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Sagnac effect in relativistic and nonrelativistic physics

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A simple special-relativistic derivation of the Sagnac effect, which reconciles the earlier general-relativistic and nonrelativistic derivations, is given. A distinction is made between the "classical Sagnac effect" and the "quantum Sagnac effect." A new group-theoretic derivation of these effects is also given. It is pointed out that there must exist a phase shift due to the Thomas precession in the interference of particles with intrinsic spin. The group-theoretic treatment also elucidates the connection between relativistic and nonrelativistic physics, on a classical and quantum level, with and without gravity. A formulation of the principle of equivalence, which is related to the Sagnac effect, is given in relativistic and nonrelativistic physics in terms of the respective invariance groups. New experiments are proposed to test the Sagnac effect in superfluid helium. The possible use of the general-relativistic Sagnac effect to measure the curvature tensor and in particular to detect gravitational waves is suggested.

I. INTRODUCTION

The phase shift in the interference of two coherent light beams due to the rotation of the apparatus was first observed by Sagnac and Michelson.[1] The same effect in the interference of neutron beams due to the Earth's rotation has been observed by Werner, Staudenmann, and Colella[2] by means of an experiment proposed by the author.[3] There have been two apparently unrelated explanations for this effect. One is general relativistic[3-6] and the other is nonrelativistic.[2,7,8] Unfortunately, the relativistic treatments have not made sufficiently explicit, in our view, the essential physical and geometrical meaning of this effect, which, as we shall see, is also necessary to understand the precise connection between the relativistic and nonrelativistic treatments, although the nonrelativistic result can, of course, be obtained as a limiting case of the relativistic result.[3]

In Sec. II of this paper, we therefore give a clearer derivation of this effect, which has been called the Sagnac effect, within the framework of special relativity, in order to bring out its essential feature. This treatment will make it clear why this effect depends only on the frequency of the beams with respect to the rotating apparatus, and is independent of the mass and hence the momenta of the particles relative to the apparatus, when the apparatus is rotating rigidly with constant angular velocity. It will also, of course, refute the often-made assertion that special relativity cannot treat the Sagnac effect, presumably because the rotating frame is an accelerated frame.[9] We make a distinction between the "classical Sagnac effect" and the "quantum Sagnac effect." Even though these two effects are essentially the same in relativistic physics, this distinction seems to be necessary because nonrelativistically the former effect is zero while the latter is nonzero.

In Sec. III, we examine the group-theoretical meaning of the relativistic and nonrelativistic Sagnac effects. This throws light on their relationship, both on classical and quantum-mechanical levels. This consideration, in our view, provides the deepest reason for the existence of the Sagnac effect and also yields a phase shift due to the Thomas precession in the interference of particles with intrinsic spin. The Sagnac effect is generalized to arbitrary space-time groups. It then becomes clear that the reason why the nonrelativistic quantum Sagnac effect cannot be obtained geometrically, unlike in the relativistic case, is because the symmetry group of nonrelativistic quantum mechanics is different from the automorphism group of its geometry. In the presence of a gravitational field, this treatment will still be valid because the symmetry groups are valid locally. This enables us to formulate a principle of equivalence, in relativistic and nonrelativistic physics, at the quantum-mechanical and classical levels, in terms of the respective symmetry groups, which is related to the Sagnac effect.

The Sagnac effect for quantum fluids is discussed in Sec. IV. Two experiments are suggested to test this effect in superfluid helium by means of Josephson interferometers. A general-relativistic theory of the interaction of superfluid helium with the gravitational field is outlined. It is pointed out that the general-relativistic Sagnac effect can, in principle, be used to measure the curvature tensor and to detect gravitational waves.

II. GEOMETRICAL TREATMENT OF THE SAGNAC EFFECT

The rotating body has had a fascination of its own for the physicist. It has also been a source

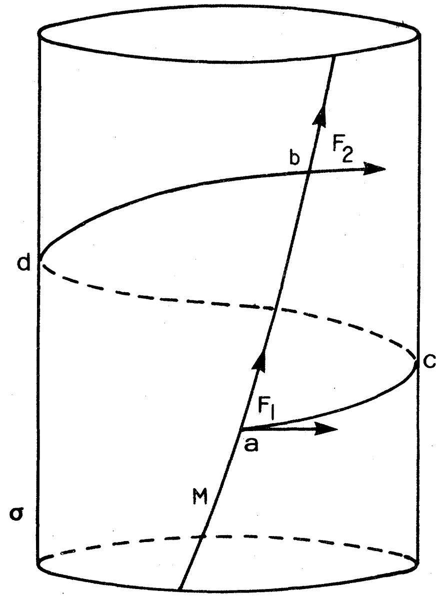

FIG. 1. A circle rotating about a fixed point in a fixed plane in the space of an inertial frame is represented by a cylinder σ in space-time. The curve M on σ is the world line of a particle on the circle. The helical curve $acdb$ on σ consists of events that are locally simultaneous with respect to inertial frames attached to all the points on the rotating circle. The "time lag" Δt in synchronizing clocks around the circle is the proper time along the segment ab of the curve M. The nonvanishing of Δt due to rotation leads to the relativistic Sagnac effect.

of some misunderstanding. Einstein,[10] for instance, has argued that the ratio of the circumference to the diameter of a rotating disk, as measured by measuring rods at rest with respect to the disk, is greater than π because of Lorentz contraction. Many authors have concluded from this that these rods determine a non-Euclidean geometry, with respect to the disk. However, the "circumference" measured by these rods is really the length of a helical curve in space-time consisting of events which are locally simultaneous with respect to the instantaneous inertial frames attached to the periphery of the disk (Fig. 1). Thus there is no violation of Euclidean geometry as a result of the rotation if the gravitational field due to the disk is neglected. Hence Einstein's argument is at best a heuristic one which enables him to guess correctly that the gravitational field, which is analogous to the inertial fields experienced by an observer on the disk, must modify the geometry of space-time.[11]

The time difference Δt between the events at the end points of the helical curve mentioned above can be experimentally determined in the following way: Consider a circular toroidal tube with perfectly smooth internal walls, which is rotating about its axis with constant angular velocity. Suppose from a point M inside the tube two particles are thrown tangentially with equal and opposite velocities with respect to an inertial frame which is instantaneously at rest relative to the tube at this event. Then, clearly, as the particles travel along the tube, at each point, they will have the same speed v with respect to the tube. Let τ_R and τ_N be, respectively, the relativistic and nonrelativistic time difference between their arrivals at M after going once around the tube. Then, clearly, $\tau_N = 0$ whereas $\tau_R = 2\Delta t$. We shall call τ the classical Sagnac effect for reasons that will become clear below. It may be noted that, in both cases, τ is independent of v.

Suppose now that a beam of identical particles of fairly well-defined momentum is split into two at M so that the two beams have equal and opposite momenta, tangential to the tube, with respect to the tube at M. We ask now what the phase difference $\Delta\phi$ between the associated de Broglie waves will be when they return to M after traveling once around the tube. This $\Delta\phi$ will be called the quantum Sagnac effect. Relativistically, $\Delta\phi$ is given by

$$\Delta\phi_R = \omega\tau_R = 2\omega\Delta t \;, \tag{1}$$

where ω is the common frequency of the beams at M with respect to the tube. The Newtonian phase shift $\Delta\phi_N$, however, is *not* $\omega\tau_N = 0$. This is basically because the phase of the wave function is not a scalar in nonrelativistic quantum mechanics, whereas it is a scalar in relativistic quantum mechanics. To calculate Δt, it is convenient to idealize the ring by a cylindrical submanifold σ in space-time (Fig. 1).[12] Let t^μ be the four-velocity field, of the particles constituting the tube, which is tangent to σ at each point. Then

$$c\Delta t = \int_{ab} t_\mu dx^\mu \;,$$

where the integral is over the portion ab of the world line of the mirror M along which interference takes place. Now since t^μ is normal to the curve $acdb$ at each point we can write $c\Delta t = \oint_{abdca} t_\mu dx^\mu$. We prove below that t_μ is curl free in the submanifold σ so that

$$c\Delta t = \oint_\gamma t_\mu dx^\mu \;, \tag{2}$$

where γ is any curve around σ.

To prove that t_μ is curl free in σ, we note that there is only one independent bivector on the two-dimensional submanifold σ. Hence $\overline{\nabla}_{[\mu} t_{\nu]} = \Lambda t_{[\mu} R_{\nu]}$, where $\overline{\nabla}$ denotes gradient in the submanifold σ, Λ is a scalar function on σ, and R^μ has been chosen to be the normalized vector field which is tangent to the circles of simultaneous events in σ in the inertial frame with respect to which the tube is rotating. Then $\overline{\nabla}_\mu R_\nu = 0$. Using also $t_\mu t^\mu = 1$, we have $2t^\mu R^\nu \overline{\nabla}_{[\mu} t_{\nu]} = t^\mu R^\nu \overline{\nabla}_\mu t_\nu = t^\mu \overline{\nabla}_\mu (R^\nu t_\nu) - t^\mu t_\nu \overline{\nabla}_\mu R^\nu = 0$, where the first term vanishes because of the constant angular velocity $\vec{\Omega}$ of the tube, which makes $R^\nu t_\nu = (1 - \Omega^2\rho^2/c^2)^{-1/2}\Omega\rho/c$ constant in σ, ρ being the radius of the tube. Hence $\Lambda = 0$. It follows also that $t^\mu \overline{\nabla}_\mu t^\nu = 0$. Now, (2) can be computed by conveniently choosing γ to be an integral curve of R^μ and (1) reads

$$\Delta\phi_R = 4\left(1 - \frac{\Omega^2\rho^2}{c^2}\right)^{-1/2} \frac{\Omega A \omega}{c^2}, \tag{3}$$

where $A = \pi\rho^2$ is the area enclosed by the tube.

The above treatment can easily be generalized to curved space-time and to arbitrary geometries of the apparatus, provided there is a Killing vector field ξ^μ in the submanifold σ which is parallel to t^μ at each point on σ. (In the above flat-space-time case, the condition that the apparatus is rigidly rotating with constant angular velocity guarantees the existence of such a Killing field.) Then, $(\xi_\mu \xi^\mu)^{1/2}\omega$ is constant in σ[4,3] and it follows that $\overline{\nabla}_{[\mu} \omega t_{\nu]} = 0$. Hence,[13]

$$\Delta\phi_R = \frac{1}{c}\oint_\gamma \omega t_\mu dx^\mu, \tag{4}$$

where γ is an arbitrary curve around σ. Now ω can vary along σ, this variation being known as the gravitational red-shift. In the more general case when such a Killing field need not be present, the interference fringes will in general fluctuate. In this case $\Delta\phi_R$ at an arbitrary point p on the mirror on which the interference takes place is given by $\Delta\phi_R = \oint_{\gamma_0} k_\mu dx^\mu$, where k^μ is the wave vector[3] and γ_0 consists of the classical trajectories along σ (geodesics on the submanifold σ) which join p to points q, r of mirror M and the segment qr of the world line of M. The eikonal equation $k_\mu k^\mu = m^2c^2/\hbar^2$ may, in this case, be solved in the neighborhood of the classical trajectories, but cannot be solved on the entire submanifold σ in general,[3] which prevents γ_0 in the last equation for $\Delta\phi_R$ from being replaced by an arbitrary curve around σ if k is to satisfy the eikonal equation. We also note that, in the metric used in the integral for $\Delta\phi_R$, there would be a contribution due to the Lense-Thirring field[14] if a rotating body is present near the interferometer.[15]

Using now the Einstein-Planck law $\hbar\omega = mc^2/(1 - v^2/c^2)^{1/2}$ and taking the nonrelativistic limit ($v/c \to 0$, $\Omega\rho/c \to 0$) of (3), we obtain

$$\Delta\phi_N = 4\,\frac{m\Omega A}{\hbar}. \tag{5}$$

For the experiment of Werner *et al.*,[2] the beams interfere halfway around the apparatus and so the result is half its value in (5).[16] Equation (5) can also be confirmed by (i) noting that the interfering beams have different momenta and hence different wave numbers with respect to the nonrotating inertial frame[7,8] and (ii) deriving the Sagnac effect as that phase shift, which is necessary in a rotating frame, to give the Coriolis field in the classical limit.[17,18] So, in the nonrelativistic case, there is no classical Sagnac effect, whereas there is a quantum Sagnac effect, which is unlike the relativistic case where the two effects are essentially the same. This apparent paradox is easily resolved as will be seen in the next section.

III. GROUP-THEORETICAL TREATMENT OF THE SAGNAC EFFECT

Since the geometry of special relativity is determined by the ten-parameter Poincaré group, clearly the geometrical derivation given in Sec. II for the relativistic Sagnac effect must reflect the structure of the Poincaré group. One way to see this connection is to consider inertial frames F_1 and F_2, attached to a and b, which are related by paralled transport along the world line through a and b (Fig. 1). Then F_2 can be obtained from F_1 by performing alternative infinitesimal Lorentz boost and translations along the curve $acdb$, as well as rotations to compensate for the Thomas precessions that arise. The product of all these transformations will relate F_2 and F_1 by a translation along ab, which represents the Sagnac effect. More realistically, each beam goes around as a result of a finite number of reflections. The frame F_1 should be given an appropriate Lorentz transformation at each reflection and a translation in the direction of the reflected beam in between two consecutive reflections in order to obtain the "final" frame F_2. It is in the limit when the distances between reflections become infinitesimal and the mirrors are situated around a circle that the transformations are around the helical curve $acdb$ as mentioned above.

Alternatively, it is possible to obtain the Sagnac effect, without any reference to space-time geometry, purely by observing that the two beams can be regarded as undergoing a series of alternative Lorentz transformations and translations before they are superposed. As we shall soon see, this not only provides a general derivation of the relativistic and nonrelativistic Sagnac effects, but also yields a phase shift due to the Thomas precession.[19] For simplicity, consider the case when a given beam travels around a rectangle of sides r and s in the x-y plane of a Cartesian coordinate system. Let $\vec{i}$ and $\vec{j}$ be unit vectors in the x and y directions. Suppose that the beam travels a distance s with velocity $v_x\hat{\imath}$. Its evolution in the x direction is represented by the transformation $\exp(sT_x)$, where T_x generates translation in the x direction. The beam is then reflected by a mirror so that it has velocity $v_y\hat{\jmath}$ relative to the laboratory frame. Then the reflected beam may be regarded as related to the incident beam by a Lorentz boost, which is $\exp(-v_xK_x+v_yK_y)$ to second order in the quantities v_x and v_y, where K_x and K_y generate the Lorentz boosts in the x and y directions. It should be noted that this transformation is not $\exp(v_yK_y)\exp(-v_xK_x)$ as one might naively obtain by transforming back to the laboratory frame and then to the frame of the reflected beam. This is because the latter transformation is equivalent to a boost and a rotation whereas, relative to the rest frame of the neutron just before reflection, assuming the neutron experiences only an impulse and no torque from the mirror during reflection, it must undergo a Lorentz boost without rotation. Hence it follows that *in the laboratory frame, the neutron would undergo a Lorentz boost and a precession* during reflection. Suppose now that the neutron beam travels a distance s in the y direction and is reflected by a second mirror so that it has velocity $-v_x\hat{\imath}$ in the laboratory frame. It is then reflected by the mirrors in the other two corners of the rectangle so that it has, respectively, velocities $-v_y\hat{\jmath}$ and $v_x\hat{\imath}$ after these reflections. The final beam is then related to the original beam by a transformation g, which, to the second order in v_x and v_y, is

$$\begin{aligned} g &\simeq \exp(v_xK_x+v_yK_y)\exp(-rT_y)\exp(v_xK_x-v_yK_y)\exp(-sT_x)\exp(-v_xK_x-v_yK_y) \\ &\quad\times\exp(rT_y)\exp(-v_xK_x+v_yK_y)\exp(sT_x) \\ &\simeq 1+2sv_x[T_x,K_x]+2rv_y[T_y,K_y]-2v_xv_y[K_x,K_y]-rs[T_x,T_y]\,. \end{aligned} \tag{6}$$

Equation (6) is generally true because no assumption has been made about the Lie algebra to obtain it. Consider now the following Poincaré Lie-algebra relations:

$$\begin{aligned} &[T_i,T_j]=0,\quad [T_i,K_j]=-\frac{1}{c^2}T_0\delta_{ij}\,, \\ &[K_i,K_j]=\frac{1}{c^2}\epsilon_{ijk}J_k\,, \end{aligned} \tag{7}$$

where T_0, J_i generate time translation and rotations, respectively. Substituting (7) into (6),

$$g\simeq 1-2\left(\frac{sv_x}{c^2}+\frac{rv_y}{c^2}\right)T_0-2\frac{v_xv_y}{c^2}J_z\,. \tag{8}$$

Similarly, the other beam which travels around in the opposite sense to the first beam will be related to the original beam by the group element

$$g'\simeq 1-2\left(\frac{sv_x'}{c^2}+\frac{rv_y'}{c^2}\right)T_0+\frac{2v_x'v_y'}{c^2}J_z\,. \tag{9}$$

Here $v_x\,\hat{\imath}$ and $v_x'\,\hat{\imath}$ are the velocities of the opposing beams along opposite sides of the rectangle which is rotating with angular velocity $\vec{\Omega}$ along the z axis. Hence $v_x-v_x'=\Omega r$. Similarly, $v_y-v_y'=\Omega s$. On defining $A=rs$, the phase shift is determined by the transformation

$$g'^{-1}g\simeq 1-4\frac{\Omega A T_0}{c^2}-2\frac{(v_xv_y+v_x'v_y')}{c^2}J_z\,, \tag{10}$$

which consists of a time translation and a rotation. The time translation, represented by the second term in (10), gives a phase shift between the interfering beams, which is the same as the Sagnac effect (3) in the present approximation. This is because we can substitute $-i\omega$ for T_0 in the WKB approximation. The last term in (10) represents the Thomas precession.

This Thomas precession term will contribute to a phase shift between the interfering beams whether or not the apparatus is rotating. But it will not contribute if the particle is spinless. The precise shift in interference fringes due to this term can be determined from the general method for treating the interference of particles with arbitrary spin, given elsewhere.[7] This shift, however, is very small for thermal neutrons in the experimental arrangement considered here. But from a conceptual point of view the present application of the Thomas precession is clearer and simpler than the original application to an atomic electron,[19] firstly because an electron cannot be visualized as traveling around the nucleus since its wave function is spread out over the entire region, unlike the wave function of the neutron beam, and secondly the neutrons are not interacting with any external field between reflections, which makes the purely kinematical nature of this effect more transparent.

Returning now to (7), we note that $P_0\equiv i\hbar T_0$ and $P_j\equiv\hbar/i\,T_j$ have the interpretation of energy and momentum operators. Also $P_0^{\,2}-c^2\sum_i P_iP_i=M^2c^4$ is a Casimir operator of the Poincaré group. The nonrelativistic limit ($c\to\infty$) of (7) is therefore

$$[P_i,P_j]=0,\quad [P_i,K_j]=M\delta_{ij},\quad [K_i,K_j]=0\,. \tag{11}$$

So if P_i, K_j etc. are to generate a group, then M, which commutes with all the generators, must also be a generator of such a group. Now, define $\overline{P}_0$ to be the nonrelativistic limit of $P_0 - Mc^2$, i.e., $\overline{P}_0 = \frac{1}{2}M^{-1}\sum_i P_i P_i$. Then the nonrelativistic limit of the commutators of P_0 with the other generators is obtained by replacing P_0 by $\overline{P}_0$. Thus we end up with an eleven parameter "quantum-mechanical Galilei group."[20] $\overline{P}_0$ now generates time translations and the last relation gives Schrödinger's equation. Now, substituting (11) in (6) yields

$$g \simeq 1 + i2(sv_x + rv_y)\frac{M}{\hbar} \tag{12}$$

and hence we obtain, instead of (10),

$$g'^{-1}g \simeq 1 + i\frac{4\Omega AM}{\hbar}. \tag{13}$$

Actually we can write $g'^{-1}g = \exp(i4\Omega AM/\hbar)$ to all orders. The phase shift is then clearly $4\Omega Am/\hbar$, where m is the eigenvalue of M, which is in agreement with (5). There is obviously no Thomas precession in the nonrelativistic case.

To obtain the ten-parameter classical Galilei group from the quantum-mechanical Galilei group, we may substitute $B_j = i\hbar K_j$ in (11) and take the classical limit ($\hbar \to 0$):

$$[P_i, P_j] = 0, \quad [P_i, B_j] = 0, \quad [B_i, B_j] = 0, \tag{14}$$

where B_i correspond to the classical Galilei boosts. Clearly (14) gives $g = g' = 1$ so that there is neither a classical Sagnac effect nor Thomas precession in Newtonian physics.

This treatment makes it clear why it was not possible to obtain the nonrelativistic quantum Sagnac effect geometrically as was done in the relativistic case. This is essentially because, in the nonrelativistic case, the symmetry group of qunatum physics is different from the symmetry group of the geometry, whereas in the relativistic case the former is the covering group of the latter. The latter difference is responsible for the observed phase shift in neutron interference, when the wave function of one of the beams is rotated by 2π radians,[21] but it does not affect the Sagnac effect. We have also generalized the Sagnac effect to an arbitrary space-time group, i.e., a group that contains the space-time translations T_μ and boosts K_i, with $[T_\mu, T_\nu] = 0$, but the other commutation relations for the generators being arbitrary. The Sagnac effect, in this general case, for the gedanken experiment considered, is $2\Omega A([T_x, K_x] + [T_y, K_y])$ to the lowest order.

The above treatment of the Sagnac effect was relative to the inertial frame F with respect to which the apparatus was rotating. We now consider the Sagnac effect from the point of view of observers situated on the rotating apparatus. Consider first when the apparatus is at rest and attach at each point of the apparatus a local inertial frame (a tetrad of orthonomal vectors) such that the corresponding axes at all points are parallel. Let T_μ, $\mu = 0, 1, 2, 3$, be the generators of space-time translations which correspond to these axes, i.e., the action of the group generated by each T_μ is along the integral curves of the vector field whose value at each event is the corresponding vector of the tetrad at that event. Now suppose that the apparatus is given an angular velocity. This gives a new field of instantaneous local inertial frames attached to the rotating frame such that the inertial frame, at each point of the apparatus which has velocity $\vec{v}$ relative to F, is related to F by a Lorentz boost with velocity $\vec{v}$. If T'_μ are the generators of translations corresponding to the boosted frame, then

$$T'_0 = \exp(v^i K_i) T_0 \exp(-v^i K_i) = t^0 T_0 + t^i T_i \tag{15}$$

and

$$\begin{aligned} T'_i &= \exp(v^i K_i) T_i \exp(-v^i K_i) \\ &= T_i + \frac{v^i}{v^2}(t^j - v^j)T_j + v^i t^0 T_0, \end{aligned} \tag{16}$$

where $t^\mu = (1 - v^2/c^2)^{-1/2}(1, v^i)$ is the four-velocity field of the rotating apparatus and we have used the Poincaré Lie algebra.

Let (x'^μ) be the coordinate system obtained by meshing together the local inertial frames attached to the rotating frame, i.e., its coordinate curves are integral curves of the four-vector fields which constitute the field of local inertial frames. Then, for reasons mentioned at the beginning of Sec. II, the time coordinate of this coordinate system must be discontinuous at some point as we go around the axis of rotation, this discontinuity giving the Sagnac effect. The Sagnac phase shift is determined, when the apparatus is rotating rigidly with constant angular velocity, by the element of the Poincaré group

$$s_R = \exp\left(\int_\gamma dx'^\mu T'_\mu\right) \tag{17}$$

in the sense of Ref. 22, where γ is a curve that goes around the submanifold σ, defined in Sec. II, as many times as the sum of the number of times the two beams travel around σ before interfering. The last statement can be verified by transforming to each successive local Lorentz frame along γ by means of an infinitesimal Lorentz boost, disregarding the Thomas precession, and performing an infinitesimal translation in the new frame. It is possible to generalize (17) to the case when the apparatus is nonrigid and/or the angular velocity is not constant by choosing γ to consist of the two classical trajectories on σ from the point of interference and the line segment along the world line of the beam splitter between the points where this world line is met by the classical trajectories.

Now (17) is independent of the wave function and so may be regarded as being due to an external field, namely the Coriolis field. It was pointed out elsewhere[22] that the phase shift due to gravity and gauge fields can be determined, respectively, by Poincaré or gauge group elements associated with curves around which interference takes place. The expression (17) may therefore appear to suggest that the effect of rotation is analogous to a gravitational field. But this is true only locally. This is because, unlike the path dependent, coordinate independent Poincaré group elements associated with gravity, (17) depends on a coordinate system (x'^{μ}) which has the discontinuity mentioned above. Unlike the case of gravity and gauge fields, there are no one-form fields that can be defined on space-time which would determine the group element s_R. The situation, however, is different in the nonrelativistic limit. In this limit there is an absolute time and the discontinuity of the coordinate system (x'^{μ}) disappears. Setting $P_i \equiv (\hbar/i)T_i$ as before and taking the nonrelativistic limit of (15) and (16) yields

$$\overline{P}_0' = \frac{1}{2m}P_iP_i + v^iP_i \tag{18}$$

and

$$P_i' = P_i + Mv^i \ . \tag{19}$$

Then

$$[P_i', P_j'] = -i\hbar M(\partial_i v^j - \partial_j v^i) \ . \tag{20}$$

Taking now the nonrelativistic limit of (17) and choosing γ to be the curve along a constant-time hypersurface, we obtain

$$s_N = \exp\left(\frac{i}{\hbar}\oint_\gamma dx'^iP_i'\right) = \exp\left(\frac{iM}{\hbar}\oint dx'^iv^i\right) . \tag{21}$$

So the Sagnac phase shift $\Delta\phi_N = (m/\hbar)\oint_\gamma dx'^iv^i = (m/\hbar)\oint_\gamma dx^iv^i$. Because lengths are absolute in Newtonian physics, it does not matter whether this integral is computed in the stationary frame F (unprimed coordinates) or in the rotating frame (primed coordinates). But it follows from (19) that in the rotating frame this phase shift can be thought of as arising from minimally coupling a potential v^i into the wave equation. So v^i is similar to the electromagnetic potential and the mass m plays a role analogous to the charge e. Noting that the angular velocity $\vec{\Omega} = \frac{1}{2}\nabla x \vec{v}$ and using Stokes's theorem, $\Delta\phi_N = (2m/\hbar)\int_S \vec{\Omega}\cdot d\vec{s}$, where S is a surface spanned by γ. Thus $2\vec{\Omega}$ is like the magnetic field. But unlike the electromagnetic phase shift which is associated with the compact U(1) group (since charge is quantized), the nonrelativistic Sagnac phase shift is associated with a noncompact one-parameter group generated by M (since mass is not quantized). In the classical limit, both the relativistic and nonrelativistic Sagnac effects give the corresponding Coriolis field.[17]

The above treatment can also be generalized to nonrigid rotation. An example of this is rotating superfluid helium. In such a case a comoving inertial frame can be set up at each point of the rotating fluid and the above treatment will apply. If the angular velocity changes with time, then the commutator $[\overline{P}_0', P_i']$ will have a nonzero component in the bivector space of σ. The tangential acceleration of the particles of the apparatus is like a rotation in an appropriate timelike plane and is therefore like an electric field if the rotation in a spacelike plane is regarded as analogous to a magnetic field. This is similar to how in electromagnetism a change in magnetic flux produces an electric field around it because of Faraday's law.

Even in the presence of gravity, all the considerations of the present section are valid locally because of the principle of equivalence which is valid in relativistic and nonrelativistic physics. The weak and strong principles of equivalence can be formulated in relativistic and nonrelativistic physics, on the classical and quantum levels, in a unified manner, by stating that *in the first-order infinitesimal neighborhood around each point the laws of physics are invariant under the Poincaré group, in the case of general relativity, and the inhomogeneous Galilei group, in the case of Newtonian gravity*. These locally acting groups determine a preferred set of frames called local inertial frames in the following manner: In the case of general relativity, the vectors constituting each local inertial frame are determined by the action of generators T_μ $(\mu = 0,1,2,3)$ of the translation such that $\eta^{\mu\nu}T_\mu T_\nu$ is the Casimir operator of the Poincaré group, where $\eta^{\mu\nu}$ is the Minkowski metric. Clearly, since the last expression is invariant under the action of a Lorentz group of transformations on the translational Lie subalgebra of the Poincaré group, this Lorentz group also represents the freedom of choice of local inertial frames at an event. In nonrelativistic gravity, both in the classical and quantum theories, the spatial axes of the local inertial frame are determined by the generators P_i of spatial translations which are such that $\delta^{ij}P_iP_j$ is the Casimir operator of, respectively, the classical and quantum-mechanical Galilei groups, while the time axis is only restricted by the condition that it is not spacelike, i.e., it is associated with some time translation generator $\overline{P}_0$. Clearly the freedom in choosing such a set of P_i and $\overline{P}_0$ is represented by the six-parameter homogeneous Galilei subgroup which leaves the Casimir operator invariant. The treatment of the quantum Sagnac effect in this section, of course, shows then that this effect is zero, in relativistic and nonrelativistic physics, if and only if the apparatus is nonrotating relative to such a local inertial frame. This provides the group-theoretic meaning of the observation that the Sagnac effect gives a local criterion for rotation in general relativity.[4,3] This observation, as we have just seen, can also be extended to Newtonian gravity which, like general relativity, can be given a curved space-time description.[23] Thus, even though it is commonly stated that in general relativity there is only a Lorentz group of symmetry at each event, it is actually the local validity of the ten-parameter Poincaré group that gives the principle of equivalence, a similar statement being valid in Newtonian gravity. This is beautifully illustrated by the Sagnac effect, when interpreted in the above group-theoretic manner.

It is pointed out elsewhere[24] that Newtonian gravity can be represented by elements of the quantum-mechanical Galilei group associated with closed curves in space-time. Since the nonrelativistic quantum Sagnac effect can also be represented by elements of this group associated with closed curves, as we saw earlier, this suggests that the effect of rotation and gravity can be treated as "gauge fields" corresponding to this group. Thus even through Einstein's conception of rotation being like a gravitational field, which as mentioned in Sec. II, cannot be implemented in general relativity, the effects of gravity and rotation can be treated in a unified manner in nonrelativistic quantum mechanics.

IV. SAGNAC EFFECT IN QUANTUM FLUIDS

The relevance of the Sagnac effect to superfluid helium has been pointed out before.[7] This observation was motivated by the fact that the Sagnac effect, in the nonrelativistic limit, is similar to the electromagnetic phase shift in quantum interference so that the quantization of vortices in helium II can be treated in a manner analogous to the quantization of magentic flux in a superconductor. But the quantization of magnetic flux may be regarded as a special case of the Josephson effect.[25] This can be realized by considering a superconducting ring, with a single Josephson junction, which encloses a magnetic flux F. Then F need not be quantized. However, it is known that there is then a Josephson current across the junction given by $J_0 \sin\Delta\phi$ where the phase difference $\Delta\phi$ across the junction is $(qF/\hbar)$, q being the charge of the Cooper pair. So there is no Josephson current, when $F=\pi n(\hbar/q)$ (n is an integer), which corresponds to equilibrium. But stable equilibrium is when n is even[26] and this corresponds to the quantization of magnetic flux.

This argument suggests that it may similarly be possible to generalize the quantization of vortices to a Josephson effect in superfluid helium, due to rotation, which plays the role analogous to a magnetic field in superconductors. Many authors have considered the A. C. Josehpson effect in superfluid helium due to a pressure difference,[27] but not the Josephson effect due to the Sagnac phase shift that we shall consider now. If a circular toroidal tube containing helium II, with a single Josephson junction, is rotated then there is a phase difference

$$\Delta\phi=\frac{2m\Omega A}{\hbar} \tag{22}$$

across the junction, where m is the mass of the helium atom, A is the area enclosed by the tube, and Ω is its angular velocity. We should therefore expect a Josephson current $I_0\sin(2m\Omega A/\hbar)$ through the junction.

Another well-known Josephson interferometer consists of two superconductors separated by two Josephson junctions.[28] In this case it is possible to apply an electrical potential difference V between the two superconductors as well as a magnetic flux F in the region enclosed by them. The Josephson current through the two Josephson junctions is then

$$J=J_1\sin\frac{e}{\hbar}Vt+J_2\sin\left(\frac{e}{\hbar}Vt+\frac{e}{\hbar}F\right) , \tag{23}$$

which is an alternating current with frequency $eV/\hbar$. The superfluid analog of this experiment will consist of two canals containing helium II filled up to different heights and connected by two Josephson junctions. Then the analogous current that we may expect to flow through the junctions is

$$I=I_1\sin\frac{mgHt}{\hbar}+I_2\sin\left(\frac{mgHt}{\hbar}+\frac{2m}{\hbar}\Omega A\right) , \tag{24}$$

where H is the differences in the heights of the levels of HeII in the two regions, g the acceleration due to gravity and Ω the component of the angular velocity normal to the plane of the apparatus.

Clearly, the above arrangement can be used to determine the local inertial frame in a gravitational field. For, if it accelerates relative to the local inertial frame, then there will be an A.C. current, whereas if it rotates relative to the local inertial frame, then there will be a D.C. current given by (24). Indeed, a local inertial frame can be defined as one in which there is no such current I for any orientation of this apparatus when it is at rest in this frame.

It seems particularly important to perform the above experiments because there does not exist yet a satisfactory microscopic theory for superfluid helium, unlike the case of superconductors. Thus the fact that the Sagnac effect has been detected in superconductors[29] does not by itself imply that such an effect must necessarily exist in superfluid helium. At present superfluid helium is described by an order parameter ψ (a complex function on space-time) which may physically be regarded as an effective wave function of the superfluid. ψ is assumed to satisfy the phenomenological time-dependent Gross-Pitaevskii equation[30]: $i\hbar\partial\psi/\partial t = -(\hbar^2/2m)\nabla^2\psi + g|\psi|^2\psi$. A general relativistic generalization of this equation is

$$\Box\Psi + \frac{m^2c^2}{\hbar^2}\psi = -\frac{2mg}{\hbar^2}|\psi|^2\psi\,, \tag{25}$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$, ∇_μ is the covariant derivative, and $g^{\mu\nu}$ is the inverse of the usual pseudo-Riemannian metric on space-time. This can be realized be noting that in the absence of gravity and in the limit of low energy, (25) yields the Gross-Pitaevskii equation after the rest mass energy is subtracted away, and then using the principle of equivalence in the presence of gravity.

Writing $\psi = \alpha e^{i\phi}$, where α and ϕ are real, and defining $v_\mu = -(\hbar/mc)\partial_\mu\phi$, the real and imaginary parts of (25) are

$$g_{\mu\nu}v^\mu v^\nu = 1 + f(\alpha)\,, \tag{26}$$

where

$$f(\alpha) = \frac{\hbar^2}{m^2c^2}\frac{\Box\alpha}{\alpha} + \frac{2g}{mc^2}\alpha^2 \quad\text{and}\quad \nabla_\mu(\alpha^2 v^\mu) = 0\,.$$

In the WKB approximation, $f(\alpha) \ll 1$. For superfluid helium contained in a toroidal tube with a Josephson junction, the phase difference $\Delta\phi$ across the junction is given by

$$\Delta\phi = \frac{mc}{\hbar}\oint_\gamma v_\mu dx^\mu\,, \tag{27}$$

where γ is a curve that goes around the tube, beginning and ending at the junction. Using (26) and (27), $\Delta\phi$ and hence the Josephson current $I_0 \sin\Delta\phi$ can be obtained for arbitrary gravitational and inertial perturbations, if v^μ is given at the boundary. Equation (27) is like the Aharonov-Bohm effect,[31] with the mass playing a role analogous to the charge.

Suppose that there is interaction between the superfluid and the container such that a component of the superfluid is dragged with the apparatus, i.e., if t^μ is the four-velocity field of the apparatus, then for this component $v^\mu = \Lambda t^\mu$ on a two-dimensional submanifold σ obtained by propagating a closed curve γ going around the tube along the integral curves of v^μ. Also, assume that the apparatus is quasirigid so that in a Fermi-normal coordinate system chosen around the world line of the center of mass of the apparatus $t^\mu = g_{00}^{-1/2}\delta_0^\mu$ to a high degree of approximation. Then neglecting second- and higher-order terms in $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$ ($\eta_{\mu\nu}$ is the Minkowski metric) and $f(\alpha)$, using (26), and choosing γ in (27) to be a curve at constant time, we obtain[32]

$$\Delta\phi = \frac{mc}{\hbar}\oint_\gamma h_{0i}dx^i \tag{28}$$

in this coordinate system.

Suppose that the apparatus is rotating with angular velocity Ω and spacetime curvature is negligible. Then (28) reduces to (22), which is valid even when Ω is changing with time, provided $d\Omega/dt \ll c^2/A$. On the other hand, if the apparatus is not given a rotation, but is subject to a gravitational field, then[33] $h_{0i} = \frac{2}{3}R_{0lim}x^l x^m + O(x^3)$, where $R_{\mu\nu\rho\sigma}$ are the components of the curvature tensor and x^l the spatial components in the chosen Fermi-nomal coordinate system. It follows that the Sagnac effect, in principle, provides an operational procedure for determining the curvature tensor by measuring the Sagnac phase shift with the apparatus in different orientations and in different states of motion. In particular, a Lense-Thirring field such as due to the earth would give a nonzero contribution to (28), which can, in principle, be detected by the Josephson current. Also for a gravitational wave, the components R_{0lim} are as strong as the components R_{0l0m} which cause the tidal forces. This suggests the possibility of detecting gravitational waves by means of the Sagnac effect (28) instead of the tidal forces due to them. The Sagnac effect in neutron interference also, in principle, provides a method of detecting gravitational waves.[34] Specific Josephson interferometers that could act as gravitational wave antennas will be considered elsewhere.[35]

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[11]Since the topology of the events locally simultaneous on the periphery of a rotating disk (a helix) is different from that of a nonrotating disk (a circle), we may even go further than Einstein and conjecture that a gravitational field can change even the local topology of spacetime and not just the metrical structure as assumed by Einstein.

[12]There is no conflict with the uncertainty principle here even if we wish to take the width of the beams into considerations. In the WKB approximation, it is possible to define σ to be a surface such that, at each point on it, the wave vectors of the two waves are tangent to it and the considerations that follow can easily be modified to apply in this case.

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