represent the logarithm term in (A3). Its evenness in α_{01} will be symbolically written as $L=L(\alpha_{01}{}^2)$. The functional derivative of (A3) with respect to α_{01} leads to the symbolic equation

$$\bar{\sigma}_{01}=\lambda\bar{\alpha}_{01}L'(\bar{\alpha}_{01}{}^2)\,. \tag{A5}$$

The factor of λ is due to the M^2 in (A3). Equations (A4c) and (A5) generate the perturbation expansion in λ of $\bar{\alpha}_{01}$ and $\bar{\sigma}_{01}$. We immediately see that $\bar{\alpha}_{01}=0$ and $\bar{\sigma}_{01}=0$ to any finite order in λ. Any possible nonzero value of $\bar{\alpha}_{01}$ and $\bar{\sigma}_{01}$ can only occur at a finite distance away from $\lambda=0$. Since $\bar{\alpha}_{ab}$ and $\bar{\sigma}_{ab}$ are diagonal, we shall use a single index to label the diagonal entries.

The next step[12] is to show that $\bar{\alpha}$ and $\bar{\sigma}$ are r independent if $0<r<R$. Define $\mathcal{E}$ and L_ν by the equations

$$\mathcal{E}=\tfrac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_0^R dr\,\langle r|\ln[\alpha_0(r)\,\omega^2-\partial_r\alpha_1\partial_r]|r\rangle\,, \tag{A6}$$

$$L_\nu=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_0^r dr\langle r|[\omega^2\alpha_0(r)-\partial_r\alpha_1\partial_r]^{-\nu}|r\rangle\,. \tag{A7}$$

Note that the derivative of L_ν with respect to ν at $\nu=0$ gives $-2\mathcal{E}$. Let $\mathfrak{D}$ be the self-adjoint Sturm-Liouville operator

$$\mathfrak{D}=-\frac{1}{\alpha_0(r)}\frac{d}{dr}\alpha_1(r)\frac{d}{dr}\,. \tag{A8}$$

This operator has positive eigenvalues $\omega_n{}^2$, and eigenfunctions $\phi_n(r)$ orthogonal with respect to the weight function $\alpha_0(r)$:

$$\mathfrak{D}\phi_n=\omega_n{}^2\phi_n\,, \tag{A9}$$

$$\int_0^R dr\,\alpha_0(r)\phi_n(r)\phi_m(r)=\delta_{nm}\,, \tag{A10}$$

$$\phi_n(0)=\phi_n(R)=0\,. \tag{A11}$$

The completeness of the eigenfunctions allows L_ν to be expressed as

$$L_\nu=\int\frac{d\omega}{2\pi}\int_0^R dr\sum_n\frac{\alpha_0(r)\,\phi_n(r)^2}{[\alpha_0(r)\,\omega^2+\alpha_0(r)\,\omega_n{}^2]^\nu} \tag{A12}$$

$$=\int\frac{d\omega}{2\pi}\int_0^R dr\sum_n\frac{\alpha_0(r)^{1-\nu}\phi_n(r)^2}{[\omega^2+\omega_n{}^2]^\nu} \tag{A13}$$

$$=(4\pi)^{-1/2}\frac{\Gamma(\nu-\frac{1}{2})}{\Gamma(\nu)}\sum_n\int_0^R dr\,\alpha_0(r)^{1-\nu}\phi_n(r)^2\omega_n{}^{1-2\nu}. \tag{A14}$$

The last expression may be split into two pieces:

$$L_\nu=(4\pi)^{-1/2}\frac{\Gamma(\nu-\frac{1}{2})}{\Gamma(\nu)}\sum_n \omega_n^{1-2\nu}\int_0^R dr\,\alpha_0(r)\phi_n(r)^2+(4\pi)^{-1/2}\frac{\Gamma(\nu-\frac{1}{2})}{\Gamma(\nu)}\sum_n\omega_n^{1-2\nu}\int_0^R dr[\alpha_0^{-\nu}-1]\,\alpha_0\phi_n^2. \tag{A15}$$

The second term is of order ν^2 and does not contribute to $\mathcal{E}$. We conclude that

$$\mathcal{E}=\tfrac{1}{2}\sum_n\omega_n, \tag{A16}$$

where the sum is understood to be analytically regularized as in (A15). Equation (A16) states that the zero-point energy is $\hbar\omega/2$ for each mode. Equation (A3) may be written

$$S_N/T=M^2\int_0^R dr[(1+\sigma_0)^{1/2}(1+\sigma_1)^{1/2}-\tfrac{1}{2}\alpha_0\sigma_0-\tfrac{1}{2}\alpha_1\sigma_1]+\tfrac{1}{2}D\sum_n\omega_n[\alpha]\,. \tag{A17}$$

The variational equations obtained by differentiating (A17) with respect to α require the knowledge of the derivatives of the ω_n's with respect to α. This may be obtained by first-order perturbation theory about the operator $\mathfrak{D}$. The Sturm-Liouville problem defined by Eqs. (A9)–(A11) may be derived from a variational principle for ϕ *and* ω^2:

$$J[\phi,\omega^2]=\int_0^r dr\,\tfrac{1}{2}\alpha_1(r)\left(\frac{d\phi}{dr}\right)^2+\tfrac{1}{2}\omega^2\left[1-\int_0^R dr\,\alpha_0(r)\,\phi^2(r)\right] \tag{A18}$$

with $\phi(0)=\phi(R)=0$. If ϕ_n is an eigenfunction then $J[\phi_n,\omega_n^2]=\omega_n^2/2$. The variational principle will be used to determine the change in ω when one perturbs α_0 and α_1. If ϕ_n is the eigenfunction for the eigenvalue problem defined by α_0 and α_1, and if $\phi_n+\epsilon\overline{\phi}_n$ is the eigenfunction for the problem defined by $\alpha+\epsilon\delta\alpha$, then the perturbed eigenvalues may be obtained by the relation

$$I[\phi_n+\epsilon\overline{\phi}_n,\ \alpha+\delta\alpha]=\tfrac{1}{2}(\omega_n+\delta\omega_n)^2. \tag{A19}$$

By using the variational principle one can show that $\delta\omega_n$ is determined by the $\delta\alpha$'s, and by the unperturbed eigenfunctions and the unperturbed eigenvalues.

It is convenient to change to a new independent variable τ defined by

$$\tau=\int_0^r dr'/\alpha_1(r'), \tag{A20}$$

$$L=\int_0^R dr'/\alpha_1(r'). \tag{A21}$$

τ ranges from zero to L. Note that L depends on α_1. The variational principle may be reformulated in the form

$$J=\int_0^L d\tau\,\tfrac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2+\tfrac{1}{2}\omega^2\left(1-\int_0^L d\tau\,\alpha_0\alpha_1\phi^2\right), \tag{A22a}$$

$$\phi(0)=\phi(L)=0. \tag{A22b}$$

The Euler-Lagrange equations for the above are

$$-\frac{d^2\phi}{d\tau^2}-\omega^2\alpha_0\alpha_1\phi=0, \tag{A23a}$$

$$1-\int_0^L d\tau\,\alpha_0\alpha_1\phi^2=0. \tag{A23b}$$

According to Eqs. (A4), $\overline{\alpha}_0(r)\overline{\alpha}_1(r)=1$ at the stationary point. The zeroth-order eigenvalue problem reduces to a harmonic oscillator.

In computing the effect of the perturbation on the eigenvalues, it is important to remember that L depends on α_1. The dependence of the variational principle on L is computed by using methods similar to those used to derive the Hamilton-Jacobi equation from a variational principle. The changes on the eigenvalues due to the perturbations are given by

$$\delta\omega_n/\delta\alpha_0(\tau)=-\tfrac{1}{2}\omega_n\alpha_1(\tau)\phi_n(\tau)^2, \tag{A24a}$$

$$\delta\omega_n/\delta\alpha_1(\tau)=-\tfrac{1}{2}\omega_n\alpha_0(\tau)\phi_n(\tau)^2+[2\omega_n\alpha_1(\tau)]^{-1}(d\phi_n/d\tau)_L^2. \tag{A24b}$$

To derive Euler-Lagrange equations for S_N it is necessary to change variables from r to τ in (A17). The results are

$$0=\frac{D}{2M^2}\sum_n\frac{\delta\omega_n}{\delta\overline{\alpha}_0(\tau)}-\tfrac{1}{2}\overline{\alpha}_1(\tau)\overline{\sigma}_0(\tau), \tag{A25a}$$

$$0=\frac{D}{2M^2}\sum_n\frac{\delta\omega_n}{\delta\overline{\alpha}_1(\tau)}-\tfrac{1}{2}\overline{\alpha}_1(\tau)\overline{\sigma}_1(\tau)-\frac{\alpha_1(L)}{\alpha_1(\tau)}[(1+\overline{\sigma}_0)^{1/2}(1+\overline{\sigma}_1)^{1/2}-\tfrac{1}{2}\overline{\alpha}\cdot\overline{\sigma}]_L+[(1+\overline{\sigma}_0)^{1/2}(1+\overline{\sigma}_1)^{1/2}-\tfrac{1}{2}\overline{\alpha}\cdot\overline{\sigma}]_\tau\,. \tag{A25b}$$

The zeroth-order Sturm-Liouville problem is defined by

$$-d^2\phi/d\tau^2-\omega^2\phi=0, \tag{A26a}$$

$$1=\int_0^L d\tau\,\phi(\tau)^2, \tag{A26b}$$

$$\phi(0)=\phi(L)=0. \tag{A26c}$$

The eigenvalues and eigenfunctions are given by

$$\omega_n = n\pi/L, \quad n = 1, 2, 3, \ldots \tag{A27a}$$

$$\phi_n(\tau) = (2/L)^{1/2} \sin\omega_n\tau. \tag{A27b}$$

Inserting (A24a) and (A27) into (A25a) yields the expression

$$\overline{\sigma}_0(\tau) = -\frac{D\pi}{M^2L^2}\sum_{n=1}^{\infty} n\sin^2\omega_n\tau. \tag{A28}$$

Inserting (A24b) and (A27) into (A25b) leads to the equation

$$\begin{aligned}0 = \frac{\pi D}{2M^2L^2}\sum_{n=1}^{\infty} n\cos^2\omega_n\tau - \tfrac{1}{2}\overline{\alpha}_1(\tau)^2\overline{\sigma}_1(\tau)\\ - \overline{\alpha}_1(L)[(1+\overline{\sigma}_0)^{1/2}(1+\overline{\sigma}_1)^{1/2} - \tfrac{1}{2}\overline{\alpha}\cdot\overline{\sigma}]_L\\ + \overline{\alpha}_1(\tau)[(1+\overline{\sigma}_0)^{1/2}(1+\overline{\sigma}_1)^{1/2} - \tfrac{1}{2}\overline{\alpha}\cdot\overline{\sigma}]_\tau.\end{aligned} \tag{A29}$$

The solution of (A28) and (A29) requires an understanding of the formal functions

$$C(\tau) = \sum_{n=1}^{\infty} n\cos^2\omega_n\tau, \tag{A30a}$$

$$S(\tau) = \sum_{n=1}^{\infty} n\sin^2\omega_n\tau. \tag{A30b}$$

In Appendix B it is demonstrated that within the context of analytic regularization, the above are defined by

$$C(\tau) = \begin{cases} -\frac{1}{12}, & \tau = 0, L \\ -\frac{1}{24}, & 0<\tau<L, \end{cases} \tag{A31a}$$

$$S(\tau) = \begin{cases} 0, & \tau = 0, L \\ -\frac{1}{24}, & 0<\tau<L. \end{cases} \tag{A31b}$$

This result can be substituted into (A27). One obtains an expression for $\overline{\sigma}_0$ which still depends on $\overline{\alpha}_1$ through L. Equations (A4a) and (A4b) can be used to express $\overline{\alpha}_0$ and $\overline{\sigma}_1$ in terms of $\overline{\alpha}_1$ and $\overline{\sigma}_0$:

$$\overline{\alpha}_0(\tau) = \overline{\alpha}_1(\tau)^{-1}, \tag{A32a}$$

$$\overline{\sigma}_1(\tau) = (1+\overline{\sigma}_0)\,\overline{\alpha}_1^{\,-2} - 1. \tag{A32b}$$

Inserting the remarks given above into (A29) yields an equation involving only $\overline{\alpha}_1$:

$$2\overline{\alpha}_1(\tau)^2 - \overline{\alpha}_1(L)^2 - 1 - \pi D/(12M^2L^2) = 0. \tag{A33}$$

It follows that $\overline{\alpha}_1(\tau)$ must be a constant $\hat{\alpha}_1$ for $0<\tau<L$. Evaluating the above at $\tau = L$ gives the result that $\overline{\alpha}_1(L)^{-2} = 1 - 2\lambda$. We conclude that

$$\overline{\alpha}_1(\tau) = \overline{\alpha}_0(\tau)^{-1} = (1-2\lambda)^{-1/2}, \quad 0\leq\tau\leq L. \tag{A34}$$

Substituting this result into (A28) and (A32b) leads to the conclusion

$$\sigma_0(r) = \begin{cases} 0, & r = 0, R \\ \dfrac{\lambda}{1-2\lambda}, & 0<r<R, \end{cases} \tag{A35}$$

$$\overline{\sigma}_1(r) = \begin{cases} -2\lambda, & r = 0, R \\ -\lambda, & 0<r<R. \end{cases} \tag{A36}$$

The discontinuities in $\overline{\sigma}$ at the end points do not affect the calculation of the static potential. The ansatz used in Sec. III is justified.

APPENDIX B

The solution of the Nambu model led us to the functions

$$C(x) = \sum_{n=1}^{\infty} n\cos^2 n\pi x, \tag{B1}$$

$$S(x) = \sum_{n=1}^{\infty} n\sin^2 n\pi x. \tag{B2}$$

These functions arose from the $\sum\omega$ term in (A17). The analytic regularization of $\sum\omega$ requires that (B1) and (B2) be interpreted as analytic continuations in the variable s of the functions

$$C(x,s) = \sum_{n=1}^{\infty} n^{-s}\cos^2 n\pi x, \tag{B3}$$

$$S(x,s) = \sum_{n=1}^{\infty} n^{-s}\sin^2 n\pi x. \tag{B4}$$

The variable x is always taken to be real. The series in (B3) and (B4) are uniformly convergent if $\mathrm{Re}\,s>1$. It immediately follows that

$$C(x,s) + S(x,s) = \zeta(s), \tag{B5}$$

$$C(x,s) - S(x,s) = \mathrm{Re}\,E(x,s), \tag{B6}$$

where

$$E(x,s) = \sum_{n=1}^{\infty} n^{-s}\exp(2\pi i n x). \tag{B7}$$

The function E is periodic in x with period 1, and reduces to $\zeta(s)$ for integer x. The strategy is to find an integral representation for E which allows an analytic continuation to $s = -1$. If $\mathrm{Re}\,s>1$, then the following manipulations are allowed:

$$\begin{aligned}E(x,s) &= \sum_{n=1}^{\infty} e^{2\pi i n x}\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}e^{-nt}\\ &= \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\sum_{n=1}^{\infty}\exp[n(2\pi i x - t)]\\ &= \frac{1}{\Gamma(s)}\int_0^\infty dt\,\frac{t^{s-1}}{\exp(t-2\pi i x)-1}\\ &= -\frac{\Gamma(1-s)}{2\pi i}\int_K dz\,\frac{(-z)^{s-1}}{\exp(z-2\pi i x)-1}.\end{aligned} \tag{B8}$$

The contour K is shown in Fig. 3. The denomi-

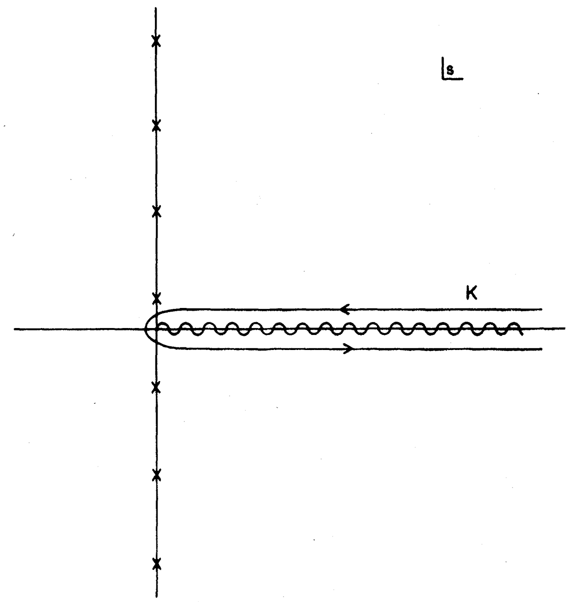

FIG. 3. The contour K for the integral representation of $E(x,s)$. The crosses are the poles at $z=2\pi i(x+n)$. The wavy line is the branch cut of $(-z)^{s-1}$.

nator of the integrand has simple zeros at $z=2\pi i(x+n)$, where n is an integer. For noninteger x, the contour does not enclose any of the zeros. For integer x it only encloses the zero at the origin. Contour-integral representation (B8) allows us to analytically continue $E(x,s)$ to the whole complex s plane.

Assume s is fixed, and let $0<x<1$. As $x\to 0$, the singularity in the integrand at $z=2\pi ix$ must cross the contour. We conclude that $E(x,s)$ is continuous in x except for a possible discontinuity at $x=0$. By periodicity this statement is extendable: $E(x,s)$ is continuous in x except for possible discontinuities at integer values of x.

The analyticity properties in s are also obtainable from the contour representation. If x is an integer, then the contour representation reduces to the contour representation of $\zeta(s)$ as required. The Riemann ζ function is analytic in the entire complex plane except for a simple pole at $s=1$. If x is not an integer, then $E(x,s)$ is analytic in the entire complex plane. The residue is zero at the potential poles located at $s=1,2,3,\ldots$. The change in the analyticity properties at integer x is a consequence of singularities crossing the contour of integration.

Evaluating (B8) at $s=-1$ leads to the results

$$C(x,-1)=\begin{cases}-\frac{1}{12}, & x=\text{integer}\\ -\frac{1}{24}, & x\neq\text{integer}\,,\end{cases}$$

$$S(x,-1)=\begin{cases}0, & x=\text{integer}\\ -\frac{1}{24}, & x\neq\text{integer}\,.\end{cases}$$

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