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Singular-perturbation–strong-coupling field theory and the moments problem

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Motivated by recent work of Bender, Cooper, Guralnik, Mjolsness, Rose, and Sharp, a new technique is presented for solving field equations in terms of singular-perturbation–strong-coupling expansions. Two traditional mathematical tools are combined into one effective procedure. Firstly, high-temperature lattice expansions are obtained for the corresponding power moments of the field solution. The approximate continuum-limit power moments are subsequently obtained through the application of Padé techniques. Secondly, in order to reconstruct the corresponding approximate global field solution, one must use function-moments reconstruction techniques. The latter involves reconsidering the traditional "moments problem" of interest to pure and applied mathematicians. The above marriage between lattice methods and moments reconstruction procedures for functions yields good results for the ϕ^4 field-theory kink, and the sine-Gordon kink solutions. It is argued that the power moments are the most efficient dynamical variables for the generation of strong-coupling expansions. Indeed, a momentum-space formulation is being advocated in which the long-range behavior of the space-dependent fields are determined by the small-momentum, infrared, domain.

INTRODUCTION

In a recent publication,[1] Bender, Cooper, Guralnik, and Sharp successfully defined a strong-coupling expansion for ϕ^4 quantum field theory by generating a high-temperature lattice expansion which, when combined with Padé resummation techniques, allowed them to recover the continuum limit of the theory. In a succeeding publication,[2] they, together with H. Rose, applied this formalism to singular perturbation theory for classical fields. In the latter case, the determination of global approximate solutions is possible in principle; however, in practice it is very involved and complicated to implement. Indeed, they only evaluate such things as the derivative of the solution at the origin.

In this paper an alternate approach is taken which also defines a strong-coupling expansion but has the advantage of yielding good global approximate solutions in a very efficient manner. Indeed, the method presented here is complementary to that developed by the aforementioned individuals. As will be elucidated below, the full strong-coupling expansion of a theory is built around singular perturbation theory. Singular perturbation theory can be regarded as an expansion around the origin in momentum space. Thus, in the alternate scheme to be presented, one expands around the large-scale behavior of the theory. In this manner, the greater the order of the calculation the better are our small-scale (i.e., derivative) results. Thus, our method gives good global results, while theirs gives good local results. Although the examples to be presented are of a classical nature, the principles should still hold for quantum field theory.

I. GENERAL PRINCIPLES

Given any field equation(s), be they classical or of the quantum-field-theoretical Schwinger-Dyson type, it is necessary to generate strong-coupling expansions whenever the coupling strengths are large. Usually the generation of such expansions involves the implicit use of singular perturbation theory. To illustrate this point, we will work with a simple field-theory model, the one-space-dimension ϕ^4 theory. Its equation of motion is

$$\partial_x{}^2\phi(x)+m^2\phi(x)=g\phi^3(x)\,. \tag{1.1}$$

For large g we are interested in derivative-term expansions. Dividing by g, and regarding $\lambda^2=1/g$ as an independent variable, Eq. (1.1) becomes

$$\lambda^2\partial_x{}^2\phi_\lambda(x)+\frac{m^2}{g}\,\phi_\lambda(x)=\phi_\lambda{}^3(x)\,. \tag{1.2}$$

The *full large-g*, strong-coupling expansion of (1.1) entails both a λ expansion (i.e., singular perturbation theory) as well as an m^2/g expansion.

It is readily apparent from scaling that the solution to (1.2) satisfies

$$\begin{aligned}&\phi_\lambda(x)\equiv\phi(x;\lambda)\,,\\&\phi(x;\lambda)=\phi(x/\lambda;1)\,.\end{aligned} \tag{1.3}$$

Thus, any formal expansion of (1.3) in powers of λ will be ill-defined or highly singular at best. Alternatively, if we "test" $\phi_\lambda(x)$ on any suitable test function $T_\alpha(x)$, where α is some parameter indexing the function, then things become less singular. Define

$$\begin{aligned}F_\alpha(\lambda)&=\int dx\;T_\alpha(x)\,\phi_\lambda(x)\\&=\int dx\;T_\alpha(x)\,\phi_1(x/\lambda)\,.\end{aligned}$$

Upon performing a change of variables, a finite-λ expansion (each order finite) may be defined provided $\phi_\lambda(x)$ is well-behaved at infinity:

$$F_\alpha(\lambda)=\lambda\int dx\, T_\alpha(\lambda x)\phi_1(x)\,. \tag{1.4}$$

From

$$T_\alpha(x)=\sum_{p=0}^{\infty}\mathcal{T}_\alpha(p)x^p/p!\ ,$$

where the $\mathcal{T}_\alpha$ are known, it follows that (1.4) has the expansion

$$F_\alpha(\lambda)=\lambda\sum\lambda^p\mathcal{T}_\alpha^{(p)}\frac{\mu^{(p)}}{p!}\,, \tag{1.5}$$

$$\mu^{(p)}\equiv\int dx\, x^p\phi_1(x)\,. \tag{1.6}$$

We may solve our field equation either in the $\phi(x)$ representation, or in the F_α representation. Clearly (1.5) is singular perturbation theory (SPT) in the F_α representation. SPT for the ϕ-space representation will be defined as SPT in F_α space combined with that transformation which relates both representations.

A *full strong-coupling expansion* entails the combination of singular perturbation theory together with some g^{-1} type expansion for the power moments.

The F_α dynamical variables are linearly related to the moments. The latter may, therefore, be regarded as dynamical variables too. From the preceding, it is evident that in the F_α representation a full strong-coupling expansion is far less singular than an analogous expansion in ϕ space. This is why the moments (or equivalently, the F_α's) form a more suitable representation in which to discuss strong-coupling theory.

As mentioned above, in order to define an SPT expansion in ϕ space, it is necessary to know how to transform configurations in F_α space into their counterparts in ϕ space. In terms of moments, we are asking how does one reconstruct a function from its moments. Is the solution unique? What is the best function that satisfies a finite set of given moment values? All these considerations are part of the general mathematics problem known as the "moments problem."[3] We will describe below the manner in which a function may be reconstructed from its moments. Contained in this prescription is also the answer to the last question raised above. A fuller discussion may be found in Refs. 4 and 5. Finally, the moments are the Taylor expansion coefficients for the Fourier space transform of the field. It is evident that the proper setting in which to discuss the strong-coupling character of a theory, intimately related to its long-range behavior, is in terms of an expansion around the origin in momentum space.

Consider a given ϕ-field configuration. Its Fourier space representation is

$$\tilde\phi(k)=\frac{1}{\sqrt{2\pi}}\int dx\,\exp(-ik\cdot x)\phi(x)$$
$$=\frac{1}{\sqrt{2\pi}}\sum_{p=0}^{\infty}(-i)^p\frac{\mu^{(p)}}{p!}k^p\,. \tag{1.7}$$

We have implicitly assumed that $\phi(k)$ dies off fast enough to cause no problems, otherwise subtractions would be necessary. If we know the first $P+1$ moments, how well may we approximate $\phi(x)$? Clearly, the polynomial

$$\frac{1}{\sqrt{2\pi}}\sum_{p=0}^{P}(-i)^p\frac{\mu^{(p)}}{p!}k^p$$

has no Fourier transform; however, we may use Padé techniques on this finite series and Fourier transform the resultant expression back into ϕ space. Specifically, we must solve (1.8) for $\{\alpha_\nu,\beta_\nu\}$. The series on the right is the Padé representation for that on the left. The manner of solution for the α's and β's is discussed in Ref. 6:

$$\frac{1}{\sqrt{2\pi}}\sum_{p=0}^{P}(-i)^p\frac{\mu^{(p)}}{p!}k^p=\frac{1}{\sqrt{2\pi}}\sum_{\nu=1}^{T}\frac{\alpha_\nu}{1+i\beta_\nu k}+O(k^{P+1}) \tag{1.8}$$

or

$$\frac{\mu^{(p)}}{p!}=\sum_{\nu=1}^{T}\alpha_\nu\beta_\nu{}^p\,. \tag{1.9}$$

The solvability criterion is

$$1+P=2T\,. \tag{1.10}$$

The Fourier transform of the right side of (1.8) becomes

$$\frac{1}{\sqrt{2\pi}}\int dk\,\exp(ik\cdot x)\{\text{Padé}\}=\begin{cases}\displaystyle\sum_\nu^{(+)}\frac{\alpha_\nu}{\beta_\nu}\exp(-x/\beta_\nu), & x>0\\[2ex] \displaystyle-\sum_\nu^{(-)}\frac{\alpha_\nu}{\beta_\nu}\exp(-x/\beta_\nu), & x<0\end{cases} \tag{1.11}$$

where

$$\sum_{\nu}^{(\pm)} \equiv \text{sum all terms for which } \mathrm{Re}(1/\beta_\nu) \gtrless 0\,.$$

In Ref. 4, Chisolm shows that the proper interpretation of the very formal and singular expression

$$\phi(x) = \int dk \frac{e^{ikx}}{\sqrt{2\pi}} \left\{ \tilde{\phi}(k) = \sum \frac{(-i)^p}{\sqrt{2\pi}} \frac{\mu^{(p)}}{p!} k^p \right\},$$

or

$$\phi(x) = \sum_{p=0}^{\infty} (-1)^p \frac{\mu^{(p)}}{p!} \frac{\partial^p \delta(x)}{\partial x^p}$$

is precisely that which has been presented. That is, if we use Padé techniques on the Fourier series and then transform back into $\phi(x)$ space, in the limit as more and more moments are taken, we recover the desired field configuration solution, provided $\tilde{\phi}(k)$ is of a Stieltjes nature.

II. STRONG-COUPLING EXPANSION FOR THE MOMENTS

A. The necessity for a lattice approach

If one proceeded to generate a naive λ expansion for the $\phi(x)$ solution of Eq. (1.2), "step-function"-like solutions would appear in the zeroth-order term. Each successive order in the expansion would entail an appropriate order of differentiation of such a step function, resulting in highly singular expressions. The necessity for the lattice as a regularizing procedure is therefore readily appreciated.

One may also consider singular perturbation theory in momentum space. In the Fourier representation, the counterpart to a lattice approach would be the introduction of an effective momentum-space cutoff.

So far there is no apparent preference for working in either one of the above two representations. We may approach the problem either in terms of lattice field variables (to be defined) or in terms of the power-moment dynamical variables. If we proceed with the latter, as motivated by the argument in Sec. I, then we are compelled to obtain the dynamical equations for the moments. Upon doing so, one realizes that the generation of a singular perturbation expansion in terms of the cutoff parameter is made readily unattainable by the presence of a convolution kernel due to the ϕ^3 interaction term in (1.2). Only by working with the lattice field variables may we transform these dynamical equations into a separable form which expedites the generation of the desired singular expansion for the power moments. Thus we use the lattice field variables to generate the high-temperature lattice expansion for the power moments; and, it is through the power moments that a consistent singular perturbation theory expansion for the global field solution may be realized.[6]

B. The lattice high-temperature expansions: Obtaining the true strong-coupling series for the moments

As was argued in the preceding subsection, we will work with lattice space variables, $\phi_a(l)$, and generate an $(a^2g)^{-1}$ expansion. Afterwards, we construct the lattice power moments.

The continuum equation of motion (1.1) on the lattice becomes[6]

$$(a^2g)^{-1}[\phi_a(l+1) + \phi_a(l-1) - 2\phi_a(l)] + \frac{m^2}{g}\phi_a(l) = \phi_a{}^3(l)\,. \quad (2.1)$$

Letting $\alpha = a^2g$, we define

$$\phi_a(l) = \sum_{\omega=0}^{\infty} \phi_l^{(\omega)} \alpha^{-\omega}\,. \quad (2.2)$$

There then results a recusion relation of the form

$$\frac{m^2}{g}\phi_l^{(0)} = (\phi_l^{(0)})^3\,, \quad (2.3a)$$

$$\phi_l^{(\omega)} = \frac{(\phi_{l+1}^{(\omega-1)} + \phi_{l-1}^{(\omega-1)} - 2\phi_l^{(\omega-1)}) - \sum'_{\omega_1+\omega_2+\omega_3=\omega} \phi_l^{(\omega_1)}\phi_l^{(\omega_2)}\phi_l^{(\omega_3)}}{3(\phi_l^{(0)})^2 - m^2/g}\,. \quad (2.3b)$$

The summation $\sum'_{\omega_1+\omega_2+\omega_3=\omega}$ does not include $\omega_j = \omega$.

We are interested in recovering the "kink" solution to Eq. (1.1), $(m/\sqrt{g})\tanh[(m/\sqrt{2})x]$, an odd solution. Accordingly, we will use a reasonable zeroth-order term of the form

$$\phi_l^{(0)} = \begin{cases} m/\sqrt{g} & \text{for } l = +1, +2, \ldots\,, \\ 0 & \text{for } l = 0\,, \\ -m/\sqrt{g} & \text{for } l = -1, -2, \ldots\,. \end{cases}$$

It may be easily shown that all higher-order terms are odd; consequently, the denominator in

(2.3b) may be set at $2m^2/g$.

The lattice kink solution generated through (2.3b) from the zeroth-order input value chosen above will satisfy the following: For each $\omega \neq 0$ there exists an L_ω such that

$$\phi_l^{(\omega)} = 0\,, \quad \text{if } |l| > L_\omega\,.$$

Accordingly, the lattice moments constructed are done so with respect to a properly subtracted lattice field solution ϕ^S. We choose to subtract as follows:

$$\phi_a^S(l) \equiv \phi_a(l) - (m/\sqrt{g})[\theta(l) - \theta(-l)]\,,$$

thus, $(\phi_l^S)^{(0)} \equiv 0$:

$$\mu_a^{(p)} = a \sum_l (al)^p \phi_a^S(l)$$

$$= g^{-(1+p)/2} \alpha^{(1+p)/2} \sum_\omega \alpha^{-\omega} C_p(\omega)\,, \qquad (2.4)$$

$$C_p(\omega) \equiv \begin{cases} \sum_l l^p \phi_l^{(\omega)} & \text{for } \omega \neq 0\,, \\ 0 & \text{for } \omega = 0\,. \end{cases} \qquad (2.5)$$

Having solved (2.2) up to some finite order W, we must try to find some way of approximating the continuum moment $\mu^{(p)}, a \to 0$, from the available information for $\mu_a^{(p)}$ as given by (2.4). The g^{-1} dependence of the continuum approximant will define the actual strong-coupling expansion for the moments.

Because the terms of (2.2) are odd, it follows that the even moments are identically zero. For $p = \text{odd}$, (2.4) becomes $\mu_a^{(p)} = \alpha^{\text{integer}} \times$ polynomial (α^{-1}). We will construct Padé approximants for such expressions, and then take the $\alpha \to 0$ limit, which will define the continuum approximant. Let

$$\frac{1+p}{2} = p_I\,, \quad \text{an integer,}$$

$$\frac{g^{p_I} \mu_\alpha^{(p)}}{\alpha^{p_I}} = \sum_{\omega=0}^{W} \alpha^{-\omega} C_p(\omega) \qquad (2.6a)$$

$$= \left[\frac{\sum_{\eta=0}^{p_I+T} N(\eta) \alpha^{-\eta}}{\sum_{\delta=0}^{T} D(\delta) \alpha^{-\delta}} \right] + O(\alpha^{-(1+W)})\,. \qquad (2.6b)$$

Equation (2.6b) is the Padé approximant for (2.6a). Things are defined so that as $\alpha \to 0$, $\lim[\alpha^{p_I}\,\text{Padé}(\alpha)] \to$ exists. The philosophy adopted is that the lattice has given us a small α^{-1} expansion which must be continued in some reasonable way to the $\alpha^{-1} = \infty$, the continuum limit. If (2.6b) is to be solvable, then the number of knowns, the $C_p(\omega)$'s, must equal the number of unknowns, that is, all the N's and all but one of the D's. We may choose $D(0) = 1$. The solvability criteria are

$$1 + W = 1 + p_I + T + T\,, \qquad W = 2T + p_I\,, \qquad (2.6c)$$

$$\lim_{\alpha \to 0} \mu_\alpha^{(p)} = (g^{-p_I}) \frac{N(p_I + T)}{D(T)} \equiv \mu_{CA}^{(p)}\,. \qquad (2.6d)$$

It is expected that as $T \to \infty$ the evaluation of the ratio in (2.6d) will converge to the actual continuum moment. In the examples discussed in Sec. III, there is oscillatory convergence. In Ref. 6, the details of solving for the N's and D's are given.

It should be apparent from (2.1) that the N's and D's, dependent on the C's, also depend on m^2/g. It therefore follows from (2.6d) that the continuum approximant for $\mu^{(p)}$, $\mu_{CA}^{(p)} = g^{-p_I} \times$ (series in m^2/g). Upon reconstructing the approximant field solution, through the method discussed in Sec. I, the α's and β's given by [see (1.9)]

$$\sum_{p=0}^{P} (-i)^p \frac{\mu_{CA}^{(p)}}{p!} k^p = \sum_{\nu=1}^{T} \frac{\alpha_\nu}{1 + i\beta_\nu k} + O(k^{1+P})\,,$$

are seen to satisfy

$$\mu_{CA}^{(p)} = \hat{\mu}_{CA}^{(p)}/(\sqrt{g})^{1+p}\,,$$

$$\alpha_\nu = \hat{\alpha}_\nu/\sqrt{g}\,,$$

$$\beta_\nu = \hat{\beta}_\nu/\sqrt{g}\,,$$

where $\hat{\mu}_{CA}^{(p)}$, $\hat{\alpha}_\nu$, and $(\hat{\beta}_\nu)^{-1}$ involve expansions

TABLE I. Moments $\mu(p)$ for the ϕ^4 subtracted kink.

p	Padé of lattice moments	Actual moments
1	−1.64	−1.64
3	−5.63	−5.68
5	−61.66	−59.16
7	−1 298	−1 260
9	−44 724	−45 360

TABLE II. Reconstructed ϕ^4 kink.

x	Padé approx.	Actual kink
3.0	0.972	0.972
2.0	0.888	0.888
1.0	0.608	0.609
0.5	0.328	0.340
0.2	0.099	0.140
0.1	0.014	0.071
−1.0	−0.608	−0.609
−2.0	−0.888	−0.888
−3.0	−0.972	−0.972

TABLE III. Moments $\mu(p)$ for the SG subtracted kink.

p	Padé $\left(\frac{\mu^{(p)}}{p!}\right)$	Actual $\left(\frac{\mu^{(p)}}{p!}\right)$
1	−2.5	−1.94
3	−2.1	−1.99
5	−2.2	−2.00
7	−2.2	−2.00
9	−2.2	−2.00

in $m/\sqrt{g}$. Fourier transforming the above Padé expression gives

$$\phi_{\mathrm{CA}}(x)=\sum_{\nu}{}^{(+)}\frac{\hat{\alpha}_{\nu}}{\hat{\beta}_{\nu}}\exp(-\sqrt{g}\,x/\hat{\beta}_{\nu}),\ \text{for } x>0$$

(similarly for $x<0$), where the coefficients and the $\hat{\beta}^{-1}$ quantities involve expansions in $m/\sqrt{g}$. Thus, in the above configuration-space representation, the actual strong-coupling expansion parameter is $m/\sqrt{g}$. There is no g^{-1} dependence.

A. ϕ^4 kink, numerical results

For the ϕ^4 kink, all the terms of the expansion in (2.2) were generated up to $W=70$. In general, good agreement between the actual moments and the continuum approximamts were obtained from $W=30$ and on. The approximants form an oscillatory converging sequence around the true value. In Table I, the actual moment values and the Padé-approximant moment values are compared. Because of the satisfactory nature of the data we proceeded to use the actual moments to test the reconstruction methods of Sec. I. It will be noted in Table II that for small x values there is a slight disagreement between the actual kink and the approximate kink solutions. This is consistent with our formalism, which is expected to yield good results for the long-range behavior of the solution, and gradually gives better results for the short-range behavior as we increase the order of the calculation. Table II involved the actual moment values corresponding to $\mu^{(0)},\ldots,\mu^{(31)}$.

B. The sine-Gordon kink

The sine-Gordon equation is

$$-\partial^2\phi+c\sin(\phi)=0\,.$$

Its corresponding kink solution is

$$\phi_{\mathrm{kink}}(x)=4\arctan[\exp(x\sqrt{c})]\,.$$

In the following discussion we shall let $c=1$.

One may apply a similar analysis to the SG kink as was done for the ϕ^4 kink. The details are given in Ref. 6. Only the numerical results are quoted in Tables III and IV. The computational program

TABLE IV. Reconstructed SG kink.

x	Padé approx.	Actual kink
10.0	1.57	1.57
7.0	1.57	1.57
3.0	1.52	1.52
1.0	1.22	1.22
0.1	0.83	0.84
0.01	0.78	0.79
−0.01	0.79	0.78
−0.1	0.83	0.84
−1.0	0.35	0.35
−3.0	0.05	0.05

for the SG kink limited us to only 18 orders in the $\phi^{(\omega)}$ expansion for the lattice field configuration. The results are satisfactory for the few orders available. Table III gives the Padé results for the first nine moments. Using the actual moment values in Table IV, the reconstructed approximate kink solution is given.

SUMMARY

In the preceding the author has endeavored to formulate a consistent technique for solving systems of field equations in terms of singular-perturbation–strong-coupling expansions. Although the examples discussed are simple, the principles should still apply for more complicated problems, including quantum field theory. Indeed, one may consider quantum field theory as a set of coupled differential Green's functions equations and apply the technique to them.

The work presented here is a brief summary of the essentials of Ref. 6. In subsequent publications, the extension of the method to periodic solutions in space and time (i.e., breather modes) will be considered.

The author has attempted to combine two traditional tools into one effective procedure for solving strong-coupling field theory on the lattice. The first of these is the use of Padé techniques as applied to high-temperature lattice expansions. This was motivated by Refs. 1 and 2. The second of these, function-moments reconstruction methods, although of concern to mathematical physicists,[3-5,7] has also been of concern to some researchers in their work on deep-inelastic structure functions.[8,9]

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