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Four-body Efimov effect in a Born-Oppenheimer model

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The possibility of a zero-energy Efimov effect is investigated in a model consisting of three identical heavy particles and a lighter one, when the light-heavy interaction leads to a zero-energy bound state for the two-heavy–one-light subsystem. The model is solved in the Born-Oppenheimer approximation with the light-heavy interaction taken to be a separable s-wave potential of Yamaguchi form. The heavy-heavy interaction is short range and, if attractive, is to be taken weak enough to support no two- or three-heavy-particle bound states. The relevant parameter is the potential strength λ of the light-heavy interaction which for $\lambda=\lambda_c$ supports a single zero-energy light-heavy bound state. If the first zero-energy bound state of the two-heavy–one-light subsystem occurs for $\lambda=\lambda'$ such that $|\lambda'|<|\lambda_c|$, there are no Efimov four-body bound states. On the contrary, if the chosen heavy-heavy potential is repulsive enough to prevent the existence of three-body bound states for $|\lambda|<|\lambda_c|$, then as $\lambda\to\lambda_c$ a few four-body Efimov states may emerge but their number remains finite. These states disappear for $|\lambda|\gtrsim|\lambda_c|$ as the three-body cut overrides them.

I. INTRODUCTION

It was first suggested by Efimov[1] that the number of bound states for three particles interacting through short-range potentials may grow to infinity as some of the pair interactions increase to just bind two particles and then decrease for stronger binding. A detailed proof of the occurrence of this phenomena as well as a justification for the disappearance of three-body bound states with increasing potential strength have appeared in the literature.[1,2] The Efimov effect can be considered as a long-range effect in hyperspace[1] or alternatively, as Amado and Noble have shown,[2] as an infrared divergence of the Faddeev kernel in momentum space which is responsible for the divergence of the trace of the kernel. A long-range effect in hyperspace and an infrared divergence of the three-body kernel in momentum space are both complicated mathematical concepts difficult to visualize. Recently, Fonseca, Redish, and Shanley[3] and Ouchinnikov and Sigal[4] used the Born-Oppenheimer (BO) approximation[5] to study the Efimov effect in a model consisting of two heavy particles and a lighter one interacting by means of short-range potentials. When the light-heavy interaction is strong enough to support a single zero-energy bound state, they demonstrated that the Efimov effect in this case can be considered as a long-range effect in the physical coordinate space. Because of the "large size" of the light-heavy bound state in the Efimov limit, the two heavy particles feel an effective long-range potential of the R^{-2} type at large separation by exchanging the light particle between them. This long-range potential is responsible for the occurrence of an infinite number of bound states for the system. In this work we use an extended version of the BO method in a four-body molecular problem in order to shed light on the physical aspects of the problem.

The possibility of occurrence of a four-body Efimov effect has been previously investigated by Amado and Greenwood[6] in a four-identical-boson model and they concluded that such a system could not show an Efimov effect. They pointed out that the value $\lambda=\lambda_c$ of the two-body coupling strength that supports a single zero-energy two-body bound state cannot lead to an infinite number of four-body bound states, because at $\lambda=\lambda_c$ a three-body bound state should already exist with a finite binding energy ϵ_3. The resulting scattering threshold at $E=\epsilon_3$ in the four-body problem makes it impossible for any four-body bound state to emerge at $E=0$ in the $\lambda=\lambda_c$ limit. Therefore, if an infinite number of Efimov bound states is to be found in the four-body problem, it should emerge for the value of the two-body coupling strength $\lambda=\lambda'$ ($|\lambda'|<|\lambda_c|$) which leads to the first zero-energy three-body bound state. In their work, Amado and Greenwood looked for an infrared divergence of the four-body kernel in momentum space and found that the singularity of the connected three-body amplitude at $\lambda=\lambda'$ was not strong enough to make the trace of the four-body kernel diverge at $E=0$ and hence produce an infinite number of four-body bound states.

In this work we make use of elementary quantum-mechanical arguments to show in a simple model that a four-body molecular system consisting of three heavy particles and a lighter one interacting by short-range pair interactions cannot have an infinite number of bound states when the two-heavy–one-light system has a zero-energy bound state. In the four-boson model of Amado and Greenwood all pair potentials are attractive in nature. Therefore, as explained above, four-body Efimov states cannot occur for $\lambda = \lambda_c$ and the only possible threshold to look for them is at $\lambda = \lambda'$. In the present model the heavy-heavy and heavy-light potentials are independent of each other and by choosing them conveniently we have two distinct situations under which a four-body Efimov effect may take place. The first one is similar to that studied by Amado and Greenwood; the heavy-heavy potential, though weak enough to support no two- or three-heavy-particle bound state, is such that the heavy-light potential strength $\lambda = \lambda'$ that leads to the first zero-energy two-heavy–one-light bound state satisfies $|\lambda'| < |\lambda_c|$ where $\lambda = \lambda_c$ leads to the first zero-energy light-heavy bound state. The second one has not been studied in the past and involves a repulsive heavy-heavy potential such that no zero-energy two-heavy–one-light bound state occurs for $|\lambda'| < |\lambda_c|$. The relevant limit to study the four-body Efimov effect is, in this case, $\lambda = \lambda_c$, where in the language of Amado and Greenwood the infrared divergence of the four-body kernel in momentum space leads to a divergence of the trace of the kernel. It is in this latter case that a finite number of four-body Efimov states may possibly appear for $\lambda = \lambda_c$, which will eventually be run over by the three-body cut for increasing potential strength λ. The model is solved in the framework of the Born-Oppenheimer approximation which allows one to develop an intuitive understanding of why there should be no zero-energy Efimov effect in a four-body molecular problem or for that matter in any N-body molecular system.

In Sec. II we describe the model and in Sec. III we solve the appropriate equations and show why there is no Efimov effect in a four-body molecular system. Finally in Sec. IV we give a brief discussion of our findings.

II. THE MODEL

Although at the present time there are several formulations of the four-body problem that allow for an exact calculation of the four-body binding energy, we find it convenient to study the possibility of a four-body Efimov effect through the molecular approach, in particular, making use of the Born-Oppenheimer approximation. For a system of three identical heavy particles of mass M and a light particle of mass m interacting by means of short-range potentials, one must solve the equation

$$H\Psi(\vec{r},\vec{\rho},\vec{R}) = E\Psi(\vec{r},\vec{\rho},\vec{R}), \tag{1}$$

where (in units of $\hbar = 2m = 1$)

$$H = -\frac{1}{\mu}\vec{\nabla}_r^2 - \frac{3}{2\mathfrak{M}}\vec{\nabla}_\rho^2 - \frac{2}{\mathfrak{M}}\vec{\nabla}_R^2 + \sum_i v_i + \sum_i V_i,$$

$$\mu = \frac{3\mathfrak{M}}{3\mathfrak{M}+1} \tag{2}$$

and $\mathfrak{M} = M/m$ is the heavy-light mass ratio. The coordinates $\vec{r}$, $\vec{\rho}$, and $\vec{R}$ are shown in Fig. 1 and the wave function Ψ is subject to the boundary condition that it approaches zero when either r, ρ, or R go to infinity. The potential v_i represents the interaction between the light particle and the heavy particle labeled by the index i, and V_i is the heavy-heavy potential in the "odd-man-out" notation common to three-body work. The heavy-heavy potential is short range and weak enough to support no heavy-heavy bound state or for that matter any bound state of the three-heavy-particle system. The light-heavy interaction is chosen to be a separable s-wave potential

$$v = \lambda |f\rangle\langle f|, \tag{3}$$

where the form factor $|f\rangle$ is of Yamaguchi form[7]

$$\langle \vec{p}|f\rangle = (p^2+\beta^2)^{-1}. \tag{4}$$

In the presence of a light-heavy bound state the two-body coupling strength λ is related to the binding energy ϵ_0 in the following way ($\epsilon_0 < 0$):

$$\lambda^{-1} = \int d^3p \,\frac{f^2(p)}{\epsilon_0 - p^2/\nu'}, \quad \nu' = \mathfrak{M}/(\mathfrak{M}+1). \tag{5}$$

Defining $\kappa_0^2 = -\nu'\epsilon_0$ and substituting (4) into (5) we obtain

$$\lambda^{-1} = -\pi^2\nu'/\beta(\beta+\kappa_0)^2. \tag{6}$$

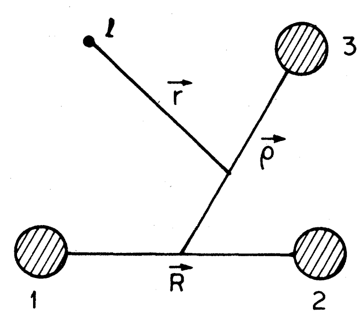

FIG. 1. Jacobi coordinates for the four-body molecular system.

For $\epsilon_0 = 0$ then $\lambda = \lambda_c = -\beta^3/\pi^2\nu'$ is the critical coupling strength above which there is no light-heavy bound state. Equations (5) and (6) are only defined for $|\lambda| \geq |\lambda_c|$ since for $|\lambda| < |\lambda_c|$ Eq. (5) has no negative-energy solution.

It is well known that for very large mass ratio $\mathfrak{M}$ ($\mathfrak{M} \simeq 2000$) molecular systems can be treated accurately by the BO approximation. In recent studies,[8] it has been shown that the BO method may be also used with reasonable accuracy to describe molecular three-body systems with small heavy-light mass ratio ($\mathfrak{M} \geq 5$) if short-range potentials are used between pairs. The BO approximation is therefore used to solve the Schrödinger equation defined in (1). This amounts to assuming that the four-body wave function can be approximately written as the product of two terms

$$\Psi(\vec{r},\vec{\rho},\vec{R}) \simeq \psi(\vec{r},\vec{\rho},\vec{R})\,\Phi(\vec{\rho},\vec{R}), \tag{7}$$

where $\psi(\vec{r},\vec{\rho},\vec{R})$ is the wave function describing the motion of the light particle when the three heavy particles are taken as fixed centers in a configuration defined by the vectors $\vec{\rho}$ and $\vec{R}$. The light-particle equation is

$$\left(-\vec{\nabla}_r^2/\mu + \sum_i v_i\right)\psi(\vec{r},\vec{\rho},\vec{R}) = \tilde{\epsilon}(\vec{\rho},\vec{R})\psi(\vec{r},\vec{\rho},\vec{R}), \tag{8}$$

where $\tilde{\epsilon}(\vec{\rho},\vec{R})$ is the lowest negative-energy eigenvalue of the three-center problem that depends parametrically on ρ, R, and the angle between $\vec{\rho}$ and $\vec{R}$. Substituting (7) in (1) and making use of (8) one readily obtains the heavy-particle equation

$$\left[-\frac{3}{2\mathfrak{M}}\vec{\nabla}_\rho^2 - \frac{2}{\mathfrak{M}}\vec{\nabla}_R^2 + \sum_i V_i + \tilde{\epsilon}(\vec{\rho},\vec{R})\right]\Phi(\vec{\rho},\vec{R}) = E\Phi(\vec{\rho},\vec{R}), \tag{9}$$

after neglecting the terms resulting from $\vec{\nabla}_\rho^2$ or $\vec{\nabla}_R^2$ acting on $\psi(\vec{r},\vec{\rho},\vec{R})$. This is the three-body Schrödinger equation for the movement of the heavy particles, where $\tilde{\epsilon}(\vec{\rho},\vec{R})$ plays the role of an effective three-body force that is due to the presence of the light particle. Although the separation of the four-body Schrödinger equation (1) into two separate equations may be valid in much broader circumstances[8] it will at least be appropriate whenever the motion of the light particle is rapid compared to the motion of the heavy particles so that the dynamics of the light one may be solved while the heavies are instantaneously fixed. Formally the separation works if the kinetic energy operators $-(3/2\mathfrak{M})\vec{\nabla}_\rho^2$ and $-(2/\mathfrak{M})\vec{\nabla}_R^2$ operating on $\psi(\vec{r},\vec{\rho},\vec{R})$ are small compared to other terms in the equation which is certainly the case when $\mathfrak{M} \gg 1$.

To calculate the binding energy ϵ_3 of the three-body molecular subsystems or for that matter the coupling strength λ' ($|\lambda'| < |\lambda_c|$) that leads to the first zero-energy three-body bound state, we also make use of the three-body BO approach. Since the heavy particles are all identical this implies the solution of another set of Schrödinger-type equations. They are[3]

$$[-\vec{\nabla}_r^2/\nu + v(\vec{r} - \tfrac{1}{2}\vec{R}) + v(\vec{r} + \tfrac{1}{2}\vec{R})]\psi(\vec{r},\vec{R}) = \epsilon(R)\psi(\vec{r},\vec{R}),$$
$$\nu = 2\mathfrak{M}/(2\mathfrak{M}+1), \tag{10}$$

for the binding energy of the light particle in the potential field of any two heavy particles fixed in space at a distance R, and

$$-\frac{2}{\mathfrak{M}}\vec{\nabla}_R^2 + V(R) + \epsilon(R)\;\;\Phi(\vec{R}) = \epsilon_3\Phi(\vec{R}) \tag{11}$$

for the relative motion of the two heavy particles. The vectors $\vec{r}$ and $\vec{R}$ are defined as in Fig. 2.

III. SOLUTION OF THE BORN-OPPENHEIMER EQUATIONS

We now solve the light-particle BO equations for the three-body and four-body molecular systems. In the absence of a heavy-heavy potential we show the results of a model calculation for the value of the coupling strength $\lambda = \lambda'$ that leads to the first zero-energy three-body bound state. The lowest-energy eigenvalue of the three-center problem is calculated for a typical value of λ' such that $|\lambda'| < |\lambda_c|$.

A. Three-body system

Since the two-body light-heavy potential is nonlocal but separable, and R is a parameter, we rewrite the three-body light-particle equation (10) in operator form

$$(\vec{p}^2/\nu + v_1 + v_2)|\psi\rangle = \epsilon|\psi\rangle, \tag{12}$$

where $\vec{p}$ is the momentum operator $i\vec{\nabla}_r$ and

$$v_1(\vec{r},\vec{r}') = \langle\vec{r}' - \tfrac{1}{2}\vec{R}|f\rangle\lambda\langle f|\vec{r} - \tfrac{1}{2}\vec{R}\rangle,$$
$$v_2(\vec{r},\vec{r}') = \langle\vec{r}' + \tfrac{1}{2}\vec{R}|f\rangle\lambda\langle f|\vec{r} + \tfrac{1}{2}\vec{R}\rangle. \tag{13}$$

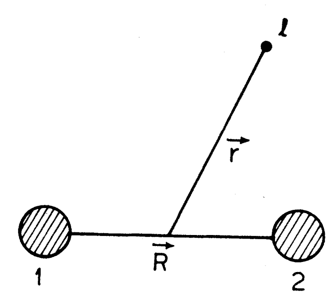

FIG. 2. Jacobi coordinates for the two-heavy–one-light subsystem.

A detailed account of how to solve the two-center problem with separable potentials has already appeared in the literature.[9] Here we present a brief review which is appropriate to the present work. Introducing the translation operator

$$\Theta(\vec{a}) = \exp(i\vec{p}\cdot\vec{a}), \tag{14}$$

we have

$$\Theta(\tfrac{1}{2}\vec{R})|\vec{r}\rangle = |\vec{r}+\tfrac{1}{2}\vec{R}\rangle, \tag{15}$$

$$\Theta^\dagger(\vec{a}) = \Theta(-\vec{a}) = \Theta(\vec{a})^{-1}. \tag{16}$$

Since Eq. (12) is invariant with respect to $\vec{R}\to-\vec{R}$ reflection we find it convenient to define the operator

$$\Theta^\pm(\vec{a}) = \frac{1}{\sqrt{2}}[\Theta(\vec{a})\pm\Theta(-\vec{a})], \tag{17}$$

which satisfies the relation

$$[\Theta^\pm(\vec{a})]^\dagger = \pm\Theta^\pm(\vec{a}). \tag{18}$$

Making use of (14), (15), and (17) in Eq. (12) and inverting the negative-definite operator $(\epsilon-\vec{p}^2/\nu)$ after moving $\vec{p}^2/\nu$ to the right-hand side, we get

$$|\psi\rangle = G_0(\Theta^+|f\rangle\lambda\langle f|\Theta^+ - \Theta^-|f\rangle\lambda\langle f|\Theta^-)|\psi\rangle, \tag{19}$$

$$G_0 = (\epsilon-\vec{p}^2/\nu)^{-1}, \tag{20}$$

where we suppressed the argument $\frac{1}{2}\vec{R}$ in all the operators Θ. Defining

$$g^\Lambda = \langle f|\Theta^\Lambda|\psi\rangle, \tag{21}$$

and noting that

$$\langle f|\Theta^\Lambda G_0\Theta^{\Lambda'}|f\rangle = \delta_{\Lambda\Lambda'}\langle f|\Theta^\Lambda G_0\Theta^\Lambda|f\rangle, \tag{22}$$

with Λ, $\Lambda'=\pm$, we get a separate equation for each Λ,

$$[1-\Lambda\lambda\langle f|\Theta^\Lambda G_0\Theta^\Lambda|f\rangle]g^\Lambda = 0. \tag{23}$$

Since we are interested in $g^\Lambda\neq 0$, for each R there is a solution for ϵ that makes the coefficient in square brackets vanish. Defining $\kappa^2=-\nu\epsilon$ and the function $J(R)$ as

$$J(R) = -\lambda\nu\int d^3p\,\frac{f^2(p)}{\kappa^2+p^2}e^{i\vec{p}\cdot\vec{R}}, \tag{24}$$

we get

$$1-\Lambda\lambda\langle f|\Theta^\Lambda G_0\Theta^\Lambda|f\rangle = 1-J(0)-\Lambda J(R) = 0. \tag{25}$$

The solutions are denoted $\epsilon_g(R)$ for $\Lambda=+$ and $\epsilon_u(R)$ for $\Lambda=-$ and the corresponding wave functions are symmetric and antisymmetric under $\vec{R}\to-\vec{R}$ reflection. This problem is solved in detail in Ref. 3 and here we discuss the relevant results.

We consider a large enough $\mathfrak{M}$ to justify the BO approach. As was pointed out in Ref. 3, for $|\lambda| > |\lambda_c|$ both ϵ_g and ϵ_u converge to ϵ_0 as $R\to\infty$ and in the limit $\lambda=\lambda_c$ $\epsilon_g(R)$ has the R^{-2} behavior for large R that is responsible for the three-body Efimov effect. Here we study the eigenfunctions of the two-center problem for $|\lambda|<|\lambda_c|$ and find that there is no ϵ_u solution for all R and that ϵ_g only exists up to a certain value of $R=R_0$ that depends on λ. The weaker the potential strength λ, the smaller the value of R_0 above which the light-particle BO equation has no bound-state solution. In Fig. 3 we show ϵ_g for several values of $\eta=(\lambda/\lambda_c)<1$. Although the shape of these curves depends on the nature of the light-heavy interaction, the nonexistence of a negative-energy solution of the two-center problem for a separation R greater than some finite R_0 is potential independent. This characteristic results exclusively from the knowledge that the eigenvalues of the light-particle equation at $R=\infty$ coincide with the negative-energy solutions of the asymptotic two-body problems—the absence of a light-heavy bound state ($|\lambda|<|\lambda_c|$) implies the absence of a bound state of the two-center problem in the separated atom limit. If at $R=\infty$ there is no solution, and at $R=0$ there is one, and $\epsilon(R)$ is a smooth function[5] of R then there is a separation R_0 above which the two heavy particles fixed in space can no longer sustain a bound state of the light particle.

Although for $|\lambda|<|\lambda_c|$ and $R>R_0$ we cannot define the BO wave function, we can nevertheless estimate the three-body binding energy by solving Eq. (11) with $\epsilon(R)=\epsilon_g(R)$ up to $R=R_0$ and $\epsilon(R)=0$ thereafter. This procedure allows the calculation of the coupling strength $\lambda'(|\lambda'|<|\lambda_c|)$ that leads to the first zero-energy three-body bound state ($\epsilon_3=0$). The accuracy of this estimate has been tested against the results of an exact Faddeev calculation[8] for the same system in a model where there is no heavy-heavy interaction. The values of λ' obtained in this case are shown in Fig. 4 as a function of the mass ratio $\mathfrak{M}$. The BO pre-

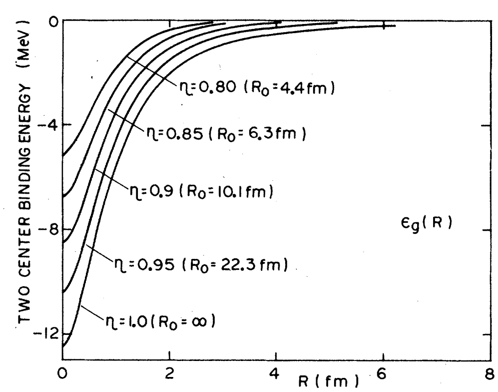

FIG. 3. Two-center binding energy vs R for different values of $\eta=\lambda/\lambda_c<1$.

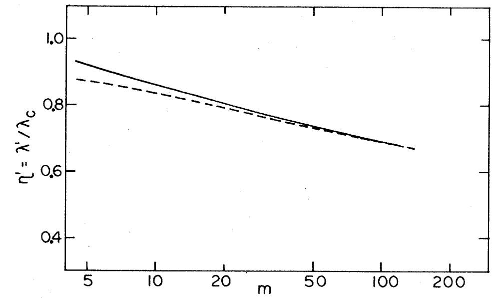

FIG. 4. Ratio λ'/λ_c vs $\mathfrak{M}$ in a model where the heavy particles do not interact. The solid line corresponds to the exact Faddeev result while the dashed line corresponds to the BO prediction.

diction for λ' approaches the exact result within 2% for $\mathfrak{M} > 15$. In general the value λ' that is responsible for the first zero-energy three-body state depends on the sign and strength of the heavy-heavy interaction. An attractive heavy-heavy potential that supports no two- or three-heavy-particle bound state favors values of λ' such that $|\lambda'| < |\lambda_c|$, while a repulsive heavy-heavy potential requires a stronger heavy-light interaction. If the first zero-energy three-body bound state emerges for $\lambda = \lambda'$ ($|\lambda'| < |\lambda_c|$), we are in a situation similar to that studied by Amado and Greenwood where the relevant limit to investigate the existence of a four-body Efimov effect is $\lambda \to \lambda'$. On the contrary if the short-range heavy-heavy potential is strongly repulsive, it may happen that the first zero-energy two-heavy–one-light bound state to appear is an Efimov state that at $\lambda = \lambda_c$ is supported by the R^{-2} tail[3] of $\epsilon_g(R)$. In this last situation the relevant limit to study the four-body Efimov effect is $\lambda \to \lambda_c$ where both the two-body and three-body subsystems are near a zero-energy bound state. For $|\lambda| > |\lambda_c|$ the two-heavy–one-light Efimov states disappear and there are no three-body bound states left until the light-heavy interaction is strong enough to overcome the heavy-heavy repulsion.

For $|\lambda'| < |\lambda_c|$ the effective potential $\epsilon(R)$ is of the type shown in Fig. 3. The absence of an effective potential $\epsilon(R)$ for $R > R_0$, though it inhibits us from making any simple assumption on the asymptotic behavior of the BO wave function, suggests that any bound state of the three-body system is "localized" in space, even in the three-body zero-energy limit. In the absence of two-body-subsystem bound states the asymptotic behavior of the three-body wave function is given by[10]

$$\psi(\vec{r}, \vec{R}) \underset{\xi\to\infty}{\sim} \frac{e^{-(-\epsilon_3)^{1/2}\xi}}{\xi^{5/2}}, \tag{26}$$

where $\xi = (\nu r^2 + \mathfrak{M} R^2/2)^{1/2}$ is the usual hyper-radius and r and R are shown in Fig. 2. Equation (26) shows that in the zero-energy limit the three-body wave function decays with inverse $\frac{5}{2}$ power of distance whereas in the case of the zero-energy two-body problem the wave function decays with the inverse of distance. The BO wave function cannot represent this complicated asymptotic behavior but the absence of a light-particle bound state for $R > R_0$ indicates that the two heavy particles have to stay predominantly within a finite region of space if the two heavy particles and the light one are to form a three-body bound state.

B. Four-body system

In this subsection we indicate how to solve the three-center problem defined in Eq. (8) with the light-heavy interaction given by Eq. (3). Making use of Eqs. (14)–(18) we can write Eq. (8) in operator form,

$$\begin{aligned}(\bar{\epsilon} - \vec{p}^2/\mu)|\psi\rangle = [&\mathcal{O}(-\tfrac{1}{3}\vec{\rho})\mathcal{O}^+(\tfrac{1}{2}\vec{R})|f\rangle\lambda\langle f|\mathcal{O}^+(\tfrac{1}{2}\vec{R})\mathcal{O}(\tfrac{1}{3}\vec{\rho}) \\ &-\mathcal{O}(-\tfrac{1}{3}\vec{\rho})\mathcal{O}^-(\tfrac{1}{2}\vec{R})|f\rangle\lambda\langle f|\mathcal{O}^-(\tfrac{1}{2}\vec{R})\mathcal{O}(\tfrac{1}{3}\vec{\rho}) \\ &+\mathcal{O}(\tfrac{2}{3}\vec{\rho})|f\rangle\lambda\langle f|\mathcal{O}(-\tfrac{2}{3}\vec{\rho})]|\psi\rangle,\end{aligned} \tag{27}$$

where ρ, R, and the angle between $\vec{\rho}$ and $\vec{R}$ are parameters. Inverting the negative-definite operator $(\bar{\epsilon} - \vec{p}^2/\mu)$ and defining

$$\begin{aligned}g^+ &= \langle f|\mathcal{O}^+(\tfrac{1}{2}\vec{R})\mathcal{O}(\tfrac{1}{3}\vec{\rho})|\psi\rangle, \\ g^- &= \langle f|\mathcal{O}^-(\tfrac{1}{2}\vec{R})\mathcal{O}(\tfrac{1}{3}\vec{\rho})|\psi\rangle, \\ g &= \langle f|\mathcal{O}(-\tfrac{2}{3}\vec{\rho})|\psi\rangle,\end{aligned} \tag{28}$$

we get a set of three coupled equations for g^+, g^-, and g:

$$\begin{aligned}&[1 - \lambda\langle f|\mathcal{O}^+(\tfrac{1}{2}\vec{R})G_0\mathcal{O}^+(\tfrac{1}{2}\vec{R})|f\rangle]g^+ \\ &\qquad - \lambda\langle f|\mathcal{O}^+(\tfrac{1}{2}\vec{R})G_0\mathcal{O}(\vec{\rho})|f\rangle g = 0, \\ &[1 + \lambda\langle f|\mathcal{O}^-(\tfrac{1}{2}\vec{R})G_0\mathcal{O}^-(\tfrac{1}{2}\vec{R})|f\rangle]g^- \\ &\qquad - \lambda\langle f|\mathcal{O}^-(\tfrac{1}{2}\vec{R})G_0\mathcal{O}(\vec{\rho})|f\rangle g = 0, \\ &[1 - \lambda\langle f|G_0|f\rangle]g - \lambda\langle f|\mathcal{O}(-\vec{\rho})G_0\mathcal{O}^+(\tfrac{1}{2}\vec{R})|f\rangle g^+ \\ &\qquad + \lambda\langle f|\mathcal{O}(-\vec{\rho})G_0\mathcal{O}^-(\tfrac{1}{2}\vec{R})|f\rangle g^- = 0.\end{aligned} \tag{29}$$

Defining $\delta^2 = -\mu\bar{\epsilon}$ we get

$$\lambda\langle f|\mathcal{O}^\pm(\tfrac{1}{2}\vec{R})G_0\mathcal{O}(\vec{\rho})|f\rangle = [J(X) \pm J(Y)]/\sqrt{2}, \tag{30}$$

where

$$\begin{aligned}X &= (R^2/4 + \rho^2 + R\rho\cos\theta)^{1/2}, \\ Y &= (R^2/4 + \rho^2 - R\rho\cos\theta)^{1/2},\end{aligned} \tag{31}$$

and θ is the angle between $\vec{R}$ and $\vec{\rho}$. We can also write

$$1 \mp \lambda\langle f|\mathcal{O}^\pm G_0\mathcal{O}^\pm|f\rangle = 1 - J(0) \mp J(R), \tag{32}$$

where $J(R)$ is given by Eq. (24) with ν and κ substituted by μ and δ, respectively. For fixed R, ρ, and θ, Eq. (29) is a homogeneous algebraic equation which has a solution for the values of $\tilde{\epsilon}<0$ that makes the matrix determinant vanish,

$$[1-J(0)-J(R)][1-J(0)+J(R)][1-J(0)] - [1-J(0)+J(R)][J(X)+J(Y)]^2/2 - [1-J(0)-J(R)][J(X)-J(Y)]^2/2=0\,. \quad (33)$$

Since $J(\infty)=0$, at $\rho=\infty$ the above equality reduces to

$$[1-J(0)-J(R)][1-J(0)+J(R)][1-J(0)]=0\,. \quad (34)$$

Therefore, taking note of Eqs. (5) and (25), the solutions of the three-center problem at $\rho=\infty$ are $\epsilon_g(R)$, $\epsilon_u(R)$, and ϵ_0 for $|\lambda|>|\lambda_c|$ but only $\epsilon_g(R)$ for $|\lambda|<|\lambda_c|$. In Figs. 5–7 we show the lowest eigenvalue $\tilde{\epsilon}(R,\rho,\theta)$ versus R for several values of ρ and $\theta=0°$, 30°, and 90°, respectively. The coupling strength $\lambda=\eta\lambda_c$ has been set at the value corresponding to $\eta=0.8$ but the curves look qualitatively the same for any other value of $\eta<1$. Again there is only a finite region in $\vec{R}$ space where we can find a negative-energy solution of the three-center problem since, for a given ρ and as long as $\eta<1$, the effective three-body potential $\tilde{\epsilon}(R,\rho,\theta)$ is only defined for finite values of R. As λ approaches λ_c the region of space where there is a solution to Eq. (33) increases and as shown in Figs. 8 and 9 at $\lambda=\lambda_c$ there is a negative-energy solution of the three-center problem for all values of R and ρ. As shown in the Appendix, at $\lambda=\lambda_c$ the effective potential presents for all ρ an asymptotic R^{-2} behavior similar to that found in Refs. 3 and 4 for $\epsilon_g(R)$. Using our model problem we also compare in Figs. 8 and 9 $\tilde{\epsilon}(R,\rho,\theta)$ with

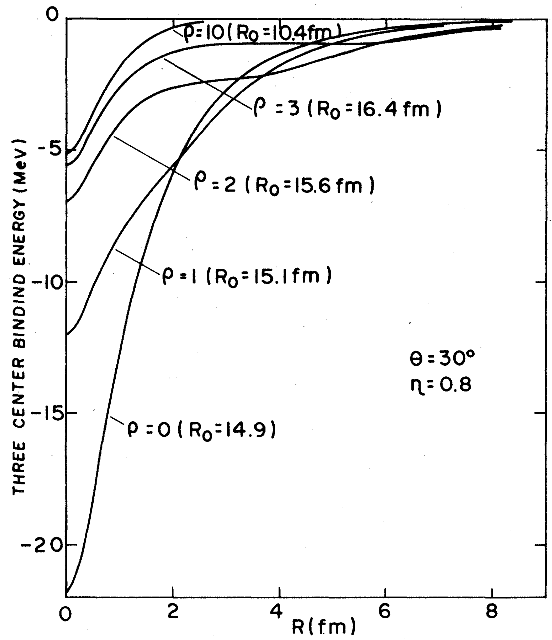

FIG. 6. Three-center binding energy vs R for $\eta=0.8$, $\theta=30°$, and different values of ρ.

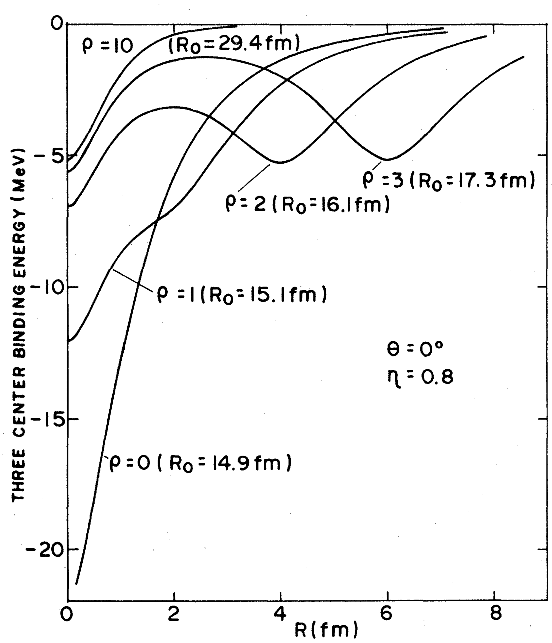

FIG. 5. Three-center binding energy vs R for $\eta=0.8$, $\theta=0$, and different values of ρ.

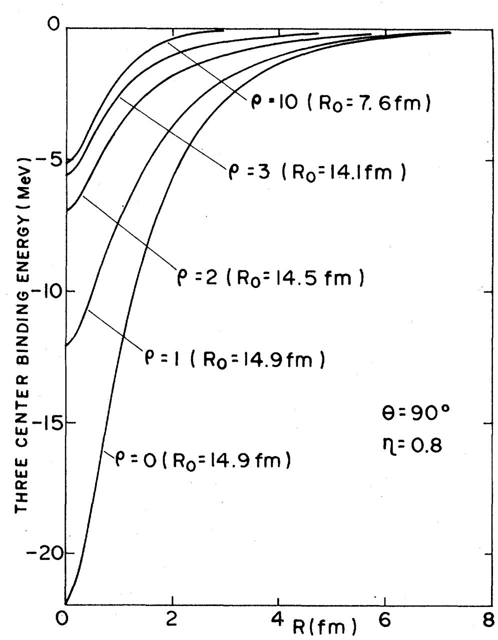

FIG. 7. Three-center binding energy vs R for $\eta=0.8$, $\theta=90°$, and different values of ρ.

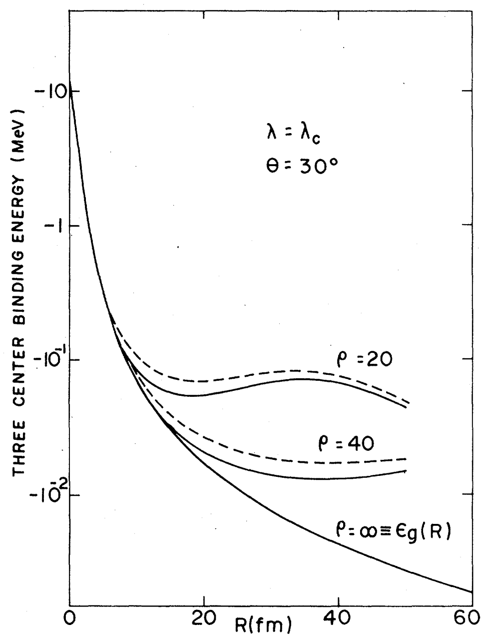

FIG. 8. Three-center binding energy vs R for $\eta=1$, $\theta=30_0$, and different values of ρ. The solid line corresponds to $\tilde{\epsilon}$ and the dashed line to W_3.

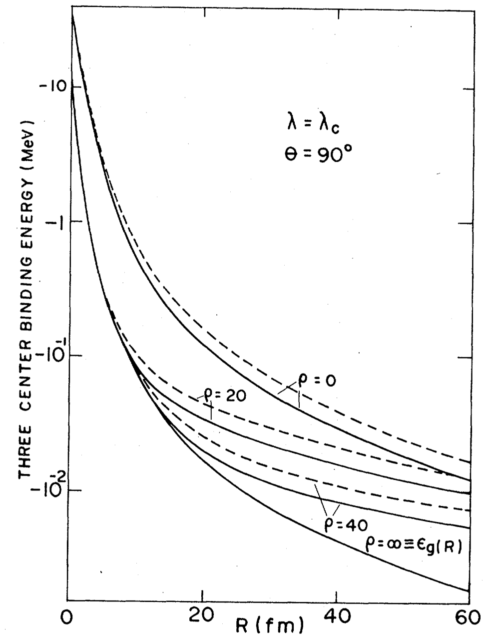

FIG. 9. Three-center binding energy vs R for $\eta=1$, $\theta=90°$, and different values of ρ. The solid line corresponds to $\tilde{\epsilon}$ and the dashed line to W_3.

$$W_3(R,\rho,\theta)=\epsilon_g(R)+\epsilon_g(|\tfrac{1}{2}\vec{R}+\vec{\rho}|)+\epsilon_g(|\tfrac{1}{2}\vec{R}-\vec{\rho}|)\,. \tag{35}$$

We find that for large separation between all three heavy particles the ratio between $\tilde{\epsilon}$ and W_3 is constant though slightly dependent on ρ and θ. Since in the $\lambda=\lambda_c$ limit[3]

$$\lim_{R\to\infty}\epsilon_g(R)=-A^2/R^2\,,\quad A=e^{-A}=0.567\,143\ldots\,, \tag{36}$$

for large interheavy particle separation we can write

$$\tilde{\epsilon}(R,\rho,\theta)\simeq -A'^2(R^{-2}+|\tfrac{1}{2}\vec{R}+\vec{\rho}|^{-2}+|\tfrac{1}{2}\vec{R}-\vec{\rho}|^{-2})\,, \tag{37}$$

where A' depends weakly on ρ and θ. At $\rho=\infty$, $A'=A$ and at $\rho=0$ we find that $A'=B/3$ where (see the Appendix)

$$B=8e^{-B}(B-e^{-B})^{-1}=1.473\,374\,5\cdots\,. \tag{38}$$

For intermediate values of ρ and $0<\theta<\pi/2$ we have $B/3<A'<A$ which indicates that in the asymptotic region and for $\lambda=\lambda_c$, $\tilde{\epsilon}$ can be approximately expressed as the sum of three $-A'^2/r_{ij}^2$ pair potentials whose net effect is weaker than the sum of the corresponding long-range tails of $\epsilon_g(R)$. In other words $\tilde{\epsilon}$ can be written as the sum of three ϵ_g pair potentials plus a repulsive three-body force that weakens the effective two-body potentials for large but finite values of ρ and R. Unlike Eq. (36) which has been proved to be valid for any short-range light-heavy interaction,[11] owing to the nonlinearity of both (25) and (33) we cannot prove (37) in a general way, nor even for the separable interaction used here. Nevertheless, this shows that the large size of the light-heavy bound state in the $\lambda=\lambda_c$ limit is again responsible for an effective long-range interaction between all three heavy particles.

We are now in a position where we can study the possibility of a four-body Efimov effect in a four-particle molecular system. As discussed in Sec. III A, depending on the strength and sign of the heavy-heavy potential, we have to consider in our model two distinct limits which may lead to an Efimov effect. If the zero-energy bound state of the two-heavy–one-light subsystem occurs for $\lambda=\lambda'$ such that $|\lambda'|<|\lambda_c|$ then the relevant limit is $\lambda\to\lambda'$ and we are in a situation where $\tilde{\epsilon}(R,\rho,\theta)$ is confined to a finite region of space. In the absence of a long-range component for the effective three-body potential in the $\lambda=\lambda'$ limit, the heavy-particle equation (9) has only a finite number of bound states and there is no four-body Efimov effect. As we have seen in the previous subsection the three-body zero-energy bound state of two heavy and one light particles is "localized" in space. When the third heavy particle stays infinitely separated from this zero-energy bound state it cannot feel its effect through the exchange of the light particle, and hence there is no long-range force between the heavies that may lead to an Efimov effect. On the contrary if there is a repulsive short-range heavy-heavy potential that prevents the existence of non-Efimov three-body bound states for $|\lambda| < |\lambda_c|$, then the relevant limit for the Efimov effect is $\lambda \to \lambda_c$. The large size of the heavy-light bound state is now responsible for a long-range force between the three heavy particles that could lead to a four-body Efimov effect. This does not happen because the Efimov bound states of the two-heavy–one-light subsystem that emerge as $\lambda \to \lambda_c$ produce two-body cuts in the four-body problem that prevent an accumulation of four-body bound states at $E=0$. Nevertheless if the long-range tail of the effective potential $\tilde{\epsilon}(R,\rho,\theta)$ in the $\lambda = \lambda_c$ limit is strong enough to support four-body states whose binding energy is lower than the lowest two-heavy–one-light Efimov state, there may exist a few Efimov bound states that disappear for $|\lambda| > |\lambda_c|$ when the three-body cut overrides them. The general pattern for the analytic structure of such four-body molecular problems in the $\lambda \simeq \lambda_c$ limit is sketched in Fig. 10. As $\lambda \to \lambda_c$ a few Efimov-type four-body bound states emerge at $E=0$. Their number remains finite because as soon as the first two-heavy–one-light Efimov state comes to exist the resulting two-body cut in the four-body sector prevents any new four-body bound state from appearing at $E=0$.

Within the framework of the BO method this structure can be easily understood. As shown in Ref. 3 the binding energy of the two-heavy–one-light Efimov bound states is obtained by solving the two-heavy-particle BO equation (11) and is related to the $-A^2/R^2$ tail of ϵ_g. In the present work the binding energy of the four-body Efimov bound states is obtained by solving the three-heavy-particle BO equation (9) and is related to the strength of the long-range $\sum_{i<j}^{3} - A'^2/r_{ij}$ behavior of $\tilde{\epsilon}$. It is well known[12] that the potential strength that binds two identical particles with binding energy B_2 also binds three with deeper binding $B_3 < B_2$. Since for all ρ and θ, A' is contained between $B/3 = 0.491\,124\ldots$ and $A = 0.567\,143\ldots$, the long-range component of $\tilde{\epsilon}$ is, on the average, only slightly weaker than the sum of three $-A^2/r_{ij}^2$ pair potentials. Owing to the complicated nature of the problem we find it difficult to predict, without going through lengthy numerical calculations, the exact binding energy of the lowest four-body Efimov states compared to the lowest two-heavy–one-light Efimov bound state. Nevertheless, taking note of the above considerations concerning the strength of the long-range component of $\tilde{\epsilon}$ compared to the strength of the long-range tail of ϵ_g, we find it possible that a few Efimov four-body bound states may exist below the lowest two-particle threshold. These states disappear for $|\lambda| > |\lambda_c|$ when the three-particle cut due to the light-heavy bound state overrides them. For increasing coupling strength $\lambda(|\lambda| > |\lambda_c|)$ both $\tilde{\epsilon}$ and ϵ_g become short range. Once the four-body and three-body Efimov states disappear there are no four-body bound states left until the light-heavy interaction becomes strong enough to overcome the short-range repulsion.

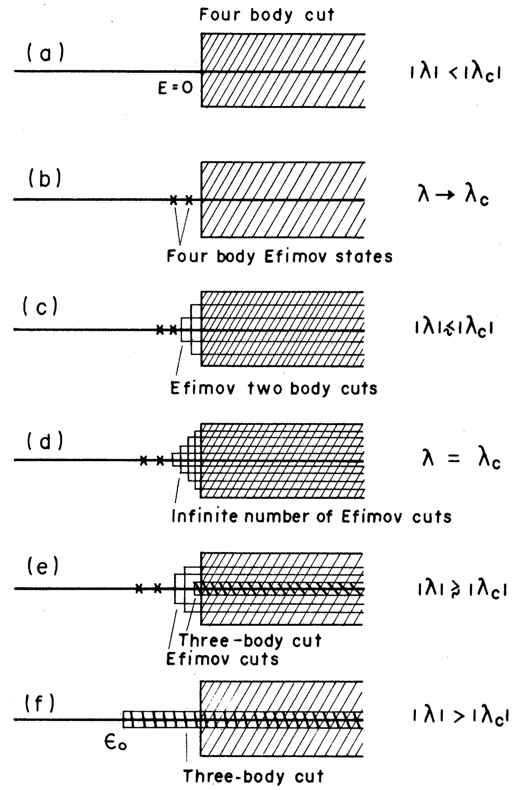

FIG. 10. Analytic structure of the four-body molecular system in the $\lambda \to \lambda_c$ limit.

Adding an extra heavy particle and looking for a five-body Efimov effect when the four-body molecular subsystem has a zero-energy bound state is not going to bring anything new. If the zero-energy four-body state results for $\lambda = \lambda''$ where $|\lambda''| < |\lambda'| < |\lambda_c|$ the effective four-body potential is confined to an even smaller region of space and there is no Efimov effect. On the contrary if we manage to avoid subsystem bound states until $\lambda \simeq \lambda_c$ we are then left with the possibility of having a few Efimov bound states but never an infinity.

IV. SUMMARY

In this work we consider the possibility of the occurrence of a zero-energy Efimov effect in a four-body BO model consisting of three heavy and one light particles. In the model the four-body wave function is written as the product of two terms which, under certain approximate condition, leads to the splitting of the four-body Schrödinger equation into the light-particle equation and a separate three-body Schrödinger equation for the movement of the heavy particles. The solution of the light-particle equation yields an effective three-body potential to be used in the equation for the three heavy particles. We find that the effective potential is either short ranged and hence the Schrödinger equation for the three heavy particles always has a finite number of bound states or it has a long-range component that may lead to the existence of a finite number of Efimov bound states. Therefore, we find that a four-body molecular system cannot show an infinite number of bound states that could be described as an Efimov effect. Our work confirms the conclusions of Amado and Greenwood though it does not exclude the possibility of finding a few Efimov states in the four-body problem that emerge for the specific value of the two-body coupling strength that leads to a zero-energy light-heavy bound state, and that disappear otherwise. The same conclusion remains true in general for the zero-energy Efimov effect in an N-body molecular system with a single light particle. If there is more than one light particle the system becomes too cumbersome to be discussed within the framework of such a simple method and more elaborate mathematics may be required. Nevertheless, because the Born-Oppenheimer approximation holds independently of the number of light particles, the existence of Efimov states boils down to the existence of a long-range force between the heavy particles in the Efimov limit. Since a zero-energy Efimov effect requires the absence of subsystem bound states we expect no new scenario to emerge that is any different from the two discussed above.

The possibility of finding Efimov states in the He-trimer has been reported recently.[13] They use a three-body model with an effective He-He potential that is not strong enough to bind the He-dimer. These states are presumably N-body Efimov states but further work is still needed on the finite-energy Efimov effect to find out their true nature.

Recently Kröger and Perne[14] have shown in a model consisting of four identical particles that a separable representation of the 1+3 subamplitudes in the four-body kernel reduces the four-body equations to effective three-body equations that can show an Efimov effect in the $\lambda=\lambda'$ limit where the three-body subsystem is having a zero-energy bound state. In their model they freeze a continuous degree of freedom in momentum space and because of this the trace of the kernel of their approximate four-body equation develops an infrared divergence in the $\lambda=\lambda'$ limit. Since in their work they do not discuss the validity of their approximation in the limit where the three-body subsystem has a zero-energy bound state, we find it difficult to believe, in the light of our present work and the related work of Amado and Greenwood,[6] how an infinite number of Efimov bound states can emerge in the $\lambda=\lambda'$ limit.

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APPENDIX

Considering that $\mathfrak{M}$ is large enough such that $\mu=\nu=\nu'\simeq 1$ we now study the long-range behavior of $\mathcal{E}(R,\rho,\theta)$ in the $\lambda=\lambda_c$ limit. Substituting (4) in (24) and making use of (6) with $\kappa_0=0$ we get

$$J(Z)=\frac{\beta^2}{(\beta+\delta)^2}\left[\frac{2\beta}{(\beta-\delta)^2}\frac{e^{-\delta Z}-e^{-\beta Z}}{Z} -\frac{\beta+\delta}{\beta-\delta}e^{-\beta Z}\right], \tag{A1}$$

$$J(0)=\frac{\beta^2}{(\beta+\delta)^2}, \tag{A2}$$

where $\delta^2=-\mu\mathcal{E}$. Since $\mathcal{E}$ goes to zero for large ρ and R we are interested in $J(Z)$ for large Z and $\delta\ll\beta$. Therefore,

$$[1-J(0)\pm J(R)]_{R\gg 1,\,\delta\ll\beta}\simeq\frac{2}{\beta}\frac{1}{R}(\delta R\pm e^{-\delta R}), \tag{A3}$$

$$[1-J(0)]_{\delta\ll\beta}\simeq\frac{2}{\beta}\delta. \tag{A4}$$

Defining

$$X=R[\tfrac{1}{4}+(\rho/R)^2+(\rho/R)\cos\theta]^{1/2}=Rx, \tag{A5}$$

$$Y=R[\tfrac{1}{4}+(\rho/R)^2+(\rho/R)\cos\theta]^{1/2}=Ry, \tag{A6}$$

together with $\delta R=B$ we obtain from (33)

$$(B-e^{-B})(B+e^{-B})B-\tfrac{1}{2}(B+e^{-B})\left(\frac{e^{-Bx}}{x}+\frac{e^{-By}}{y}\right)^2 -\tfrac{1}{2}(B-e^{-B})\left(\frac{e^{-Bx}}{x}-\frac{e^{-By}}{y}\right)^2=0 \quad \text{(A7)}$$

which, for fixed ρ and θ is a transcendental equation for B. If $\rho=\infty$ then $x=y=\infty$ and (A7) reduces to

$$(B-e^{-B})(B+e^{-B})B=0\,, \quad \text{(A8)}$$

which has a nontrivial solution for $B=e^{-B}=A=0.567\,143\ldots$. For $\rho=0$ then $x=y=\frac{1}{2}$ and (A7) reduces to

$$(B-e^{-B})B=8e^{-B}\,, \quad \text{(A9)}$$

which has a solution for $B=1.473\,374\ldots$. Taking note that for $\rho=0$ the separation between all the three heavy particles is $r_{13}=R/2$, $r_{23}=R/2$, and $r_{12}=R$ (see Fig. 1) we can write

$$\bar{\epsilon}(R,\rho=0,\theta)=-\frac{B^2}{R^2}=-A'^2\left(\frac{4}{R^2}+\frac{4}{R^2}+\frac{1}{R^2}\right). \quad \text{(A10)}$$

such that $A'=B/3=0.491\,124\,8\ldots$. For intermediate values of ρ and $0<\theta<\pi/2$ our numerical studies indicate that we can always write

$$\bar{\epsilon}(R,\rho,\theta)=-A'^2\sum_{i<j}^{3} r_{ij}^{-2} \quad \text{(A11)}$$

with $B/3<A'<A$, and r_{ij} is the distance between all three heavy particles.

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