Monte Carlo simulations of lattice models with finite subgroups of SU(3) as gauge groups

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A Monte Carlo study of four-dimensional gauge theory on five crystal-like subgroups of SU(3) is carried out using the Wilson action. All these models are found to have a first-order freezing (order-disorder) phase transition due to the discrete nature of the groups. This transition occurs before the continuum, scaling limit is approached. Hence, it is not possible to use any of these groups to approximate the continuum limit of the SU(3) gauge theory with the Wilson action.

Lattice gauge theories provide a very convenient regularization of the corresponding continuum theories and are also interesting in their own right. On a lattice the gauge group does not have to be continuous, but may just as well be finite.

Monte Carlo simulations of systems having the Z_N finite subgroups of U(1) as gauge groups[1-3] as well as models with finite non-Abelian subgroups of SU(2) as gauge groups[4-8] have been studied in the literature. In either case, the dynamics of the system with continuous gauge groups was well approximated with a finite group for all physically interesting values of the coupling constant. The advantages of using the finite groups in a Monte Carlo computation are then quite relevant. With finite groups it is possible to multiply group elements by a composition table rather than by arithmetic operations, pack more than one lattice variable in a single computer word, and process several variables simultaneously.[9] This allows a considerable reduction in computer time, permitting simulations on larger lattices with good statistics.

SU(3) is the relevant gauge group for the strong interactions. However, direct SU(3) computer simulations are rather slow because of the excessive algebra needed for the group multiplication. The possibility of improving the speed of computations by using finite subgroups of SU(3) as gauge groups is therefore quite intriguing.

For this reason we have performed a Monte Carlo investigation of the properties of a variety of models, having finite subgroups of SU(3) as gauge groups. We recall from studies[2,4-6] of the subgroups of U(1) and SU(2) that systems with a finite gauge group undergo a transition to an ordered, broken-symmetry phase at some critical value β_c of the parameter which controls the strength of the self-interaction. For values of β smaller than β_c the finite-group model generally simulates the corresponding continuous-group system quite well. The relevant question is whether the domain of interesting physical effects corresponds to β lower than β_c or not. For the SU(2) gauge theory the relevant dynamical effects occur near $\beta \approx 2$ (Ref. 10) and the 120-element icosahedral subgroup with $\beta_c \sim 6$ (Refs. 5 and 6) can represent the physics of SU(2) rather well.

In the SU(3) gauge theory the transition to the scaling behavior and other physically interesting effects occur at $\beta \approx 6$ (with the normalization we shall use).[11,12] However, our analysis shows that the model with the largest finite subgroup of SU(3) exhibits a transition at $\beta_c \approx 3.6$.

Hence, the continuum limit of the SU(3) theory cannot be approximated with any of these groups, at least with the Wilson action.

We now describe our analysis in detail.

The finite, non-Abelian subgroups of SU(3) are of two kinds.[13,14] There are crystal-like groups, which can be considered a generalization to SU(3) of the polyhedral subgroups of the rotation groups, and there are the analogs of the dihedral groups.[15] We have studied systems having crystal-like groups of 60, 108, 216, 648, and 1080 elements as gauge groups. The last four groups contain the center of SU(3); the 60-element subgroup is expressed in terms of real matrices and does not contain Z_3. We shall denote these groups by $S(60)$, $\tilde{S}(108)$, $\tilde{S}(216)$, $\tilde{S}(648)$, and $\tilde{S}(1080)$. The corresponding notation in Ref. 13 is $\Sigma(60)$, $\Sigma(36\phi)$, $\Sigma(72\phi)$, $\Sigma(216\phi)$, and $\Sigma(360\phi)$, where ϕ equals 3 or 1 depending on whether or not the center of SU(3) is contained in the group.

The gauge theory is formulated in the manner of Wilson.[16] The link variables U_{ij} are 3×3 matrices and the action is

$$S=\beta \sum_{\Box} [1-\tfrac{1}{6}\,\mathrm{tr}(U_\Box + U_\Box^\dagger)], \quad (1)$$

where $U_\Box$ represents the oriented product of the four U_{ij} on the boundary of an elementary square (plaquette). By Monte Carlo simulations we have measured

$$E(\beta)=\langle 1-\tfrac{1}{6}\,\mathrm{tr}(U_\Box+U_\Box^\dagger)\rangle\,. \qquad (2)$$

We use a generalized "Metropolis" algorithm[17] to equilibrate the lattice. Thermal (hysteresis) cycles and iterations starting from mixed configurations (half-ordered, half-disordered) are used to pinpoint the critical value of β.[1,2,4] We also use multiple storage, masking operations, and parallel processing of variables to speed up the computations.[9]

When the group is too large, it is not practical to store a multiplication table in the memory of the computer. We have implemented the group operation for $\tilde{S}(648)$ and $\tilde{S}(1080)$ in the following way: First, all group elements are written as products

$$U=AB\,, \qquad (3)$$

where A belongs to an Abelian subgroup of the group and the B's are fixed representatives of the cofactor sets. Then, the result of a group multiplication is reconstructed in terms of three tables of lower size, which encode the following mappings:

$$(A,B)\to(A',B'),\quad \text{with } A'B'=BA\,, \qquad (4)$$

$$(B,B')\to(A,B''),\quad \text{with } BB'=AB''\,, \qquad (5)$$

$$(A,A')\to(A''),\quad \text{with } A''=AA'\,. \qquad (6)$$

A generic product $ABA'B'$ can then be reexpressed in the form $A''B''$ by the sequence of operations

$$A(BA')B'=AA''(B''B')=(AA''A''')B'''=A''''B''' \qquad (7)$$

which are all defined by the mappings of Eqs. (4)–(6).

Our results are illustrated in Figs. 1–4 and summarized in Tables I and II. Figures 1–3 reproduce the values found for $E(\beta)$ in the simulation of thermal cycles for the groups $S(60)$, $\tilde{S}(648)$, and $\tilde{S}(1080)$. All the cycles display wide hysteresis loops, which signal a phase transition. The solid lines represent strong-coupling expansions appropriate for the group under consideration, whereas

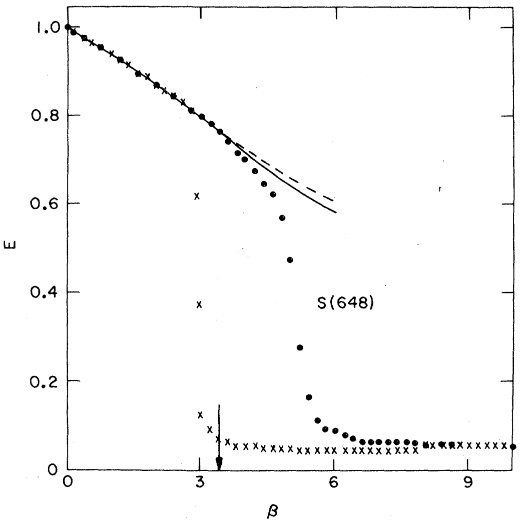

FIG. 2. Thermal cycle for the $\tilde{S}(648)$ model on a $4^3\times 8$ lattice.

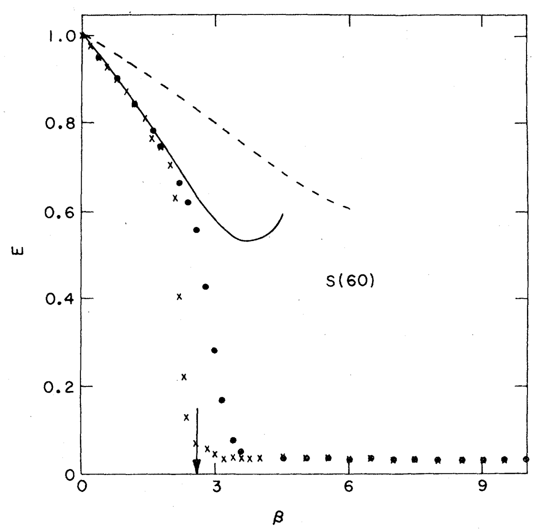

FIG. 1. Thermal cycle for the $S(60)$ model on a $4^3\times 8$ lattice.

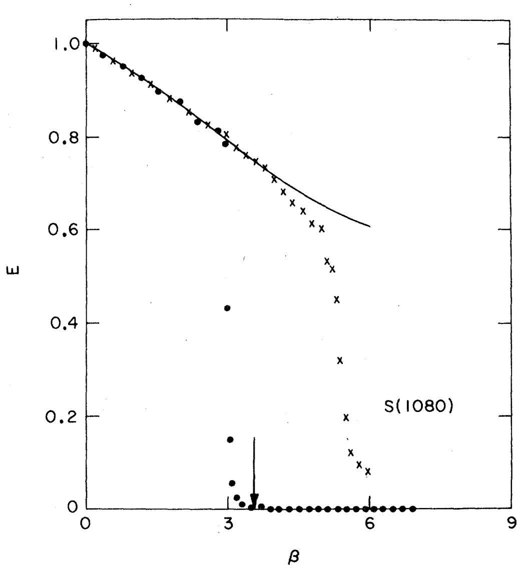

FIG. 3. Thermal cycle for the $\tilde{S}(1080)$ model on a 4^4 lattice.

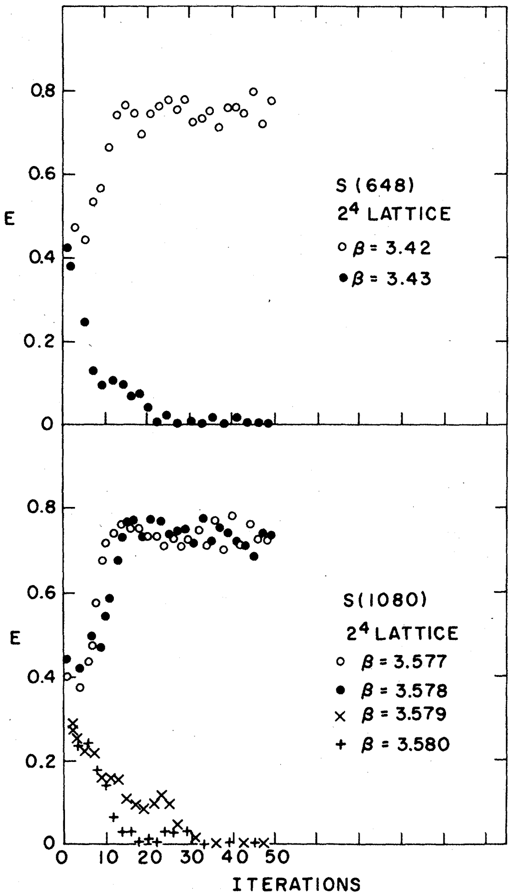

FIG. 4. Mixed phase runs for $S(648)$ and $S(1080)$.

the broken line gives the analogous expansion for the SU(3) system. In Fig. 1 the disagreement between the two expansions even to order β is due to the real nature of the group. In Fig. 3 only the results of the strong-coupling expansion for the $\tilde{S}(1080)$ model are shown, because this curve is indistinguishable from the one for the SU(3) system within the range of the plot.

In Fig. 4 we illustrate the results of mixed phase runs for the $\tilde{S}(648)$ and $\tilde{S}(1080)$ systems. Mixed phase runs of this type were performed for all the models considered, to determine the nature and location of the phase transition.

In Table I we present some relevant information about the subgroups of SU(3) as well as the values we find for β_c. The phase transitions appear to be of the first order for all the models, with the possible exception of the $S(60)$ model, where the results of the mixed phase runs give only a marginal indication of a discontinuous change in $E(\beta)$. The second column in Table I gives the value $E_{\min}$ of $1-\frac{1}{6}\,\mathrm{tr}(U_\Box + U_\Box^\dagger)$ for the neighbors of the identity. N is the number of these neighbors, n the number of upgrades done for each U_{ij} variable before proceeding to the next link in a Monte Carlo iteration, and β_c is the critical coupling.

In Table II, we reproduce the strong-coupling expansions for $E(\beta)$ to order β^5 for the models with finite gauge groups and the SU(3) system.

From our results it is clear that even the model with the largest subgroup of SU(3) is not adequate to simulate the SU(3) system in the interesting physical region.

The reason why β_c remains relatively small although the highest subgroups of SU(3) have a very large number of elements can be traced to the fact that the action gap, i.e., the value of $1-\frac{1}{6}\,\mathrm{Tr}(U_\Box + U_\Box^\dagger)$ for the neighbors of the identity, never becomes sufficiently small. This parameter sets the scale for the critical temperature $T_c = 1/\beta_c$ where the transition occurs. Although entropy effects due to the large number of neighbors of the identity tend to reduce T_c, none of the subgroups have a

TABLE I. Some parameters of the various groups studied.

Subgroup	$E_{\min}$	N	n	β_c
$\tilde{S}(60)$	$\frac{5-\sqrt{5}}{6} \simeq 0.4607$	12	6	2.7 ± 0.2
$\tilde{S}(108)$	$\frac{2}{3}$	18	9	2.5 ± 0.2
$\tilde{S}(216)$	$\frac{2}{3}$	54	20	3.2 ± 0.1
$\tilde{S}(648)$	$1-\frac{1}{3}\left(\cos\frac{\pi}{9}+\cos\frac{2\pi}{9}\right) \simeq 0.4314$	24	12	3.43 ± 0.02
$\tilde{S}(1080)$	$\frac{5-\sqrt{5}}{6} \simeq 0.4607$	72	20	3.58 ± 0.02

TABLE II. $O(\beta^5)$ strong-coupling expansion for $E(\beta)$ for the groups studied and for SU(3).

Gauge group	$E(\beta)$
$S(60)$	$1-\frac{\beta}{9}-\frac{\beta^2}{54}+\frac{\beta^4}{1454}+\frac{119\beta^5}{1\,180\,980}$
$\tilde{S}(108)$	$1-\frac{\beta}{18}-\frac{\beta^2}{216}-\frac{\beta^3}{1296}-\frac{5\beta^4}{31\,104}-\frac{413\beta^5}{18\,895\,680}$
$\tilde{S}(216)$	$1-\frac{\beta}{18}-\frac{\beta^2}{216}-\frac{139\beta^5}{9\,447\,840}$
$\tilde{S}(648)$	$1-\frac{\beta}{18}-\frac{\beta^2}{216}+\frac{5\beta^4}{93\,312}+\frac{13\beta^5}{7\,558\,272}$
$\tilde{S}(1080)$	$1-\frac{\beta}{18}-\frac{\beta^2}{216}+\frac{5\beta^4}{93\,312}+\frac{373\beta^5}{75\,582\,720}$
SU(3)	$1-\frac{\beta}{18}-\frac{\beta^2}{216}+\frac{5\beta^4}{93\,312}+\frac{400\beta^5}{75\,582\,720}$

small enough action gap to make β_c exceed the value $\beta \approx 6$, where the strong coupling-weak coupling crossover takes place.

Recently it has been found in the SU(2) case that the position of this crossover depends markedly on the specific form of the action.[7] If a different choice of action for the SU(3) model could lower the crossover value of β then systems with finite subgroups of SU(3) as gauge groups might be useful for physically relevant computations.

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