PHYSICAL REVIEW D VOLUME 24, NUMBER 2 15 JULY 1981

Limitations on quantum measurements. I. Determination of the minimal amount of nonideality and identification of the optimal measuring apparatuses

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(Received 27 February 1981)

The problem of determining the minimal deviation from the ideal scheme in a measurement, compatible with the existence of additive conserved quantities, is reconsidered. In the case of the measurement of a spin component of a spin-1/2 particle, we rederive in a simple and rigorous way the bound previously obtained by Yanase. The procedure allows the derivation of definite equations characterizing an optimal measuring apparatus. A detailed discussion of the possible malfunctioning of the apparatuses is also given.

I. INTRODUCTION

In nature there are invariance principles which are thought to be valid for any type of phenomena: In particular they must then govern also the interaction between the quantum systems and the measuring apparatuses. Associated with invariance there are conservation laws and, from the point of view of the quantum theory of measurement, an important role is played by the additive conserved quantities. In fact, as is well known,[1-4] the existence of additive conserved quantities puts severe limitations on the quantum measurement processes. In particular, any quantum-mechanical observable which does not commute with the operator representing an additive conserved quantity (at least when this operator is bounded, see, e.g., Ref. 3) can be measured only in an "approximate" way. The argument leading to this conclusion, in the case in which the measured operator M constitutes by itself a complete set of commuting observables, can be sketched as follows. Let

$$M|m\rangle = m|m\rangle \tag{1.1}$$

be the eigenvalue equation for M, $\{|m\rangle\}$ being a complete orthonormal set in the Hilbert space $\mathcal{H}^S$ of the measured system. The ideal measurement scheme[5] assumes then the existence of a normalized state $|A_0\rangle$ in the Hilbert space $\mathcal{H}^A$ of the apparatus and of a unitary operator U of $\mathcal{H}^A \otimes \mathcal{H}^S$, representing the effect of the system-apparatus interaction, such that

$$U|A_0 m\rangle = |A_m m\rangle , \tag{1.2}$$

where $|A_m\rangle$ are the final states of the apparatus and are assumed to be eigenstates of an observable whose further detection yields the desired information about the result of the measurement. It is now easy to prove, by taking into account the orthogonality of the states $|A_m\rangle$ corresponding to different results of the measurement, that the existence of an additive conserved quantity

$$\Gamma^A + \Gamma^S \equiv \Gamma = U^\dagger \Gamma U \tag{1.3}$$

of the system plus apparatus contradicts Eq. (1.2), unless M commutes with Γ^S (and therefore with Γ). In fact

$$\begin{aligned}\langle m'|[\Gamma^S, M]|m\rangle &= \langle m' A_0|[\Gamma, M]|A_0 m\rangle \\ &= (m - m')\langle m' A_0|\Gamma|A_0 m\rangle \\ &= (m - m')\langle m' A_{m'}|\Gamma|A_m m\rangle \\ &= (m - m')(\langle m'|m\rangle\langle A_{m'}|\Gamma^A|A_m\rangle \\ &\quad + \langle A_{m'}|A_m\rangle\langle m'|\Gamma^S|m\rangle) = 0 ,\end{aligned} \tag{1.4}$$

so that

$$[\Gamma^S, M] = 0 . \tag{1.5}$$

If (1.5) is not true, M cannot be measured according to the ideal scheme (1.2).[6] We must then resort to a more general measurement scheme.

The above remarks are relevant from a conceptual point of view and show, for instance, that one cannot have an ideal measurement process for the spin components of a quantum system. To resort to a nonideal measurement scheme means to introduce finite probabilities of getting a wrong or ambiguous answer, or else of altering the state of the system even when it is initially in an eigenstate of the measured quantity. One is then faced with the problem of investigating how small these probabilities can be made.

It has been pointed out that the malfunctioning probability can be made very small provided $|A_0\rangle$ contains a large amount of the conserved quantity.[1,2,4] In particular, Yanase[7] has dealt with the problem of determining, in the case of the measurement of a spin component of a spin-$\frac{1}{2}$ particle, the minimum probability for the malfunctioning of the apparatus consistent with a given mean value of the squared conserved quantity, obtaining the result that such a minimum probability is inversely proportional to this mean value. The derivation of Ref. 7, however, requires a very cumbersome playing with the angular momentum components of the apparatus states. Moreover, the derivation involves a treatment of the values of the angular momentum components as continuous variables, an assumption whose implications are not clear, since for physical reasons low angular momentum states must be present in the apparatus.

In this paper we reconsider this problem. In Sec. II we discuss, making reference to the case of a spin-$\frac{1}{2}$ particle, the physical meaning of the various terms appearing in a nonideal measurement scheme. The main part of the paper is Sec. III, where, using symmetry considerations, we rederive the Yanase bound in a simple and rigorous way. The method allows us to obtain the equations characterizing an optimal measuring apparatus. This is done in Sec. IV.

II. NONIDEAL MEASUREMENT SCHEMES

Let us focus our attention on the case of the measurement of the third component of the spin $\vec{S}$ of a spin-$\frac{1}{2}$ particle. The eigenvalues and eigenvectors of S_3 satisfy the equation

$$S_3|u_\pm\rangle = \pm\tfrac{1}{2}\hbar|u_\pm\rangle . \tag{2.1}$$

Let $\vec{L}$ be the angular momentum of the measuring apparatus. Since S_1+L_1 and S_2+L_2 are conserved and S_3 does not commute with S_1 and S_2, it follows that S_3 cannot be measured according to the ideal scheme

$$U|A_0u_\pm\rangle = |A_\pm u_\pm\rangle,\quad \langle A_-|A_+\rangle = 0 . \tag{2.2}$$

This can be seen most directly by noting that, since S_1+L_1 is conserved, one must have

$$\langle A_0u_-|(S_1+L_1)|A_0u_+\rangle = \langle A_0u_-|U^+(S_1+L_1)U|A_0u_+\rangle . \tag{2.3}$$

But the left-hand side of Eq. (2.3) is equal to $\hbar/2$ while the right-hand side, using Eq. (2.2), turns out to be zero. A similar contradiction is obtained using S_2+L_2 in place of S_1+L_1.

Quite in general we can write, in place of (2.2),

$$U|A_0u_\pm\rangle = |\alpha_\pm u_\pm\rangle + |\epsilon_\pm u_\mp\rangle , \tag{2.4}$$

with $|\alpha_\pm\rangle$ and $|\epsilon_\pm\rangle$ properly defined states of the apparatus. Obviously, since at the right-hand side of (2.4) the most general state of $\mathcal{H}^S\otimes\mathcal{H}^A$ appears, Eq. (2.4) is certainly compatible with the existence of additive conserved quantities. However, if we want that the evolution given by (2.4) corresponds to a reasonable description of a measurement, we have to make specific assumptions on the states $|\alpha_\pm\rangle$ and $|\epsilon_\pm\rangle$ appearing in it. To this purpose we assume that the two states $|\alpha_\pm\rangle$ differ very little from two states $|A_+\rangle$ and $|A_-\rangle$ of the apparatus which are eigenstates belonging to different eigenvalues of an apparatus observable $\mathcal{A}$, so that $\langle A_-|A_+\rangle = 0$. Isolating the parts of $|\alpha_\pm\rangle$ orthogonal to $|A_\pm\rangle$, respectively, we can write

$$|\alpha_\pm\rangle = |A_\pm\rangle + |\eta_\pm\rangle . \tag{2.5}$$

In order that (2.4) be an acceptable basis for the description of a measurement, we must assume that the norms of $|\eta_\pm\rangle$ and $|\epsilon_\pm\rangle$ are much smaller than the norms of $|A_\pm\rangle$. We rewrite (2.4) by isolating for each state appearing in it the components on $|A_+\rangle$, $|A_-\rangle$, and the orthogonal ones. We have

$$U|A_0u_+\rangle = |A_+u_+\rangle + |\eta_+^-u_+\rangle + |\eta_+'u_+\rangle + |\epsilon_+^+u_-\rangle + |\epsilon_+^-u_-\rangle + |\epsilon_+'u_-\rangle , \tag{2.6a}$$

$$U|A_0u_-\rangle = |A_-u_-\rangle + |\eta_-^+u_-\rangle + |\eta_-'u_-\rangle + |\epsilon_-^-u_+\rangle + |\epsilon_-^+u_+\rangle + |\epsilon_-'u_+\rangle , \tag{2.6b}$$

where the upper indices plus, minus, and prime of $|\eta\rangle$ and $|\epsilon\rangle$ refer to the components on $|A_+\rangle$, $|A_-\rangle$, and on the manifold orthogonal to both of them, respectively. As already stated all terms except $|A_+u_+\rangle$ and $|A_-u_-\rangle$ in Eqs. (2.6) are assumed to have a small norm. Since the right-hand sides of Eqs. (2.6) have unit norms, the apparatus vectors $|A_\pm\rangle$ also have unit norms up to terms of the order of the squared norms of the ϵ and η vectors. The ideal measurement scheme, which is forbidden by angular momentum conservation, corresponds to setting equal to zero all terms at the right-hand sides of (2.6) except the terms containing $|A_+\rangle$ and $|A_-\rangle$. We note that while in the ideal scheme the state of the system, when it is in an eigenstate of the measured quantity, is not modified by the measurement process; in (2.6) there are terms which correspond both to a final state of the system which is the same as the initial one, and terms in which the state of the system is changed. With reference to this fact we say that the measurement process is *distorting* whenever the final state of the measured system can be different from the initial one. Thus the ϵ terms are associated with distortions, while the η terms do not introduce distortions.

As we have recalled previously, the information on the system is obtained by measuring, after the interaction causing the time evolution described by U, the apparatus observable $\mathfrak{A}$. The result of the measurement is "the third spin component $+\hbar/2$" whenever the apparatus is found in the state $|A_+\rangle$; it is "the third spin component $-\hbar/2$" whenever the apparatus is found in the state $|A_-\rangle$. It is seen that, according to the scheme (2.6), the ϵ and η terms which have different upper and lower signs give rise to a nonzero probability of getting a wrong result. Furthermore, due to the primed terms, there is a nonzero probability that the apparatus is found in an eigenstate of $\mathfrak{A}$ different both from $|A_+\rangle$ and from $|A_-\rangle$. In this case the experimenter gets no answer from the measurement. Summarizing, the terms in (2.6) responsible for the various possibilities are

correct answer: $|A_+u_+\rangle$, $|A_-u_-\rangle$, $|\epsilon_+^+u_-\rangle$, $|\epsilon_-^-u_+\rangle$,

wrong answer: $|\eta_+^-u_+\rangle$, $|\eta_-^+u_-\rangle$, $|\epsilon_+^-u_-\rangle$, $|\epsilon_-^+u_+\rangle$,

no answer: primed ϵ and η terms.

We note that the presence of distortion and the possibility of a wrong answer or of no answer are quite independent.

The above discussion about the possible malfunctioning of the apparatus refers to the case in which the apparatus is used only to obtain information about the states of the system before the measurement process. However, in quantum mechanics, there is a second use of a measuring device which is of great importance, i.e., the apparatus is very often used to prepare a definite quantum state for the measured system. If the measurement takes place according to the ideal scheme, the apparatus can be used both to measure and to prepare the system. As already discussed, when we use the apparatus simply to measure the third component of the spin, the states with different upper and lower signs and the primed terms are related to a malfunctioning of the apparatus with respect to the use we are making of it. On the contrary, when the apparatus is used to prepare a state, the terms which are responsible for the malfunctioning are $|\epsilon_+^+\rangle$, $|\epsilon_-^-\rangle$, $|\eta_+^-\rangle$, $|\eta_-^+\rangle$ and $|\epsilon_+'\rangle$, $|\epsilon_-'\rangle$, $|\eta_+'\rangle$, $|\eta_-'\rangle$. In fact for the first four we would get a wrong answer for the state of the system after the measurement, while the remaining four give, as previously, no answer.

For the previously discussed reasons it is appropriate to consider as related to a malfunctioning of the apparatus all terms in Eqs. (2.6) except for the two terms $|A_+U_+\rangle$ and $|A_-U_-\rangle$ corresponding to the ideal functioning. We will then define as a measure of the malfunctioning *the total amount of nonideality* e^2 as given by the sum of the squared norms of all terms in Eqs. (2.6) except those corresponding to the ideal scheme. It is also useful to define the *total amount of distortion* ϵ^2 as the sum of the squared norms of all the ϵ terms in Eqs. (2.6).

In the next section we will be interested in getting a bound for the total amount of nonideality. To derive the bound we will first derive a corresponding bound for the total amount of distortion.

Before concluding this section we want to stress that the presence of the distorting terms in (2.4) is essential to overcome the contradiction with the conservation law. In fact, if all the terms ϵ are zero, by inserting Eq. (2.4) into Eq. (2.3) we get

$$\hbar/2 = \langle\alpha_-|\alpha_+\rangle\hbar/2 \, . \tag{2.7}$$

There follows $\langle\alpha_-|\alpha_+\rangle = 1$, implying $|\alpha_-\rangle = |\alpha_+\rangle$, which is absurd if the apparatus has to allow the identification of the different initial states of the system. A similar argument cannot be developed for the η terms, which could be dropped without any apparent contradiction.

III. LOWER BOUNDS FOR THE MALFUNCTIONING OF THE APPARATUS

This section is devoted to the rederivation of the Yanase bounds in a simple and rigorous way. We make use of the general measurement scheme (2.4). The problem to be faced is to make as small as possible the *amount of distortion*

$$\epsilon^2 = \langle\epsilon_+|\epsilon_+\rangle + \langle\epsilon_-|\epsilon_-\rangle \, , \tag{3.1}$$

compatible with the conservation law for the total angular momentum $\vec{S}+\vec{L}$. Owing to the fact that U is an operator of $\mathcal{H}^S \otimes \mathcal{H}^A$, its most general expression is

$$U = \tfrac{1}{2}\left(B + \sum_{i=1}^{3} C_i\sigma_i\right), \tag{3.2}$$

where σ_i are the Pauli matrices and B, C_i ($i=1, 2,3$) are operators of $\mathcal{H}^A$. Imposing rotational invariance, i.e.,

$$[U, S_i + L_i] = 0, \quad i = 1,2,3 \tag{3.3}$$

we obviously obtain that B is a scalar and the C_i's transform like the components of a vector:

$$[B, L_i] = 0\,, \tag{3.4a}$$

$$[C_j, L_i] = i\hbar \sum_k \epsilon_{jik} C_k\,. \tag{3.4b}$$

The unitarity condition $U^+U = 1$ gives

$$B^\dagger B + \sum_i C_i^\dagger C_i = 4\,, \tag{3.5a}$$

$$B^\dagger C_k + C_k^\dagger B + i \sum_{ij} \epsilon_{kij} C_i^\dagger C_j = 0\,, \tag{3.5b}$$

while $UU^\dagger = 1$ gives similar relations where the dagger is attached to the second operator in each term. We note that (3.5a) implies that the operators B and C_i are bounded. Inserting the expression (3.2) into Eq. (2.4) we have

$$|\alpha_+\rangle = \tfrac{1}{2}(B + C_3)|A_0\rangle\,, \tag{3.6a}$$

$$|\alpha_-\rangle = \tfrac{1}{2}(B - C_3)|A_0\rangle\,, \tag{3.6b}$$

$$|\epsilon_+\rangle = \tfrac{1}{2}(C_1 + iC_2)|A_0\rangle\,, \tag{3.6c}$$

$$|\epsilon_-\rangle = \tfrac{1}{2}(C_1 - iC_2)|A_0\rangle\,. \tag{3.6d}$$

Using the notation $\rangle = |A_0\rangle$ the parameter ϵ^2 can be written

$$\begin{aligned}\epsilon^2 &= \tfrac{1}{2}(\langle C_1^\dagger C_1\rangle + \langle C_2^\dagger C_2\rangle) \\ &= \tfrac{1}{2}(\|C_1\rangle\|^2 + \|C_2\rangle\|^2)\,.\end{aligned} \tag{3.7}$$

In order to find a lower bound for ϵ^2 we insert Eq. (2.4) into (2.3). We have

$$\begin{aligned}\hbar/2 = (\hbar/2)\langle\alpha_-|\alpha_+\rangle &+ \langle\alpha_-|L_1|\epsilon_+\rangle \\ &+ \langle\epsilon_-|L_1|\alpha_+\rangle + (\hbar/2)\langle\epsilon_-|\epsilon_+\rangle\,.\end{aligned} \tag{3.8}$$

Taking the real part of Eq. (3.8) and inserting the expressions (3.6) we find

$$\begin{aligned}\hbar(1 - \mathrm{Re}\langle\alpha_-|\alpha_+\rangle) &= \langle\alpha_-|L_1|\epsilon_+\rangle + \langle\epsilon_+|L_1|\alpha_-\rangle + \langle\epsilon_-|L_1|\alpha_+\rangle \\ &\quad + \langle\alpha_+|L_1|\epsilon_-\rangle + (\hbar/2)\langle\epsilon_-|\epsilon_+\rangle \\ &\quad + (\hbar/2)\langle\epsilon_+|\epsilon_-\rangle \\ &= \mathrm{Re}\langle C_1^\dagger L_1 B\rangle - \mathrm{Im}\langle C_2^\dagger L_1 C_3\rangle \\ &\quad + (\hbar/4)\langle C_1^\dagger C_1\rangle - (\hbar/4)\langle C_2^\dagger C_2\rangle\,.\end{aligned} \tag{3.9}$$

Use of the commutation relations (3.4) gives

$$\begin{aligned}\hbar(1 - \mathrm{Re}\langle\alpha_-|\alpha_+\rangle) &= \mathrm{Re}\langle C_1^\dagger B L_1\rangle - \mathrm{Im}\langle C_2^\dagger C_3 L_1\rangle \\ &\quad + (\hbar/4)\langle C_1^\dagger C_1\rangle \\ &\quad + (3\hbar/4)\langle C_2^\dagger C_2\rangle\,.\end{aligned} \tag{3.10}$$

We can then write

$$\begin{aligned}\hbar(1 - \mathrm{Re}\langle\alpha_-|\alpha_+\rangle) &\leq |\langle C_1^\dagger B L_1\rangle| + |\langle C_2^\dagger C_3 L_1\rangle| \\ &\quad + (\hbar/4)\langle C_1^\dagger C_1\rangle \\ &\quad + (3\hbar/4)\langle C_2^\dagger C_2\rangle\end{aligned} \tag{3.11}$$

and, using the Schwarz inequality,

$$\begin{aligned}\hbar(1 - \mathrm{Re}\langle\alpha_-|\alpha_+\rangle) &\leq \|C_1\rangle\|\;\|BL_1\rangle\| + \|C_2\rangle\|\;\|C_3L_1\rangle\| \\ &\quad + (\hbar/4)\langle C_1^\dagger C_1\rangle + (3\hbar/4)\langle C_2^\dagger C_2\rangle\,.\end{aligned} \tag{3.12}$$

Since for any given set of four positive numbers a, b, c, d it is $ab + cd \leq (a^2 + c^2)^{1/2}\,(b^2 + d^2)^{1/2}$, we get

$$\hbar(1 - \mathrm{Re}\langle\alpha_-|\alpha_+\rangle) \leq (\langle C_1^\dagger C_1\rangle + \langle C_2^\dagger C_2\rangle)^{1/2}\,(\langle L_1 B^\dagger B L_1\rangle + \langle L_1 C_3^\dagger C_3 L_1\rangle)^{1/2} + (\hbar/4)\langle C_1^\dagger C_1\rangle + (3\hbar/4)\langle C_2^\dagger C_2\rangle\,. \tag{3.13}$$

Adding $\langle L_1 C_1^\dagger C_1 L_1\rangle + \langle L_1 C_2^\dagger C_2 L_1\rangle$ in the second square root and using Eqs. (3.7) and (3.5a), we find

$$\begin{aligned}\hbar(1 - \mathrm{Re}\langle\alpha_-|\alpha_+\rangle) &\leq \epsilon(8\langle L_1^2\rangle)^{1/2} + (\hbar/4)\langle C_1^\dagger C_1\rangle \\ &\quad + (3\hbar/4)\langle C_2^\dagger C_2\rangle\,.\end{aligned} \tag{3.14}$$

From Eq. (3.7) we also get immediately

$$(\tfrac{1}{4})\langle C_1^\dagger C_1\rangle + (\tfrac{3}{4})\langle C_2^\dagger C_2\rangle \leq (\tfrac{3}{2})\epsilon^2\,. \tag{3.15}$$

From Eq. (2.5), using $\langle A_-|A_+\rangle = 0$ and the Schwarz inequality, we get

$$|\mathrm{Re}\langle\alpha_-|\alpha_+\rangle| \leq 2\eta + \eta^2\,, \tag{3.16}$$

where

$$\eta^2 = \langle\eta_+|\eta_+\rangle + \langle\eta_-|\eta_-\rangle\,. \tag{3.17}$$

Therefore, keeping only the leading terms in both sides of (3.14) it follows that

$$\epsilon^2 \geq \hbar^2/8\langle L_1^2\rangle\,. \tag{3.18}$$

Similarly, starting from the conservation of $S_2 + L_2$, we would obtain

$$\epsilon^2 \geq \hbar^2/8\langle L_2^2\rangle\,. \tag{3.19}$$

Of course, in the case of full rotational invariance, the more restrictive of conditions (3.18) and (3.19) must be satisfied. The conservation of the third component of the angular momentum does not give any limitation since $S_3 + L_3$ commutes with the measured quantity.

It is seen from Eqs. (3.14) and (3.16) that the presence of the η terms does not change in an essential way the lower bounds for the amount of distortion. Obviously if we define the *total amount of nonideality*

$$e^2 = \epsilon^2 + \eta^2\,, \tag{3.20}$$

this quantity too is larger than $\hbar^2/8\langle L_1^2\rangle$ and $\hbar^2/8\langle L_2^2\rangle$.

The values of (3.18) and (3.19) are equal to those obtained by Yanase in a completely different way in the framework of a measurement scheme without η terms. We note that in the previous chain of inequalities we have repeatedly increased the right-hand sides. To keep the distortion as small as possible, compatible with the value of $\langle L_1^2\rangle$ or $\langle L_2^2\rangle$, we must require that the equality sign holds in all the equations. We will discuss this point in the next section.

IV. OPTIMAL MEASURING APPARATUS

We shall now investigate which conditions have to be satisfied in order that the lower bounds for the amount of distortion obtained in Sec. III be actually attained. Of course, if the two bounds are different, only the larger one can be reached. Let us suppose that it is the bound (3.18). It is easily seen that a necessary and sufficient condition, in order that the equality sign holds in Eqs. (3.11), (3.12), and (3.13), is

$$C_1|A_0\rangle=(\beta^2/\hbar)BL_1|A_0\rangle\,,\qquad C_2|A_0\rangle=i(\beta^2/\hbar)C_3L_1|A_0\rangle\,, \tag{4.1}$$

where β^2 is a real positive number. If Eqs. (4.1) are satisfied, we get in place of (3.14) the equation

$$\hbar(1-\mathrm{Re}\langle\alpha_-|\alpha_+\rangle)=\epsilon(8\langle L_1^2\rangle-2\langle L_1C_1^\dagger C_1L_1\rangle-2\langle L_1C_2^\dagger C_2L_1\rangle)^{1/2}+(\hbar/4)\langle C_1^\dagger C_1\rangle+(3\hbar/4)\langle C_2^\dagger C_2\rangle\,. \tag{4.2}$$

If we further require that

$$C_1L_1|A_0\rangle=0\,,\qquad C_2L_1|A_0\rangle=0\,, \tag{4.3}$$

recalling Eqs. (3.15) and (3.16), we find

$$\hbar\simeq\epsilon(8\langle L_1^2\rangle)^{1/2} \tag{4.4}$$

up to higher-order infinitesimal quantities. But the equations (4.1) and (4.3), as they stand, are incompatible with the unitarity relations. In fact, from (3.5b), we have

$$B^\dagger C_1+C_1^\dagger B+iC_2^\dagger C_3-iC_3^\dagger C_2=0\,. \tag{4.5}$$

Multiplying this equation by $\langle A_0|$ on the left and $L_1|A_0\rangle$ on the right and using Eqs. (4.1) and (4.3), we get

$$\langle C_1^\dagger C_1\rangle+\langle C_2^\dagger C_2\rangle=0\,, \tag{4.6}$$

i.e., $\epsilon^2=0$, which is absurd, as we know. However, since in any case the result (4.4) should hold in the limit in which $\langle L_1^2\rangle$ becomes very large, it is seen that we can drop the condition (4.3) substituting it with the weaker condition that the vectors at the left-hand sides of Eqs. (4.3) remain finite when $\langle L_1^2\rangle\to\infty$. Then the square root in Eq. (4.2) shall behave like $(8\langle L_1^2\rangle)^{1/2}$, so that again Eq. (4.4) is valid up to higher-order infinitesimal quantities.

The parameter β^2 appearing in Eqs. (4.1) cannot have an arbitrary value. In fact, substituting Eqs. (4.1) into Eq. (3.7) and using Eq. (3.5a) together with the result (4.4) one gets

$$\beta^2\simeq\hbar^2/4\langle L_1^2\rangle\,. \tag{4.7}$$

The equation (4.4) is deduced directly from the real part of Eq. (3.8). We note that the equation coming from the imaginary part of Eq. (3.8) is verified to the same degree of accuracy as Eq. (4.4). In fact, taking the imaginary part of (3.8) and using (4.1), one gets

$$0=\hbar\,\mathrm{Im}\langle\alpha_-|\alpha_+\rangle-(\hbar/4)(\langle C_2^\dagger C_1\rangle+\langle C_1^\dagger C_2\rangle)\,. \tag{4.8}$$

It is easily seen that the moduli of the two terms at the right-hand side are smaller than $\hbar(2\eta+\eta^2)$ and $\hbar\epsilon^2$, respectively. Terms of the same type have been neglected in (4.4).

Summarizing, we have proved that, to make the total amount of nonideality e^2 small, one has first to choose an apparatus for which the η terms are not present and the mean values $\langle L_1^2\rangle$ and $\langle L_2^2\rangle$ are very large. Secondly, for given values of $\langle L_1^2\rangle$ and $\langle L_2^2\rangle$, supposing that $\langle L_1^2\rangle<\langle L_2^2\rangle$, the minimum amount of distortion, which now coincides with the minimum amount of nonideality, is obtained when the operators B and C_i, which define through Eq. (3.2) the evolution of the system and the apparatus interacting together, satisfy (up to higher-order terms) the relations

$$C_1|A_0\rangle=(\hbar/4\langle L_1^2\rangle)BL_1|A_0\rangle\,,\qquad C_2|A_0\rangle=i(\hbar/4\langle L_1^2\rangle)C_3L_1|A_0\rangle\,, \tag{4.9}$$

while the norms of $C_1L_1|A_0\rangle$ and $C_2L_1|A_0\rangle$ remain small with respect to $(\langle L_1^2\rangle)^{1/2}$. The measurement process gives then the minimum of nonideality compatible with the conservation of angular momentum and with the given values of $\langle L_1^2\rangle$ and $\langle L_2^2\rangle$.

ACKNOWLEDGMENTS

This work was supported in part by the Istituto Nazionale di Fisica Nucleare, Sezioni di Trieste e Pavia.

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