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Solutions with both magnetic and electric charges in a $(4+N)$-dimensional geometric field theory

Toshihiko Tajima
Toyama Technical College, 13 Hongo-machi, Toyama 930-11, Japan
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We study classical solutions with both magnetic and electric charges in a $(4+N)$-dimensional geometric field theory. Then we choose the group SO(3,1) for the internal symmetry and try to find the solutions under the ansatz that they are invariant with respect to the diagonal subgroup of $SO(3,1)_M \otimes SO(3,1)_I$ in order to get relativistically covariant solutions, where $SO(3,1)_M$ and $SO(3,1)_I$ represent SO(3,1) for rotations of the Minkowski space and for the internal symmetry, respectively. Thus we obtain solutions having a covariant expression for the electromagnetic field associated with magnetic and electric charges moving with constant velocity. Particles with both magnetic and electric charges are usually referred to as dyons. Hence our solutions give a specific realization of dyons in a uniform motion. As a result of the incorporation of the internal symmetry $SO(3,1)_I$, furthermore, it is found that they have a new pole in addition to the magnetic and electric charges. The gauge fields associated with this pole contribute negative energy. Then there exists a relation among the three kinds of charges, which gives a lower bound to the absolute value of the electric charge. Finally, we discuss the stability of our solutions.

I. INTRODUCTION

It is well known that an SO(3) gauge theory with a Higgs triplet exhibits classical solutions with magnetic monopoles, namely, 't Hooft-Polyakov solutions.[1,2] Furthermore, Julia and Zee found classical solutions having both magnetic and electric charges in this theory.[3] These solutions are of great interest from the viewpoint that they represent specific realizations of particles with magnetic monopoles suggested by Dirac[4] and dyons which were discussed by Schwinger[5] and Zwanziger[6] as particles with both magnetic and electric charges, respectively.

In this paper we try to find classical magnetic monopole or dyon solutions in a $(4+N)$-dimensional geometric field theory developed by Kaluza *et al.*[7,8] This theory has recently acquired renewed interest in providing a unified gauge principle that gives rise to gravity and a Lie group as an internal symmetry.[9] That is, it is shown that an $(4+N)$-dimensional metric tensor naturally provides the gravitational field $g_{\mu\nu}(x)$, N gauge fields $A_\mu^a(x)$ $(a=5,6,\ldots,4+N)$ for a Lie group, and a metric tensor $g_{ab}(x)$ $(a=5,6,\ldots,4+N)$ which plays the counterpart of the Higgs field. In order to find classical solutions, we will follow the technique used by 't Hooft and Polyakov. However, we will then choose the group SO(3,1) for the internal symmetry instead of SO(3). The 't Hooft-Polyakov solutions are constructed under the ansatz that they are invariant with respect to the diagonal subgroup $SO(3)_S+SO(3)_I$ of $SO(3)_S \otimes SO(3)_I$,[2] where $SO(3)_S$ and $SO(3)_I$ represent SO(3) for spatial rotations and for the internal symmetry, respectively. Therefore, their solutions have the expression written only in terms of the spatial coordinates whose origin, i.e., the center of a magnetic monopole, is at rest. When we try to get relativistically covariant solutions, we find that one of the possible ways is to replace $SO(3)_I$ by $SO(3,1)_I$ and construct the solutions invariant with respect to the diagonal subgroup of $SO(3,1)_M \otimes SO(3,1)_I$ where $SO(3,1)_M$ and $SO(3,1)_I$ represent SO(3,1) for rotations of the Minkowski space and for the internal symmetry, respectively. Then the problem is how to understand the symmetry $SO(3,1)_I$ in the actual world. Although we now have no definite answer, one of our speculations is that it might be the symmetry which gives rise to the symmetry $SU(2)_L \otimes SU(2)_R$ (Ref. 10) because they are locally equivalent to each other. As a result of the incorporation of the symmetry $SO(3,1)_I$, we can start with assuming a covariant expression for classical solutions.

In this paper we present the form of the classical solutions for a dyon moving with constant velocity.

Then it is found that they have a new pole in addition to the magnetic and electric charges. The gauge fields associated with this pole contribute with negative energy. This reflects the fact that the metric of the internal space is no longer definitely spacelike because of the symmetry $SO(3,1)_I$. Furthermore, we find that the gauge fields associated with this new charge play an essential role in providing the localized structure of our solutions. At the same time a relation among the three kinds of charges is derived. This gives a lower bound to the absolute value of the electric charge.

We start with presenting the Lagrangian of $(4+N)$-dimensional geometric field theory in Sec. II. In Sec. III, relativistically covariant solutions of dyons are obtained in the theory with the internal symmetry $SO(3,1)_I$. Their Euler-Lagrange equations are investigated in Sec. IV. Then we show that the equations indeed exhibit spatially localized solutions. In the concluding remarks in Sec. V, we discuss the stability of our solutions, the mass and the electric charge of dyons.

II. LAGRANGIAN OF THE $(4+N)$-DIMENSIONAL GEOMETRIC FIELD THEORY

The $(4+N)$-dimensional geometric field theory starts with geometrics of a $(4+N)$-dimensional space with coordinates z^A $(A=1,2,\ldots,4+N)$ and a metric tensor γ_{AB} (z). After a few basic assumptions are introduced,[7,8] the metric tensor can be described in terms of three kinds of fields: the gravitational field $g_{\mu\nu}(x)$ $(\mu,\nu=1,2,3,4,)$, gauge fields $A_\mu^a(x)$ $(a=5,6,\ldots,4+N)$, and a metric tensor $g_{ab}(x)$ $(a,b=5,6,\ldots,4+N)$ in the N-dimensional internal space. The Lagrangian of these fields is given by the invariant integration of a $(4+N)$-dimensional curvature scalar $R_{(4+N)}$. In this paper we neglect the effect of gravitation. Then the Lagrangian is[8]

$$L=\int d^4x\,\mathscr{L}(x)\ , \tag{2.1}$$

$$\mathscr{L}(x)=[\det(g_{ab})]^{1/2}\left[\frac{e^2\kappa^2}{16\pi G}\mathscr{L}_V+\frac{1}{16\pi G}\mathscr{L}_{S1}+\frac{\kappa^{-2}}{16\pi G}(\mathscr{L}_{S2}+\kappa^2\lambda)\right]\ , \tag{2.2}$$

$$\mathscr{L}_V=-\tfrac{1}{4}F_{\mu\nu}^aF_{\mu\nu}^b g_{ab}\ , \tag{2.3}$$

$$\mathscr{L}_{S1}=-\{\tfrac{1}{4}g^{ab}g^{cd}[(D_\mu g_{ac})(D_\mu g_{bd})-(D_\mu g_{ab})(D_\mu g_{cd})]-\tfrac{1}{2}(D_\mu g^{ab})(D_\mu g_{ab})-\tfrac{1}{2}g^{ab}D_\mu D_\mu g_{ab}\}\ , \tag{2.4}$$

$$\mathscr{L}_{S2}=-(\tfrac{1}{2}f_{ad}^c f_{bc}^d g^{ab}+\tfrac{1}{4}f_{ac}^e f_{bd}^f g^{ab}g^{cd}g_{ef})\ , \tag{2.5}$$

where f_{bc}^a are the structure constants of the group of the N-dimensional space, G is the gravitational constant, e is a gauge coupling constant, and κ and λ are constants with dimensions of length and $(\text{length})^{-2}$, respectively. Here the gauge field strength is defined as

$$F_{\mu\nu}^a=\partial_\mu A_\nu^a-\partial_\nu A_\mu^a+ef_{bc}^a A_\mu^b A_\nu^c\ , \tag{2.6}$$

D_μ indicates a covariant derivative

$$D_\mu g_{ab}=\partial_\mu g_{ab}-eA_\mu^d(f_{da}^c g_{cb}+f_{db}^c g_{ac})\ , \tag{2.7}$$

$$D_\mu g^{ab}=\partial_\mu g^{ab}+eA_\mu^d(f_{dc}^a g^{cb}+f_{dc}^b g^{ac})\ , \tag{2.8}$$

and g_{ab} and g^{bc} are related by

$$g_{ab}g^{bc}=\delta_{ac}\ . \tag{2.9}$$

In this theory $g^{ab}(x)$ is a Lorentz scalar and plays the role of the Higgs field. The potential of this field is given from (2.2) as

$$V(x)=-\frac{\kappa^{-2}}{16\pi G}[\det(g_{ab})]^{1/2}[\mathscr{L}_{S2}(x)+\kappa^2\lambda]\ . \tag{2.10}$$

The first term in $\mathscr{L}$ is the Yang-Mills Lagrangian for the gauge fields A_μ^a if the internal space is flat $(g_{ab}=\delta_{ab})$ and we choose the normalization $e^2\kappa^2/16\pi G=1$. But, as will be shown in Sec. III, the potential (2.10) has a stable minimum at $g=g_0\neq 1$ in the metric $g^{ab}=g\delta_{ab}$. Therefore, we should renormalize g_{ab} so that the flat metric $g_{ab}=\delta_{ab}$ is a stable minimum by transforming g^{ab} to $g_0 g^{ab}$.

III. COVARIANT SOLUTIONS IN THE THEORY WITH THE INTERNAL SYMMETRY SO(3,1)

There are six generators $M_{\alpha\beta}$ ($M_{\alpha\beta}=-M_{\alpha\beta}$ and $\alpha,\beta=1,2,3,4$) in SO(3,1). The structure constants f_{ab}^{c} $(=f_{\alpha\beta,\lambda\rho}^{\eta\xi})$ are given as

$$\begin{aligned} f_{\alpha\beta,\lambda\rho}^{\eta\xi} = \ & \delta_{\alpha\lambda}(\delta_{\eta\beta}\delta_{\xi\rho}-\delta_{\eta\rho}\delta_{\xi\beta}) \\ & +\delta_{\beta\rho}(\delta_{\eta\alpha}\delta_{\xi\lambda}-\delta_{\eta\lambda}\delta_{\xi\alpha}) \\ & -\delta_{\alpha\rho}(\delta_{\eta\beta}\delta_{\xi\lambda}-\delta_{\eta\lambda}\delta_{\xi\beta}) \\ & +\delta_{\beta\lambda}(\delta_{\eta\alpha}\delta_{\xi\rho}-\delta_{\eta\rho}\delta_{\xi\alpha}) \ . \end{aligned} \tag{3.1}$$

In the following a,b,. . ., indicating the indices of six-dimensional internal space, are identified with the combinations (α,β) (η,ξ), ...

To normalize the coefficient of the Yang-Mills Lagrangian in (2.2), we first take $g^{ab}=g\delta_{ab}$ $(g_{ab}=g^{-1}\delta_{ab})$. Then the potential (2.10) is

$$V(x)=-\frac{\kappa^{-2}}{16\pi G\,|\,g\,|^{3}}(6g+\kappa^2\lambda) \ , \tag{3.2}$$

which is described in Fig. 1. If λ is negative, this potential has a stable minimum at $g_0=-\kappa^2\lambda/4$. Hence we assume that λ is negative. We renormalize g^{ab} by transforming g^{ab} to $g_0 g^{ab}$ in (2.2) so that the potential takes the minimum at the flat internal metric $g_{ab}=\delta_{ab}$. Then the Lagrangian (2.2) becomes

$$\mathscr{L}(x)=[\det(g_{ab})]^{1/2}\left[\frac{e^2\kappa^2}{16\pi G g_0{}^4}\mathscr{L}_V+\frac{1}{16\pi G g_0{}^3}\mathscr{L}_{S1}\right] -V(x) \ , \tag{3.3}$$

$$V(x)=-\frac{\kappa^{-2}}{16\pi G g_0{}^2}\{\,[\det(g_{ab})]^{1/2}(\mathscr{L}_{S2}-4)-2\,\} \ , \tag{3.4}$$

where the potential is adjusted to be zero at $g^{ab}=\delta_{ab}$. From (3.3) one finds that we should take the normalization

$$\frac{e^2\kappa^2}{16\pi G g_0{}^4}=1 \ . \tag{3.5}$$

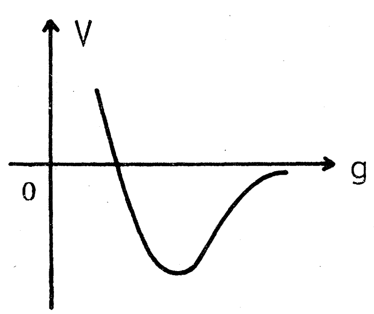

FIG. 1. The potential (3.2) vs g.

Before giving the explicit form of solutions, we provide the relative coordinate x'_μ defined as $x'_\mu=x_\mu-X_\mu(\tau)$ to express the differences of the coordinates between a point (x_μ) of an observer and a center (X_μ) of a localized solution with a proper time τ. In this paper we assume *a priori* $x'^2=0$. Then we get the following rule[11] for the differentiation of a function $f(x')$: $\partial_\mu f(x')=(\delta_{\mu\nu}-\sigma^{-1}x'_\mu U_\nu)\,\partial'_\nu f(x')$ where $\partial_\mu=\partial/\partial x_\mu$, $\partial'_\mu=\partial/\partial x'_\mu$, U_μ is a four-dimensional velocity of the center ($U_\mu=dX_\mu/d\tau$ and $U^2=-1$), and $\sigma=(x\cdot U)$. In the following we look for solutions whose center moves with constant velocity. Hence we assume that U_μ is independent of x_μ. In the rest frame of the center,[11]

$$x'_k=-r_k, \ \ x'_4=-ir \ (r=|\,\vec{r}\,|\,) \text{ and } \sigma=r \ , \tag{3.6}$$

where $\vec{r}$ is the spatial vector pointing to the point (x_μ) from the center.

As was explained in the Introduction, we will consider the solutions invariant with respect to the diagonal subgroup of $SO(3,1)_M \otimes SO(3,1)_I$. Then, in terms of the field $h_{ab}(x)$ defined as

$$g_{ab}(x)=\delta_{ab}+h_{ab}(x) \ , \tag{3.7}$$

the general form of $g_{ab}(x)$ is written as

$$\begin{aligned} h_{ab}(x)= & -h_1(\sigma)(x'_\eta U_\xi-x'_\xi U_\eta)(x'_\lambda U_\rho-x'_\rho U_\lambda)/\sigma^2 \\ & -h_2(\sigma)\epsilon_{\eta\xi\alpha\beta}x'_\alpha U_\beta\epsilon_{\lambda\rho\gamma\delta}x'_\gamma U_\delta/\sigma^2 \ , \end{aligned} \tag{3.8}$$

where $a=(\eta,\xi)$ and $b=(\lambda,\rho)$. In this paper we investigate the case that either h_1 or h_2 is equal to zero only for the purpose of making our solutions as simple as possible. Because the two terms in (3.8) differ only in the way of assigning the indices a and b, the form of our solutions does not depend essentially on which term we leave in (3.8). Hence, hereafter we choose the case $h_2=0$ and rewrite (3.8) as

$$h_{ab}(x)=-h(\sigma)(x'_\eta U_\xi-x'_\xi U_\eta)(x'_\lambda U_\rho-x'_\rho U_\lambda)/\sigma^2 \ , \tag{3.9}$$

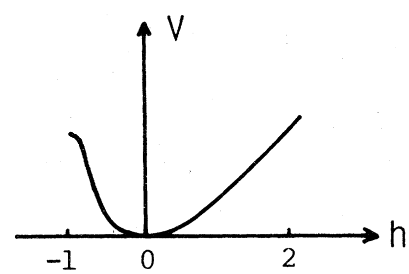

FIG. 2. The potential (3.10) vs h.

where $a=(\eta,\xi)$ and $b=(\lambda,\rho)$. From (2.9), $h^{ab}(x) = -h_{ab}(x)/[1+h(\sigma)]$. Then the potential becomes

$$V(x)=\frac{\kappa^{-2}}{16\pi G g_o{}^2}[(1+h)^{1/2}(h-2)+2] , \quad (3.10)$$

which is plotted in Fig. 2. This has a stable minimum at $h=0$, where its value is adjusted to be zero. This means that we should take the boundary condition that $h(\sigma)\rightarrow 0$ sufficiently fast as $\sigma\rightarrow\infty$ (which corresponds to $r\rightarrow\infty$ in the rest frame of the center) in order to ensure a finite-energy solution. It is quite consistent with a point of view that the internal space should be nearly flat far from the center.

Next we write the general expression for $A^a_\mu(x)$ with $a=(\lambda,\rho)$:

$$\begin{aligned} A^a_\mu(x)=&(\delta_{\mu\lambda}x'_\rho-\delta_{\mu\rho}x'_\lambda)\frac{y_1(\sigma)}{e\sigma^2}+(\delta_{\mu\lambda}U_\rho-\delta_{\mu\rho}U_\lambda)\frac{y_2(\sigma)}{e\sigma} \\ &+i\epsilon_{\lambda\rho\alpha\beta}x'_\alpha U_\beta\left[x'_\alpha\frac{q_1(\sigma)}{e\sigma^3}+U_\mu\frac{q_2(\sigma)}{e\sigma^2}\right]+(x'_\lambda U_\rho-x'_\rho U_\lambda)\left[x'_\mu\frac{p_1(\sigma)}{e\sigma^3}+U_\mu\frac{p_2(\sigma)}{e\sigma^2}\right] \\ &+\epsilon_{\mu\lambda\rho\alpha}\left[x'_\alpha\frac{z_1(\sigma)}{e\sigma^2}+U_\alpha\frac{z_2(\sigma)}{e\sigma}\right] . \end{aligned} \quad (3.11)$$

Writing the Lagrangian $L=(\pi/2)\int d\tau\int\sigma^2 d\sigma \times\mathscr{L}(\sigma)$ in terms of $h(\sigma)$, $y_1(\sigma)$, $y_2(\sigma)$, $q_1(\sigma)$, etc., and requiring it to be stationary with respect to variations of them, we get the Euler-Lagrange equations. Because the equations are very long and complicated, we do not write them here. In order to show the asymptotic form of $F^a_{\mu\nu}$ at the infinity of σ, we here, however, give the boundary conditions at infinity which are derived from requiring that the equations have the solutions localized around the center. These conditions are

$$\begin{aligned} &y_1(\sigma)\rightarrow 1,\ \ y_2(\sigma)\rightarrow 1, \\ &\sigma^2[q_1(\sigma)/\sigma]'\rightarrow 0,\ \ \sigma^2[p_1(\sigma)/\sigma]'\rightarrow 0, \\ &z_1(\sigma)\rightarrow 0 \text{ and } z_2(\sigma)\rightarrow 0 \ \text{ as } \sigma\rightarrow\infty \end{aligned} \quad (3.12)$$

and

$$3\{1-\sigma^2[p_2(\sigma)/\sigma]'\}^2+\sigma^4[q_2(\sigma)/\sigma]'^2+1\rightarrow 0 \quad \text{as } \sigma\rightarrow\infty , \quad (3.13)$$

where $f'(\sigma)=df/d\sigma$. These conditions suggest the very simple model that $y_1(\sigma)=y_2(\sigma)$ and $q_1(\sigma) =p_1(\sigma)=z_1(\sigma)=z_2(\sigma)=0$. In fact we confirmed that there is no loss of essentialities in this model in comparison with the general case. Therefore, in the next section we give the full expressions of the Lagrangian and the Euler-Lagrange equations in this simple model.

We conclude this section by presenting the asymptotic form of $F^a_{\mu\nu}$ at infinity under the conditions (3.12) and (3.13). In the rest frame of the center (3.6), only the following components of $F^a_{\mu\nu}$ survive as $r\rightarrow\infty$:

$$F^{(l,m)}_{ij}(\vec{r})\rightarrow e^{-1}y_2(\infty)\epsilon_{ijk}r_k\epsilon_{lmn}r_n/r^4 , \quad (3.14)$$

$$F^{(l,m)}_{i4}(\vec{r})\rightarrow i\mathscr{Q}r_i\epsilon_{lmn}r_n/r^4 , \quad (3.15)$$

$$F^{(l,4)}_{i4}(\vec{r})\rightarrow\mathscr{P}r_ir_l/r^4 , \quad (3.16)$$

where $e\mathscr{Q}=-\sigma^2[q_2(\sigma)/\sigma]'_{\sigma\rightarrow\infty}$, $e\mathscr{P}=1-\sigma^2[p_2(\sigma)/\sigma]'_{\sigma\rightarrow\infty}$, and $y_2(\infty)=1$ from (3.12). Equations (3.14) and (3.15) show that the components $F^{(l,m)}_{ij}$ and $F^{(l,m)}_{i4}$ associated with an $SO(3,1)_I$ rotation axis $\epsilon_{lmn}r_n/r$ are just the electromagnetic tensors of monopoles with magnetic charge e^{-1} and with electric charge $\mathscr{Q}$, respectively. On the other hand, we have another component $F^{(l,4)}_{i4}$ which is associated with an axis $(r_l/r)U_4$. From (3.16) this has another charge $\mathscr{P}$ like an electric one. The components $(l,4)$, $l=1,2,3$, of the internal space have a timelike metric so that the gauge fields associated with the new charge contribute negative energy.

In the frame of the center moving with constant velocity, a rotation axis $\epsilon_{\lambda\rho\alpha\beta}x'_\alpha U_\beta/\sigma$ corresponds to the axis $\epsilon_{lmn}r_n/r$ in the rest frame. Thus the electromagnetic tensor is defined as

$$\mathscr{F}^{\text{em}}_{\mu\nu}(x)=(-i)\sigma^{-1}\epsilon_{\lambda\rho\alpha\beta}x'_\alpha U_\beta F^{(\lambda,\rho)}_{\mu\nu} . \quad (3.17)$$

In the limit $\sigma\rightarrow\infty$ we have the following asymptotic form:

$$\begin{aligned}\mathscr{F}^{\text{em}}_{\mu\nu}(x)\rightarrow&(-i)e^{-1}\epsilon_{\mu\nu\alpha\beta}x'_\alpha U_\beta/\sigma^3 \\ &+\mathscr{Q}(x'_\mu U_\nu-x'_\nu U_\mu)/\sigma^3 .\end{aligned} \quad (3.18)$$

As we expect this is the covariant expression of the electromagnetic tensor for a dyon in a uniform motion.[11]

As to the component of the field strength associated with the charge $\mathscr{P}$, we note that the rotation axis $(x'_\lambda U_\rho - x'_\rho U_\lambda)/\sigma$ corresponds to the axis $(r_l/r)\ U_4$ in the rest frame. With the definition

$$\mathscr{F}^P_{\mu\nu}(x) = \sigma^{-1}(x'_\lambda U_\rho - x'_\rho U_\lambda) F^{(\lambda,\rho)}_{\mu\nu}(x) \ , \tag{3.19}$$

we obtain the asymptotic form

$$\mathscr{F}^P_{\mu\nu}(x) \rightarrow \mathscr{P}(x'_\mu U_\nu - x'_\nu U_\mu)/\sigma^3 \ . \tag{3.20}$$

IV. EULER-LAGRANGE EQUATIONS

We derive the Euler-Lagrange equations in the simplified case $y_1(\sigma) = y_2(\sigma)$ and $q_1(\sigma) = p_1(\sigma) = z_1(\sigma) = z_2(\sigma) = 0$ given in Sec. III. We use here the functions $K(\sigma) = 1 - y_2(\sigma)$, $Q(\sigma) = q_2(\sigma)$, and $P(\sigma) = 1 + p_2(\sigma)$. From the asymptotic form of $F^a_{\mu\nu}$ given in (3.14)–(3.16), one finds that these functions describe the spatial structure of the charges. In this section we work in the rest frame of the center. There the Lagrangian is written as

$$L = (\pi/2)\int d\tau \int r^2 dr\, \mathscr{L}(r) \ , \tag{4.1}$$

$$\mathscr{L}(r) = (1+h)^{1/2}\left[\mathscr{L}_V(r) + \frac{1}{16\pi G g_0{}^3}\mathscr{L}_{S1}(r)\right] - V(r) \ , \tag{4.2}$$

$$\mathscr{L}_V(r) = -\frac{1}{4e^2 r^4}[4r^4(K/r)'^2 + 8r^2K(K/r)' + 2(1+h)r^4(K/r - P/r)'^2 - 2r^4(Q/r)'^2$$
$$+4K^2(P^2-Q^2) - 2K^3(4P-3K) + 2] \ , \tag{4.3}$$

$$\mathscr{L}_{S1}(r) = -\frac{1}{4r^2}\frac{1}{(1+h)}[(rh')^2 + h^2K^2] \ , \tag{4.4}$$

where $V(r)$ is given in (3.10) and $f'(r) = df/dr$. Requiring the Lagrangian to be stationary with respect to variations of h, K, Q, and P, we get the following four second-order coupled differential equations:

$$\frac{2e^2}{16\pi G g_0{}^3}[r^2h'/(1+h)^{1/2}]' = -\frac{2e^2r^2}{(1+h)^{1/2}}\mathscr{L}_V + 2(1+h)^{1/2}r^2(K/r - P/r)'^2$$
$$+\frac{e^2}{16\pi G g_0{}^2}\left[\frac{1}{2g_0}\frac{3h+4}{(1+h)^{3/2}}hK^2 + 6\kappa^{-2}\frac{r^2h}{(1+h)^{1/2}}\right] \ , \tag{4.5}$$

$$8r[r^2(1+h)^{1/2}(K/r)']' + 8r[(1+h)^{1/2}K]' = (1+h)^{1/2}\left[8(P^2-Q^2)K - 16(P-K)K^2 + \frac{e^2r^2}{16\pi G g_0{}^3}\frac{2h^2}{1+h}K\right] \ , \tag{4.6}$$

$$r[r^2(1+h)^{1/2}(Q/r)']' = 2(1+h)^{1/2}K^2Q \ , \tag{4.7}$$

$$r[r^2(1+h)^{3/2}(P/r - K/r)']' = 2(1+h)^{1/2}(P-K)K^2 \ . \tag{4.8}$$

In order to see that the Euler-Lagrange equations exhibit the localized solutions, we solve them in the first approximation at large r. Let us first look at Eq. (4.5) and assume that

$$K(r) \rightarrow 0 \text{ exponentially } as\ r \rightarrow \infty \ . \tag{4.9}$$

Then if we set the boundary condition

$$3r^4[P(r)/r]'^2 - r^4[Q(r)/r]'^2 + 1 \rightarrow 0$$
$$\text{exponentially } as\ r \rightarrow \infty \ , \tag{4.10}$$

the asymptotic form of Eq. (4.5) becomes

$$h(r)'' \sim 3g_0\kappa^{-2}h(r) \ . \tag{4.11}$$

Here we wish to note that (4.9) and (4.10) correspond to the boundary conditions (3.12) and (3.13) assumed in the general case, respectively. From Eq. (4.11), Eq. (4.5) gives the following asymptotic form to $h(r)$:

$$h(r)\to Ce^{-\mu r} \text{ as } r\to\infty , \tag{4.12}$$

where C is a constant and

$$\mu=(3g_0)^{1/2}\kappa^{-1} . \tag{4.13}$$

We next proceed to Eqs. (4.7) and (4.8). Using (4.12) and the assumption (4.9), we obtain

$$Q(r)\to M_Q r+e\mathscr{Q}[1-h(r)/2\mu r] \text{ as } r\to\infty , \tag{4.14}$$

$$P(r)-K(r)\to M_P r+e\mathscr{P}[1-h(r)/2\mu r] \text{ as } r\to\infty , \tag{4.15}$$

where M_Q and M_P are parameters with the dimension of a mass. From these solutions the spatial structures of the electric and the new charge are given by

$$-e^{-1}r^2[Q(r)/r]'\to\mathscr{Q}[1-\tfrac{1}{2}h(r)] \text{ as } r\to\infty , \tag{4.16}$$

$$-e^{-1}r^2[P(r)/r]'\to\mathscr{P}[1-\tfrac{1}{2}h(r)]+e^{-1}r^2[K(r)/r]' \text{ as } r\to\infty . \tag{4.17}$$

Hence $\mathscr{Q}$ and $\mathscr{P}$ are parameters which denote the values of the charges defined in Eqs. (3.15) and (3.16).

Inserting (4.14) and (4.15) into Eq. (4.6), the asymptotic form of Eq. (4.6) becomes

$$K(r)''\sim(M_P{}^2-M_Q{}^2)K . \tag{4.18}$$

If $M_P{}^2>M_Q{}^2$, Eq. (4.6) admits the solution of the asymptotic form

$$K(r)\to De^{-\alpha r} , \tag{4.19}$$

where $\alpha=(M_P{}^2-M_Q{}^2)^{1/2}$ and D is a constant. Here one finds that this behavior is consistent with the assumption (4.9) given at the beginning. We now wish to emphasize that the localized structure of $K(r)$ like (4.19) originates from the contribution of the component of the gauge fields associated with the new charge $\mathscr{P}$. As was explained in Sec. III, this component acts with timelike metric in the internal space and the leading term $M_P r$ of $P(r)$ in Eq. (4.15) provides the mass for the field $K(r)$. Therefore, we stress that the localized structure of $K(r)$ is essentially attributed to the incorporation of the internal symmetry SO(3,1). In the SO(3) gauge theory, on the other hand, the field $K(r)$ acquires its mass by the Higgs mechanism. As stressed by Fujii,[12] the Higgs mechanism does not operate in the geometric field theory when the structure constants f^a_{bc} are totally antisymmetric. Because f^a_{bc} are totally antisymmetric in SO(3,1), the Higgs mechanism does not work in our model. This really appears in the fact that $\mathscr{L}_{S1}$ in (4.4) does not have a term like const$\times K^2/r^2$ at infinity. Hence at first sight our model looks as though it has no mechanism to provide the localized structure for the field $K(r)$. But as was explained above the gauge fields associated with the charge $\mathscr{P}$ act with timelike metric to give that structure.

Thus we could solve the Euler-Lagrange equations at large r. In order to solve them for all r, we must set the boundary conditions at $r\to0$. Although the expression of the energy is not given here, the Lagrangian (4.1) shows that we need the following boundary conditions to keep energy finite: $h(r)\to h_0+$ const$\times r^2$, $K(r)\to1+$ const$\times r^2$, $Q(r)\to$ const$\times r^2$, and $P(r)\to1+$ const$\times r^2$ as $r\to0$, where h_0 is a constant. Although we have not obtained any numerical solution, we expect to get the regular solutions which connect the forms in the above boundary conditions to the asymptotic forms (4.12), (4.14), (4.15), and (4.19) at infinity. Finally, we list the asymptotic values of the functions $h(r)$, etc.: $h(r)=h_0\to0$, $K(r)=1\to0$, $-e^{-1}r^2(Q/r)'=0\to\mathscr{Q}$, and $-e^{-1}r^2(P/r)'=1\to\mathscr{P}$ as $r=0\to\infty$.

V. CONCLUDING REMARKS

We first discuss the condition (4.10) which gives the strong relation among the charges. This comes from the fact that the Lagrangian density has the overall factor $[\det(g_{ab})]^{1/2}$ $[=(1+h)^{1/2}]$ and includes the Yang-Mills Lagrangian density of the form $F^a_{\mu\nu}F^b_{\mu\nu}g_{ab}$ instead of $F^a_{\mu\nu}F^a_{\mu\nu}$ in the ordinary gauge theory. The first term on the right-hand side of Eq. (4.5) is derived by taking the variation of h in the overall factor and the second term arises from the same variation in the Yang-Mills Lagrangian density. The condition (4.10) comes from these two terms and therefore is very characteristic of the geometric field theory. Equation (4.10) imposes the following relation among the charges:

$$3\mathscr{P}^2-\mathscr{Q}^2+e^{-2}=0 \tag{5.1}$$

where e^{-1} is the magnetic charge. One finds that the electric charge $\mathscr{Q}$ is restricted by $|\mathscr{Q}|>e^{-1}$. As was metioned in Sec. III, the gauge fields associated with the charge $\mathscr{P}$ contribute negative energy. Here it is very important to investigate whether or not this negative-energy component might cause a serious difficulty in quantum-mechanical considerations. But we have not studied it yet and this is one of our important problems in the future.

We next discuss the stability of our solutions. Here we first look at the asymptotic form of $-r^2(Q/r)'$ given in (4.16). As was explained in Sec. IV, it should change from zero to $\mathscr{Q}$ ($\neq 0$) as r goes from the origin to infinity. Therefore, one can realize that the asymptotic behavior of $Q(r)$ forbids the solution $h(r)=0$ for all r. This means that the internal metric field $g_{ab}(x)$ of our solutions is not allowed to continuously turn to be flat. From this point of view we insist that our solutions are stable. Furthermore, as discussed at the beginning of this section, the absolute value of the electric charge has a lower bound in this theory. Hence our dyon solutions are prohibited from collapsing into solutions having only magnetic monopoles.

From the solutions (4.12), dyons have the size of order μ^{-1} $[=\kappa/(3g_0)^{1/2}]$. We can roughly estimate the mass of dyons from the potential (3.10). Using the normalization condition (3.5), we get $M\sim h_0\mu/\,10e^2$. We cannot discuss here anything about its value because we have no knowledge about the values of h_0 and μ.

We conclude this paper with a summary of some important problems to be investigated in the future. The first one is to obtain the regular solutions of the Euler-Lagrange equations for all r. As was shown in Sec. IV, the equations are so complicated that it takes much time to obtain them by numerical calculations. Another interesting problem is to study whether the new charge $\mathscr{P}$ contributing negative energy will play some prominent role in particle physics or not.

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