# Magnetic-moment operator of the relativistic electron

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We consider for the Dirac electron the operator $\vec{m} = (1/2)e\vec{x}\times\dot{\vec{x}}$ in the Heisenberg representation, and prove, by separating the center-of-mass coordinate $\vec{x}_A$ and the relative coordinate $\vec{\xi}$ of the charge with respect to the center of mass (i.e., *Zitterbewegung*), that the orbital angular momentum of the *Zitterbewegung* is equal to the spin with the correct $g$ factor 2. The odd and even parts of $\vec{m}$ and the correct identification of the $\vec{L}$ and $\vec{S}$ operators are discussed.

One of the remarkable properties of the Dirac electron is that the magnetic moment $\vec{m}$ is not proportional to the total angular momentum $\vec{J} = \vec{L} + \vec{S}$, rather it is proportional to a different quantity $\vec{L} + 2\vec{S}$. Furthermore, if one uses Foldy-Wouthuysen (FW) operators, instead of the Dirac operators, the expectation value of $\vec{J}$ does not change, but that of $\vec{m}$ does, hence the question of the correct identification of orbital and spin operators arises. We shall come back to this point.

Many physicists have also wondered about the origin of the spin and the origin of the $g$ factor 2. In particular, Huang,[1] following the notion of "*Zitterbewegung*" of Schrödinger,[2] has shown, by taking special wave packets, that spin may be considered as an orbital angular momentum of the charge performing a *Zitterbewegung*. Another qualitative picture has been suggested[3] for $g > 1$, namely that for a spinning charged spherical shell the electromagnetic field might influence the translational inertia (hence cyclotron frequency $\nu_c$) differently then the rotational inertia (hence the precession frequency $\nu_\rho$).

In previous work,[4,5] we have given a detailed and precise formulation of the notion of *Zitterbewegung* and its implication for the internal group structure and geometry of the electron. We have also found exact solutions of Heisenberg equations of the complete dynamics of the electron including the case of external magnetic fields. The main characteristics of the dynamics of the relativistic electron underlying the present problem are the following.

(i) The electron has three sets of independent dynamical variables: position, velocity, and momentum (and not just two). This manifests itself in an internal degree of freedom, namely the spin. The same is true also, although perhaps not generally recognized, for the classical Lorentz-Dirac equation with the radiation reaction term proportional to $\dddot{x}$, the third derivative of the position.[6]

(ii) The much discussed and disputed problem of what is the correct position operator for the relativistic electron finds a solution by the *fact* that there are indeed *two* distinct position operators, the Dirac operator $\vec{x}$ denoting the position of the charge, and the center-of-mass operator $\vec{X}_A$ which moves according to the laws of relativistic dynamics with a velocity $\vec{p}/H$.[4,5]

Using these ideas and methods we discuss here the magnetic-moment operator

$$\vec{m} = \tfrac{1}{2}e\vec{x}\times\dot{\vec{x}}\,, \tag{1}$$

where $\vec{x}$ is the Dirac position operator of the charge in the Heisenberg representation. Because $\dot{\vec{x}}$ and $\vec{p}$ are independent, $\vec{x}\times\dot{\vec{x}}$ is different from $\vec{x}\times\vec{p}$. We shall prove algebraically that the *even part* of $\vec{m}$ is given by

$$\vec{m}_{\text{even}} = \tfrac{1}{2}e(\vec{L}_E + 2\vec{S}_E)H^{-1}\,. \tag{2}$$

We shall also give the odd part of $\vec{m}$ which is time dependent and contributes in higher-order processes. This result makes the notion that the spin is the orbital angular momentum of the charge around the center of mass more precise, and derives the $g$ factor in the Heisenberg representation, independent of wave packets or expectation values.

The proof of Eq. (2) follows from the even and odd parts of $\vec{x}$ and $\dot{\vec{x}}$. We know that[4] (in units $c = \hbar = 1$)

$$\vec{x} = \vec{a} + H^{-1}\vec{p}t + \tfrac{1}{2}i\vec{\eta}H^{-1}(\equiv \vec{X}_A + \vec{\xi})\,, \tag{3}$$

where $\vec{a} + H^{-1}\vec{p}t$ is even, $\frac{1}{2}i\vec{\eta}H^{-1}$ is odd, $\vec{a}$ is a constant operator, $\vec{X}_A$ is the center-of-mass motion, $\vec{\xi}$ is the "relative" motion, and

$$\vec{\eta} = e^{2iHt}[\vec{\alpha}(0) - H^{-1}\vec{p}]\,, \tag{4}$$

where $\vec{\alpha}(0)$ is the initial value of the operator $\vec{\alpha}(t)$.

We have also for the velocity $\dot{\vec{x}}$

$$\dot{\vec{x}} \equiv \vec{\alpha} = H^{-1}\vec{p} + \vec{\eta}(= \dot{\vec{X}}_A + \dot{\vec{\xi}})\,. \tag{5}$$

Inserting (3) and (5) into (1) we obtain

$$\vec{m} = \tfrac{1}{2}e[(\vec{a} + H^{-1}\vec{p}t)\times H^{-1}\vec{p} + \tfrac{1}{2}i\vec{\eta}H^{-1}\times\vec{\eta} + (\tfrac{1}{2}i\vec{\eta}H^{-1}\times H^{-1}\vec{p}) + (\vec{a} + H^{-1}\vec{p}t)\times\vec{p}]\,, \tag{6}$$

 

where the first line on the right-hand side is even and the second is odd. Thus

$$\vec{m}_E = \tfrac{1}{2}e[\vec{a}\times\vec{p}H^{-1} - \tfrac{1}{2}i(\vec{\eta}\times\vec{\eta})H^{-1}]$$
$$= \tfrac{1}{2}e(\vec{L}_E + 2\vec{S}_E)H^{-1}, \qquad (7)$$

where $\vec{L}_E$ and $\vec{S}_E$ are the even (and constant) parts of $\vec{L}=\vec{x}\times\vec{p}$ and $\vec{S}$, where

$$\vec{S} = \tfrac{1}{4}i\vec{\alpha}\times\vec{\alpha}. \qquad (8)$$

The spin therefore arises from the product of two odd terms which gives an even operator. Note furthermore that although $\vec{m}_E$ is a constant operator, this is not true of the odd part $\vec{m}_0$, hence $\vec{m}$ itself is not a constant of the motion for a free electron. The time dependence of $\vec{m}_0$ is quite complicated but can in principle be written down since we know the time dependence of $\vec{x}$ and $\vec{\alpha}$. We also know the time dependence of these quantities in the presence of an external magnetic field, so that one can examine $\vec{m}_E$ and $\vec{m}_0$ in this case. Equation (6) implies that if $\psi(x,t)$ is any positive- (or negative-) energy solution of the Dirac equation then for the magnetic-moment density $\vec{M}(\vec{x},t)$ we have

$$\vec{M}(\vec{x},t) = \tfrac{1}{2}\vec{x}\times\vec{j}(\vec{x},t) = \frac{e}{2}\vec{x}\times\psi^\dagger(\vec{x},t)\vec{\alpha}\psi(\vec{x},t) = \psi^\dagger\vec{m}\psi, \qquad (9)$$

giving for the total magnetic moment

$$\vec{M} = \int \vec{M}(\vec{x},t)d^3x = \int \psi^*(\vec{x},t)\vec{m}\psi(\vec{x},t)d^3x = \langle\psi|\vec{m}|\psi\rangle$$
$$= \tfrac{1}{2}e\langle\psi|(\vec{L}_E + 2\vec{S}_E)H^{-1}|\psi\rangle = \tfrac{1}{2}e\langle\psi|(\vec{L}+2\vec{S})H^{-1}|\psi\rangle. \qquad (10)$$

Here we used the fact that $\vec{m}_0$ does not have any matrix elements between positive- (or negative-) energy states, and the last identity was shown earlier.[4] Equation (10) identifies $\vec{L}$ and $\vec{S}$ with orbital angular momentum and spin, and shows at the same time in a forceful way that the $g$ factor is 2 and the spin magnetic moment for a moving electron is

$$\vec{\mu} = e\vec{S}H^{-1}. \qquad (11)$$

Finally we remark that if one looks at the expectation value of the total angular momentum alone

$$\vec{J} = \vec{L} + \vec{S} = \vec{L}_E + \vec{S}_E = \vec{L}_{FW} + \vec{S}_{FW},$$

then one cannot discriminate between the identification of $\vec{L}$ (or $\vec{L}_E$) and $\vec{L}_{FW}$ with orbital, and $\vec{S}$ (or $\vec{S}_E$) and $\vec{S}_{FW}$ with spin. However, if we also look at the expectation value of $\vec{m}$, then because

$$\langle\psi|(\vec{L}+2\vec{S})H^{-1}|\psi\rangle \neq \langle\psi|(\vec{L}_{FW}+2\vec{S}_{FW})H^{-1}|\psi\rangle,$$

we seem to come down in favor of the Dirac (or Schrödinger) operators.

In view of the recent extreme accuracy of the experimental determination of the $g$ factor of the electron,[3] it would be interesting to extend the calculations to the case of external magnetic field using the results of Ref. 5 and also to take into account the self-energy effects to encompass the anomalous magnetic moment.

In an arbitrary external vector potential $\vec{A}$ with $H=\vec{\alpha}\cdot\vec{\pi}+m\beta$, $\vec{\pi}=\vec{p}-e\vec{A}$, one can prove from (1) that

$$H\vec{m} + \vec{m}H \equiv e(\vec{x}\times\vec{\pi} + 2\vec{S}), \qquad (12)$$

so that in an energy eigenstate $|\psi\rangle$ we have

$$\vec{M} = \frac{e}{2E}\langle\psi|(\vec{x}\times\vec{\pi}+2\vec{S})|\psi\rangle, \qquad (13)$$

which generalizes Eq. (10).

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