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Coherent meson-pair states

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It is shown that the use of coherent meson-*pair* states simplifies and improves calculations in static-source theories with complicated algebras.

I. INTRODUCTION

Consider the Hamiltonian for a meson field interacting with a static source:

$$H=\int \omega(k)a^\dagger(k)\cdot a(k)dk - \rho\cdot\int [v(k)a^\dagger(k)+v^*(k)a(k)]dk\ . \qquad (1)$$

Here $a(k)$ is the annihilation operator for a meson of momentum k; the meson can be isoscalar or isovector and can interact with the source as a scalar or as a (p-wave) vector. The operator ρ is the corresponding source current operator. The two cases that will be discussed are (a) isovector scalar, abbreviated VS, where $a(k)$ is an isovector and ρ is the isospin operator τ, and (b) isovector vector or $SU_2\times SU_2$, abbreviated VV, where $a(k)$ is an isovector and a vector with components $a_{\lambda i}(k)$ and ρ is the operator $\tau_\lambda\sigma_i$ with $\rho\cdot a=\sum_{\lambda,i=1}^{3}\tau_\lambda\sigma_i a_{\lambda i}$. The VV case is the usual static model of the pion-nucleon interaction.[1] This model has recently received new attention owing to the fact that theories of the interaction of pions with composite-quark nucleons[2] lead to Hamiltonians of the form of Eq. (1). Besides the VS and VV cases, the trivial case of isoscalar scalar, abbreviated SS, where $\rho=1$, is well known, and the case of SU_3 has recently received attention[3] as well.

The static-source Hamiltonian of Eq. (1) has been widely discussed in the literature. It has been shown that for many purposes the spectrum of the corresponding single-mode Hamiltonian H^A is of interest;[4] that is, the mesons are all in the state with normalized wave function $\phi(k)$,

$$a(k)=A\phi(k)\ ,\qquad \int|\phi(k)|^2dk=1\ , \qquad (2)$$

and H is equivalent to

$$H^A=W[A^\dagger\cdot A-\gamma\rho\cdot(A+A^\dagger)] = Wh^A\ , \qquad (3)$$

where

$$\gamma=\left|\int v^*(k)\phi(k)dk\right| \Big/ \int\omega(k)|\phi(k)|^2dk\ ,\qquad W=\int\omega(k)|\phi(k)|^2dk\ . \qquad (4)$$

The eigenvalues of H^A have been connected with nucleon isobars,[5] and the ground state of H^A and the lowest isobar states have been used in calculations of meson-nucleon scattering.[6,7] The present paper gives a coherent-state method for obtaining eigenvalues and eigenvectors of h^A and H^A; the method is directed particularly at the VV case, where the algebra of the ρ operators is complicated enough that other methods,[8] which give good results in the VS case, are too difficult to apply. The improved accuracy that can be gained by using the present method will be particularly necessary in calculating the potential between two static sources in the VV case; the methods used to compute this potential in the VS case[9] are not adequate for the VV case. When the algebra is still more complicated, as in the case[3] of SU_3, the use of the present method, which is algebraically relatively simple while giving good numerical accuracy, will be even more necessary, both for the simple source and for two interacting sources.

A rather comprehensive study of H^A was made by Harlow and Jacobsohn,[5] who include states with $T\leq\frac{3}{2}, J\leq\frac{3}{2}$. It is not possible to extract the spectrum of h^A_{VV} directly from Ref. 5, which gives spectra of H^A for various values of γ, but does not give the corresponding values of W. The lowest eigenvalue of h^A has also been studied by Friedman, Lee and Christian[7] and by Halpern *et al.*;[10] the methods used in these studies are not suitable for extension to other than the ground state. Finally, Schwartz[11] used as many as 1189 orthogonal states to compute the lowest eigenvalue of H^A; his lowest eigenvalue is a useful standard of comparison.

In the isovector scalar case, it has been shown that coherent states[8] can be used to give a simple but reasonably accurate approximation to the ground state of h^A. The coherent state that was used satisfies

$$\rho \cdot A|y\rangle = \tau \cdot A|y\rangle = y|y\rangle , \quad (5)$$

where y is varied to produce the best coherent state. Equation (5) is the VS analog of the SS coherent-state condition

$$A|y\rangle = y|y\rangle \quad (6)$$

that is known to give the exact ground state of h^A when y is chosen appropriately. The relative simplicity of the algebra in the VS case makes it possible to obtain a simple closed form for the state $|y\rangle$. The VV analog of Eq. (5) is

$$\rho \cdot A|y\rangle = \sum_{i\lambda} \tau_\lambda \sigma_i A_{\lambda i}|y\rangle = y|y\rangle . \quad (7)$$

With the aid of MACSYMA,[12] I have been able to show that coherent states satisfying Eq. (7) reproduce the weak- and strong-coupling results, but the algebra is enormously complicated; the form of the coherent states of Eq. (7) is too complicated for them to be of any real use for intermediate coupling.

The work of Ref. 7 showed that states of the form $f(A^\dagger \cdot A^\dagger)|\Omega\rangle$, where $|\Omega\rangle$ is the bare source state, give a useful approximation to the ground state for weak and intermediate couplings. This leads to the idea of a coherent meson-pair (CMP) state that satisfies

$$A \cdot A|y\rangle = y|y\rangle . \quad (8)$$

The advantage of the CMP condition is that it is relatively simple algebraically even in the VV and SU_3 cases; matrix elements between CMP states are easily evaluated. Moreover, the significant improvement in energy values that is obtained by using coherent states in general also occurs when coherent meson-pair states are used.

To see how the CMP states work, the two simplest $(\frac{1}{2},\frac{1}{2})$ states in the VV case are $|\Omega\rangle$ and $\rho \cdot A^\dagger|\Omega\rangle$, where $|\Omega\rangle$ is again the bare source state. If h^A is diagonalized in the subspace spanned by these two states, its ground-state eigenvalue is -1.99 for the value 0.81476 of γ corresponding to the parameters used in Ref. 11. Each of the above states can be turned into a CMP state by multiplying by an appropriate function of $A^\dagger \cdot A^\dagger$; the two coherent states are then $g_9(\xi A^\dagger \cdot A^\dagger) \times |\Omega\rangle$ and $g_{11}(\eta A^\dagger \cdot A^\dagger)\rho \cdot A^\dagger|\Omega\rangle$, where ξ and η are parameters and the g functions will be constructed below. When the two CMP states are used as a basis for diagonalizing h^A, the ground-state eigenvalue is -2.21 for $\gamma = 0.81476$. Since W in Ref. 11 has the value $4.1m$, the improvement in the eigenvalue of H^A is of the order of 100 MeV. Similarly for the difference between the lowest $(\frac{3}{2},\frac{3}{2})$ state and the ground state when states with up to two operators $A^\dagger$ are used: going from the usual two-meson states to the two-meson states with coherent pairs changes the spacing by $0.1W$ which is of the order of 50 MeV. It is evident, therefore, that CMP states can be useful in obtaining accurate eigenvalues and eigenstates of H^A, and hence of H of Eq. (1). For example, in computing the potential between two static sources of pions, the extra accuracy obtained by using CMP states will be essential, since the potential involves the difference of energies computed for different separations of the sources.[9]

The organization of the rest of the paper is as follows: Sec. II gives the general construction for CMP states and evaluates their matrix elements, Sec. III details the VV computations, Sec. IV discusses meson-pair excitations, and Sec. V contains some comments and conclusions.

II. CONSTRUCTION AND MATRIX ELEMENTS OF COHERENT MESON-PAIR STATES

Suppose an n-meson state $|n\rangle$ is given. In addition, let $|n\rangle$ satisfy the condition

$$A \cdot A|n\rangle = 0 , \quad (9)$$

then $|n\rangle$ will be called a "basic" n-meson state. In general, if $|n\rangle$ does not satisfy (8), it can be converted to a basic n-meson state by adding a linear combination of functions of $A^\dagger \cdot A^\dagger$ times basic states with fewer than n mesons. Write the CMP state $|n, y\rangle$ as

$$|n,y\rangle = \sum_{m=0}^{\infty} c_m (yA^\dagger \cdot A^\dagger)^m |n\rangle . \quad (10)$$

The commutation relation

$$[A \cdot A, A^\dagger \cdot A^\dagger] = 4A \cdot A + 2\nu , \quad \nu = 3 \ \text{VS} , \quad \nu = 9 \ \text{VV} , \quad (11)$$

can be used together with Eq. (8), to show that the CMP condition, Eq. (7), is satisfied if

$$c_m = \frac{(2n+\nu-2)!!}{2^m m!(2n+\nu+2m-2)!!} . \quad (12)$$

Let

$$\sum c_m x^m = g_{\nu+2n}(x) , \quad (13)$$

then $g_\mu(0) = 1$ and

$$|n,y\rangle = g_{\nu+2n}(yA^\dagger \cdot A^\dagger)|n\rangle , \quad A \cdot A|n,y\rangle = y|n,y\rangle . \quad (14)$$

The $g_\mu(x)$ are related to spherical Bessel functions by multiplicative factors. The states $|\Omega\rangle$ and $\rho \cdot A^\dagger|\Omega\rangle$ are evidently basic states with $n = 0$ and $n = 1$, respectively; hence the form of the CMP states given in the Introduction.

Now by steps,

$$\begin{aligned}\langle mx|ny\rangle &= \langle m|g_{\nu+2m}(xA\cdot A)|ny\rangle \\ &= g_{\nu+2m}(xy)\langle m|ny\rangle \\ &= g_{\nu+2m}(xy)\langle m|g_{\nu+2n}(yA^\dagger\cdot A^\dagger)|n\rangle \\ &= g_{\nu+2m}(xy)\langle m|n\rangle\ . \end{aligned} \tag{15}$$

Since $A^\dagger\cdot A$ counts $A^\dagger$ operators, and the number of $A^\dagger$'s in $g(yA^\dagger\cdot A^\dagger)$ is twice the number of y's,

$$A^\dagger\cdot A|ny\rangle = \left(2y\frac{d}{dy}+n\right)|ny\rangle\ , \tag{16}$$

and

$$\langle mx|A^\dagger\cdot A|ny\rangle = \left(2xy\frac{d}{d(xy)}+n\right)g_{\nu+2m}(xy)\langle m|n\rangle\ . \tag{17}$$

The derivative relation

$$\frac{d}{dx}g_\mu(x) = \frac{g_{\mu+2}(x)}{2\mu} \tag{18}$$

follows easily from Eqs. (11) and (12), so that

$$\langle mx|A^\dagger\cdot A|ny\rangle = \left(\frac{xy}{\nu+2m}g_{\nu+2m+2}(xy) + ng_{\nu+2m}(xy)\right)\langle m|n\rangle\ . \tag{19}$$

Finally, with

$$[A\cdot A, \rho\cdot A^\dagger] = 2\rho\cdot A\ , \tag{20}$$

it follows that

$$\begin{aligned}\langle mx|\rho\cdot A|ny\rangle &= g_{\nu+2m}(xy)\langle m|\rho\cdot A g_{\nu+2n}(yA^\dagger\cdot A^\dagger)|n\rangle \\ &= g_{\nu+2m}(xy)\ \langle m|\rho\cdot A|n\rangle + \frac{y}{2(\nu+2n)}\langle m|\rho\cdot AA^\dagger\cdot A^\dagger|n\rangle \\ &= g_{\nu+2m}(xy)\ \langle m|\rho\cdot A|n\rangle + \frac{y}{\nu+2n}\langle m|\rho\cdot A^\dagger|n\rangle\ . \end{aligned} \tag{21}$$

Thus, all required matrix elements between CMP states are reduced to known functions times matrix elements (assumed known) between basic states.

III. COMPUTATIONS IN $SU_2 \times SU_2$

In $SU_2\times SU_2$, the no-meson state $|\Omega\rangle$ and the one-meson states $\{A^\dagger,\Omega\rangle\}^{TJ}$ are clearly basic states; the curly brackets will be used to indicate vector coupling of the enclosed operators

$$\{\phi^t,\chi^\tau\}^T_M = \sum_{\mu m}(tm\tau\mu|TM)\Phi^t_m\chi^\tau_\mu\ . \tag{22}$$

The index M is omitted. In $SU_2\times SU_2$, both isospin and angular momentum are vector coupled. The two-meson states $\{\{A^\dagger,A^\dagger\}^{ts},|\Omega\rangle\}^{TJ}$ are basic except for the one with $t=s=0$, which is proportional to $A^\dagger\cdot A^\dagger|\Omega\rangle$. Since the CMP function $g(yA^\dagger\cdot A^\dagger)$ describes the degree of freedom associated with the operator $A^\dagger\cdot A^\dagger$, the state $A^\dagger\cdot A^\dagger|\Omega\rangle$ is not needed when CMP states are used. It is, of course, needed when CMP states are not used. The CMP states that involve excitations of the $A^\dagger\cdot A^\dagger$ degree of freedom are discussed in Sec. IV. Computations that include such states show that they have no effect on low-lying states. Thus, such states can be ignored in CMP calculations of low-lying states. Hence, although the CMP states have the extra variational CMP parameter, there are fewer of them; for n large the ratio of numbers of states approaches 0.

Generally, a basic n-meson state is of the form

$$\{\Phi^{tl}_{n\alpha}, |\Omega\rangle\}^{TJ}, \tag{23}$$

enumeration is required only of the basic n-meson wave functions $\Phi^{tl}_{n\alpha}$. For $n=0$, there is only

$$\Phi^{00}_0 = 1\ , \tag{24}$$

for $n=1$ there is

$$\Phi^{11}_1 = A^\dagger\ , \tag{25}$$

and for $n=2$, there are

$$\Phi^{tl} = \{A^\dagger,A^\dagger\}^{tl},\quad tl=22,20,02,11\ . \tag{26}$$

These functions are normalized to

$$(\Phi^{tl}_n, \Phi^{t'l'}_{n'}) = n\delta_{nn'}\delta_{tt'}\delta_{ll'}\ ,\quad n=1,2\ . \tag{27}$$

In general,

$$(\Phi^{tl\alpha}_m, \Phi^{t'l'\beta}_n) = \delta_{tt'}\delta_{ll'}\delta_{mn}N^{tln}_{\alpha\beta}\ . \tag{28}$$

Since

$$\rho\cdot A^\dagger|\Omega\rangle = 3\{A^\dagger|\Omega\rangle\}^{1/2\,1/2}, \tag{29}$$

it follows that

$$\begin{aligned}(\{\Phi^{tl\alpha}_m,|\Omega\rangle\}^{TJ}, \rho\cdot A^\dagger\{\Phi^{t'l'\beta}_n,\Omega\rangle\}^{TJ}) &= 3(-)^{t'+l'-t-l}\mathrm{U}(t',1,T,\tfrac12;t,\tfrac12) \\ &\quad\times \mathrm{U}(l',1,J,\tfrac12;l,\tfrac12) \\ &\quad\times(\Phi^{tl\alpha}_m,\{A^\dagger,\Phi^{t'l'\beta}_n\}^{tl})\ . \end{aligned} \tag{30}$$

Thus, all matrix elements can be computed if the norms $(\Phi^{tI\alpha}, \Phi^{t'I'\beta})$ and the parentages $(\Phi^{tI\alpha}, \{A^\dagger, \Phi^{t'I'\beta}\}^{tI})$ are known. These are given in the Appendix for $m \leqq 4$. There are eight basic three-meson states and 18 basic four-meson states. In Ref. 5, states with up to three operators $A^\dagger$ were included.

IV. RESULTS FOR $SU_2 \times SU_2$

With the matrix elements of Sec. II and the Appendix, the matrices of the unit operator and of h^A_{VV} can be constructed and simultaneously diagonalized to give the corresponding eigenvalues and eigenvectors of h^A_{VV}. These can be used to discuss the following questions.

(1). How large is the coherence parameter (CP) [y of Eq. (8)]? Expansion in powers of γ shows that for weak coupling the coherence parameter is of order γ^2. It is possible to associate a different coherence parameter with each basic state; however, the numerical calculations show that it is sufficient to use the same coherence parameters for all the basic n-meson states, so that there is one coherence parameter for each possible number of mesons. Thus for the case $TJ = \frac{1}{2}, \frac{1}{2}$, the maximum number of coherence parameters used was 5, since n took the values from 0 to 4. When γ is 0.81476, the best values of these parameters were found to be 2.56, 2.32, 2.14, 2.02, and 1.77, respectively.

(2). How much difference do the coherent pairs make? Table I gives some eigenvalues for the case $\gamma = 0.81476$. It is evident that the addition of coherent pairs substantially lowers the computed ground-state eigenvalues. This is somewhat deceptive, in that in the case of no coherent pairs the nonbasic states have not been included; the same number of states has been used for the "no CP" and "with CP" computations. On the other hand, it is considerably more difficult to add the extra states than it is to add coherent pairs to a state for which matrix elements have already been calculated; in this sense, the use of CMP states is the simplest way to include nonbasic states. From Table I, it can be seen that the CMP do not affect the spacing between ground states as much as they do the spacing between states of a given T, J.

TABLE I. Eigenvalues for $\gamma = 0.814\,76$.

level	Maximum number of mesons in basic states 1	2	3	4
$(\frac{1}{2},\frac{1}{2})$g.s. no CP	−1.995	−2.532	−2.657	−2.692
$(\frac{1}{2},\frac{1}{2})$ g.s. with CP	−2.207	−2.889	−3.062	−3.115
$(\frac{3}{2},\frac{3}{2})$ g.s. no CP		−1.480	−2.019	−2.125
g.s. with CP		−1.731	−2.407	−2.551
$(\frac{1}{2},\frac{1}{2})$ first no CP	2.995	1.054	0.135	−0.445
with CP	3.619	1.469	0.285	−0.524

(3). What effects do other modes have? The eigenvalues shown in Table I for the $(\frac{1}{2},\frac{1}{2})$ ground state with CMP are for subspaces with 2, 3, 5, and 7 basic states, respectively. In terms of the states used in Ref. 11, the eigenvalue −2.207 lies between the values obtained in Ref. 11 with 16 and 25 states; the corresponding pairs of subspace dimensions for the other three eigenvalues given in Table I are (94,115), (319,364), and (634,721), respectively. The states of Ref. 11 include modes other than the single mode

$$\phi(k) = [v(k)/w(k)]_{\text{normalized}}$$

that has been included in the calculations that give the results shown in Table I. It is clear that a single mode is adequate to give the ground state energy to very good accuracy.

(4) How is the convergence with number of mesons? Table I shows that the convergence with meson number is fair for ground states and poor for excited states. Of course, for smaller values of γ the convergence is much better; for $\gamma = 0.1$ the states listed in Table I all have the same eigenvalues for meson numbers 3 and 4. However, the value of γ in Refs. 11, 5, and 7 is in the range 0.5 to 1.2; for γ of this order of magnitude, the four-meson calculation clearly gives accurate results only for the lowest state with given TJ. It would be useful to have the basic-state matrix elements for five-meson and six-meson states.

V. MESON-PAIR EXCITATIONS

The no-meson CMP state in $SU_2 \times SU_2$ is

$$|0, y\rangle = g_9(yA^\dagger \cdot A^\dagger)|\Omega\rangle .$$

A state $|0, y'\rangle$ is orthogonal to $|0, y\rangle$ if

$$\langle 0, y' | 0, y\rangle = g_9(yy') = 0 .$$

The first zero of g_9 occurs when its argument is −48.8; other zeros occur at larger negative values. The state $|0, y'\rangle$ differs from $|0, y\rangle$ only in its meson-pair wave function; it corresponds to meson-pair excitation. For y of order 2, y'^2 is very large compared to y^2, and it is easily seen from Eq. (17) that the kinetic energy expectation value in the state $|0, y\rangle$ is large. This is the reason that the meson-pair excited state has almost no effect on computed ground-state energies; numerical com-

TABLE II. States and normalizations.

α	$\Phi\alpha$	$(\Phi\alpha, \Phi\alpha)$	α	$\Phi\alpha$	$(\Phi\alpha, \Phi\alpha)$
1	1	1	15	$(22,22)^{44}$	24
			16	$(22,20)^{42}$	$\frac{28}{3}$
2	A^+	1	17	$(22,02)^{24}$	$\frac{28}{3}$
			18	$(20,20)^{40}$	$\frac{40}{3}$
3	(22)	2	19	$(02,02)^{04}$	$\frac{40}{3}$
4	(20)	2	20	$(22,11)^{33}$	8
5	(02)	2	21	$(22,11)^{32}$	8
6	(11)	2	22	$(22,11)^{23}$	8
			23	$(20,11)^{31}$	$\frac{20}{3}$
7	$(22)^{33}$	6	24	$(02,11)^{13}$	$\frac{20}{3}$
8	$(20)^{31}$	$\frac{10}{3}$	25	$(20,02)^{22}-\frac{4}{13}(22,00)^{22}$	$\frac{68}{13}$
9	$(02)^{13}$	$\frac{10}{3}$	26	$(11,11)^{22}-\frac{3}{13}(22,00)^{22}$	$\frac{152}{13}$
10	$(11)^{22}$	3	27	$(20,11)^{21}$	$\frac{20}{3}$
11	$(11)^{21}$	3	28	$(02,11)^{12}$	$\frac{20}{3}$
12	$(11)^{12}$	3	29	$(11,11)^{20}+\frac{6}{13}(20,00)^{20}$	$\frac{140}{13}$
13	$(11)^{11}-\frac{6}{11}(00)^{11}$	$\frac{25}{11}$	30	$(11,11)^{02}+\frac{6}{13}(02,00)^{02}$	$\frac{140}{13}$
14	$(11)^{00}$	6	31	$(11,11)^{11}-\frac{3}{13}(11,00)^{11}$	$\frac{100}{13}$
			32	$(11,11)^{00}-\frac{6}{11}(00,00)^{00}$	$\frac{100}{11}$

putations that include the pair excited state show that the pair excited state is at high excitation energy and does not affect the lower states.

VI. SUMMARY

Coherent meson-pair states that satisfy Eq. (8) provide a way of taking into account meson-field coherence effects without the algebraic complexities that arise when coherent states that satisfy Eq. (7) are used. Use of CMP states enables the correlation computed in Ref. 7 to be computed in a relatively simpler manner; it gives numerical results for 7 states that are better than those of Ref. 11 for 634 states.

When nonstatic sources are used, the methods of Refs. 5, 7, and 11 are not applicable. However, the simplicity of CMP states makes possible the construction of translated CMP states by the methods discussed in Ref. 13.

The results given in Sec. III show that for values of γ of order 1 accurate calculations of other than the lowest state of a given TJ do not yet exist. It seems likely that CMP states will be an essential tool for constructing such computations. Similarly, the accuracy required for computing the static potential between two sources will necessitate the use of CMP states.

TABLE III. Nonzero $P_{\alpha,\beta}$ for $\beta = 3-6$ with $a[b] \equiv a(b)^{1/2}$.

$\beta \backslash \alpha$	8	9	10	11	12	13
3	$\frac{4}{3}[5]$	$\frac{4}{3}[5]$	3	$-[5]$	$-[5]$	$\frac{5}{11}$
4	$\frac{10}{3}$			2		$-\frac{10}{11}[5]$
5		$\frac{10}{3}$			2	$-\frac{10}{11}[5]$
6			3	3	3	$\frac{25}{11}$

APPENDIX

The purpose of this appendix is to give the matrix elements necessary to construct the matrix of $h^{\mathbf{A}}_{\mathbf{VV}}$. From Sec. III it follows that only the matrices $N^{tln}_{\alpha\beta}$ and the parentages $(\Phi^{tl\alpha}_n, \{A^\dagger, \Phi^{t'l'\beta}_{n-1}\}^{tl})$ are needed. The 32 basic states that satisfy Eq. (8) with $n \leqq 4$ can be chosen as shown in Table II, where the notation is

$$(tl) \equiv \{A^\dagger, A^\dagger\}^{tl} ,$$

$$(tl)^{TL} \equiv \{A^\dagger, \{A^\dagger, A^\dagger\}^{tl}\}^{TL} ,$$

$$(sm, tl)^{TL} \equiv \{\{A^\dagger, A^\dagger\}^{sm}, \{A^\dagger, A^\dagger\}^{tl}\}^{TL} .$$

TABLE IV. Nonzero $P_{\alpha,\beta}$ for $\beta=7$–13 with $a[b]\equiv a(b)^{1/2}$.

$\alpha \backslash \beta$	7	8	9	10	11	12	13
16	$4[\frac{21}{5}]$	$\frac{28}{3}$					
17	$4[\frac{21}{5}]$		$\frac{28}{3}$				
20	8			8			
21	$4[\frac{14}{5}]$	$4[\frac{2}{3}]$		4	$4[3]$		
22	$4[\frac{14}{5}]$		$4[\frac{2}{3}]$	4		$4[3]$	
23		$\frac{20}{3}[\frac{2}{3}]$		$2[5]$	$-10[\frac{1}{3}]$		
24			$\frac{20}{3}[\frac{2}{3}]$	$2[5]$		$-10[\frac{1}{3}]$	
25	$-\frac{112}{65}$	$\frac{76}{13}[\frac{7}{15}]$	$\frac{76}{13}[\frac{7}{15}]$	$-\frac{8}{13}$	$-\frac{20}{13}[3]$	$-\frac{20}{13}[3]$	$-\frac{48}{13}$
26	$-\frac{84}{65}$	$-\frac{8}{13}[\frac{7}{15}]$	$-\frac{8}{13}[\frac{7}{15}]$	$\frac{111}{13}$	$-\frac{41}{13}[3]$	$-\frac{41}{13}[3]$	$\frac{29}{13}$
27		$\frac{4}{3}[\frac{35}{3}]$		$[5]$	$-5[\frac{1}{3}]$	$[15]$	-5
28			$\frac{4}{3}[\frac{35}{3}]$	$[5]$	$[15]$	$-5[\frac{1}{3}]$	-5
29		$\frac{8}{13}[\frac{35}{3}]$			$\frac{70}{13}[3]$		$-\frac{70}{13}$
30			$\frac{8}{13}[\frac{35}{3}]$			$\frac{70}{13}[3]$	$-\frac{70}{13}$
31				$\frac{55}{13}$	$\frac{15}{13}[15]$	$\frac{15}{13}[15]$	$\frac{25}{13}$

All $(\Phi_\alpha, \Phi_\beta)$ for $\alpha \neq \beta$ are zero except for

$$(\Phi_{25}, \Phi_{26}) = (\Phi_{26}, \Phi_{25}) = \tfrac{12}{13}\,.$$

The nonzero parentages

$$P_{\alpha,\beta} = (\Phi_\alpha, \{A^\dagger, \Phi_\beta\})$$

are given in Tables III and IV, together with

$$P_{2,1} = 1\,,$$

$$P_{3,2} = P_{4,2} = P_{5,2} = P_{6,2} = 2\,,$$

$$P_{7,3} = P_{14,6} = 6\,,$$

$$P_{18,8} = P_{19,9} = \tfrac{40}{3}\,,$$

$$P_{15,7} = 24\,,$$

$$P_{31,14} = \tfrac{100}{13}\,,$$

$$P_{32,13} = \tfrac{100}{11}\,.$$

The notation in Tables III and IV is that

$$a[b] \equiv a(b)^{1/2}\,.$$

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