Some properties of a particular static, axially symmetric space-time

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The behavior of the directional singularities of a family of Weyl solutions is examined. By examining the space-time in a different coordinate system, the directional singularities are understood. The singular points are actually extended hypersurfaces which have been collapsed to a point by an improper choice of coordinates. The singular structure is examined in the new coordinate system. The coordinate system shows the space-time to be geodesically incomplete. A completion of the symmetry axis is described.

I. INTRODUCTION

It has long been known that static axisymmetric or Weyl solutions of the Einstein equations may contain points where the space-time exhibits directional singularities. A space-time has a directional singularity at a point if the value of a curvature invariant is different when the point is approached from different directions. It is quite probable that this phenomenon indicates that the coordinate system is inappropriate in the neighborhood of the point, that these "points" really represent a higher-dimensional surface. We show that this is the case for a particular family of Weyl solutions and examine the properties of the space-time at the points where the directional singularities occur.

We will study the γ solution,[1] a Weyl solution where the source in Weyl coordinates has a mass density γ distributed uniformly and symmetrically along the axis for a length $2m$. In prolate spheroidal coordinates the γ solution is

$$ds^2=-\left(\frac{x-m}{x+m}\right)^{\gamma}dt^2+\left(\frac{x^2-m^2}{x^2-m^2y^2}\right)^{\gamma^2}\left(\frac{x+m}{x-m}\right)^{\gamma}(x^2-m^2y^2)\times\left(\frac{dx^2}{x^2-m^2}+\frac{dy^2}{1-y^2}\right)+\left(\frac{x+m}{x-m}\right)^{\gamma}(x^2-m^2)(1-y^2)d\phi^2\,, \tag{1}$$

where $m\leq x<\infty$, $-1\leq y\leq 1$, $0\leq\phi<2\pi$, $-\infty<t<\infty$.

We shall consider γ and m to be non-negative real numbers. The solution has interesting special cases. Either $\gamma=0$ or $m=0$ is a Minkowski space-time; $\gamma=1$, $m\neq 0$ represents a Schwarzschild space-time; $\gamma\to\infty$, $m\to 0$, $m\gamma\to$ constant represents a Curzon space-time. The γ solution is also the static limit of the Tomimatsu-Sato (TS) family of solutions.[2] If the rotation parameter q equals zero, the TS solutions become equal to the γ solution with the TS deformation parameter δ being equal to γ. The properties of the singularities of the TS solution for $\delta=2$ have been studied by Ernst[3] and Economou[4]; this paper may be considered an extension of their study for static TS solutions with integer or noninteger distortion parameter.

II. SINGULARITY STRUCTURE

We shall examine the singularity structure of the γ solution by examining the curvature invariants. Define a complex null tetrad $(m,\bar{m},l,k)$ in the usual way; k and l are real, m and $\bar{m}$ are complex and complex conjugates of each other:

$$\begin{aligned}&k^{\mu}l_{\mu}=-1\,,\quad m^{\mu}\bar{m}_{\mu}=1\,,\\&k^{\mu}k_{\mu}=l^{\mu}l_{\mu}=m^{\mu}m_{\mu}=k^{\mu}m_{\mu}=l^{\mu}m_{\mu}=0\,,\\&g_{\mu\nu}=2m_{(\mu}\bar{m}_{\nu)}-2k_{(\mu}l_{\nu)}\,.\end{aligned} \tag{2}$$

The complex Weyl tetrad components are defined as

$$\begin{aligned}&\psi_0\equiv C_{\mu\nu\rho\sigma}k^{\mu}m^{\nu}k^{\rho}m^{\sigma}\,,\\&\psi_1\equiv C_{\mu\nu\rho\sigma}k^{\mu}l^{\nu}k^{\rho}m^{\sigma}\,,\\&\psi_2\equiv\tfrac{1}{2}C_{\mu\nu\rho\sigma}k^{\mu}l^{\nu}(k^{\rho}l^{\sigma}-m^{\rho}\bar{m}^{\sigma})\,,\\&\psi_3\equiv C_{\mu\nu\rho\sigma}l^{\mu}k^{\nu}l^{\rho}\bar{m}^{\sigma}\,,\\&\psi_4\equiv C_{\mu\nu\rho\sigma}l^{\mu}\bar{m}^{\nu}l^{\rho}\bar{m}^{\sigma}\,.\end{aligned} \tag{3}$$

Choose the (t,x,y,ϕ) components of the null tetrad to be

$$\begin{aligned}&k^{\mu}=\left[1,\,-\left(\frac{x-m}{x+m}\right)^{\gamma}\left(\frac{x^2-m^2y^2}{x^2-m^2}\right)^{(\gamma^2-1)/2},0,0\right]\,,\\&l^{\mu}=\frac{1}{2}\left[\left(\frac{x+m}{x-m}\right)^{\gamma},\left(\frac{x^2-m^2y^2}{x^2-m^2}\right)^{(\gamma^2-1)/2},0,0\right]\,,\\&m^{\mu}=\frac{1}{\sqrt{2}}\left(\frac{x+m}{x-m}\right)^{\gamma/2}\times\left[0,0,-i\left(\frac{x^2-m^2y^2}{x^2-m^2}\right)^{\gamma^2/2}\left(\frac{1-y^2}{x^2-m^2y^2}\right)^{1/2},\right.\\&\qquad\left.(1-y^2)^{-1/2}(x^2-m^2)^{-1/2}\right]\,.\end{aligned} \tag{4}$$

We have calculated the complex Weyl tetrad components. (All conformal and curvature tensors referred to in this paper have been calculated using a REDUCE 2 program obtained from the University of South Carolina.) The complex Weyl tetrad components are

$$\psi_0 = -m^3\gamma(\gamma^2-1)\frac{x(1-y^2)}{(x^2-m^2)(x^2-m^2y^2)^2} \times\left(\frac{x-m}{x+m}\right)^{2\gamma}\left(\frac{x^2-m^2y^2}{x^2-m^2}\right)^{\gamma^2},$$

$$\psi_1 = 0,$$

$$\psi_2 = \frac{m\gamma}{2}\left(\frac{x-m}{x+m}\right)^{\gamma}\left(\frac{x^2-m^2y^2}{x^2-m^2}\right)^{\gamma^2-1} \times\frac{\gamma^2m^2x(1-y^2)+x(x^2-m^2)+(x^2-m^2y^2)(x-2\gamma m)}{(x^2-m^2)^2(x^2-m^2y^2)}, \tag{5}$$

$$\psi_3 = 0,$$

$$\psi_4 = -\frac{\gamma(\gamma^2-1)m^3}{4}\,\frac{x(1-y^2)}{(x^2-m^2)(x^2-m^2y^2)^2}\left(\frac{x^2-m^2y^2}{x^2-m^2}\right)^{\gamma^2}.$$

As they should, the tetrad components all vanish for $\gamma=0$, Minkowski space. For the Schwarzschild space, $\gamma=1$, the only nonvanishing component is ψ_2 and it is everywhere finite. For $\gamma<2$, but not equal to 0 or 1, the components are infinite for $x=m$, $-1\leqslant y\leqslant 1$. If $\gamma\geqslant 2$, the components are singular for $x=m$, $-1<y<1$. However, the north pole, $x=m$, $y=1$, and south pole $x=m$, $y=-1$ show directional behavior. If we let first $x=m$ and then go to the limit $y=\pm 1$, the components are infinite. If we let first $y=\pm1$ and then go to the limit $x=m$, the components are finite. Hence the poles exhibit directional singularities, i.e., the behavior of the conformal tensor components at the poles depends on the direction in which we approach them.

In passing we remark on the Petrov type. Following the algorithm of d'Inverno and Russell-Clark reproduced in Ref. 5, p. 64, one sees that the γ solution ($\gamma\neq 0$ and $\gamma\neq 1$) is type D on the axis and type I or general elsewhere.

III. THE GEOMETRY NEAR THE NORTH POLE

Since the poles have associated directional singularities we expect that they should be represented by surfaces rather than by points. We will examine the "north" pole, the south pole has a symmetric description. We change coordinates in the neighborhood of the north pole, using the same coordinate transformation to polar-type coordinates as Ernst used[3] (henceforth we consider only $\gamma\geqslant 2$),

$$x^2 = m^2[1+8r\cos^2(\tfrac{1}{2}\theta)], \qquad y^2 = 1-8r\sin^2(\tfrac{1}{2}\theta). \tag{6}$$

Clearly, as $r\to 0$; $x\to m$, $y\to 1$. In the new (t,r,θ,ϕ) coordinate system, the line element is given by

$$ds^2 = 4m^2\cos^{2\gamma^2}(\tfrac{1}{2}\theta)\left(\frac{[1+8r\cos^2(\frac{1}{2}\theta)]^{1/2}+1}{[1+8r\cos^2(\frac{1}{2}\theta)]^{1/2}-1}\right)^{\gamma}\left(\frac{1}{[1+8r\cos^2(\frac{1}{2}\theta)]}[\cos^2(\tfrac{1}{2}\theta)dr^2 - r\sin\theta dr\,d\theta + r^2\sin^2(\tfrac{1}{2}\theta)d\theta^2] + \frac{1}{[1-8r\sin^2(\frac{1}{2}\theta)]}[\sin^2(\tfrac{1}{2}\theta)dr^2 + r\sin\theta dr\,d\theta + r^2\cos^2(\tfrac{1}{2}\theta)d\theta^2]\right) + 4m^2r^2\sin^2(\tfrac{1}{2}\theta)\left(\frac{[1+8r\cos^2(\frac{1}{2}\theta)]^{1/2}+1}{[1+8r\cos^2(\frac{1}{2}\theta)]^{1/2}-1}\right)^{\gamma}d\phi^2 - \frac{1}{4}\left(\frac{[1+8r\cos^2(\frac{1}{2}\theta)]^{1/2}-1}{[1+8r\cos^2(\frac{1}{2}\theta)]^{1/2}+1}\right)^{\gamma}dt^2. \tag{7}$$

Here, $0\leqslant r<\infty$, $0\leqslant\theta<\pi$, $0\leqslant\phi\leqslant 2\pi$, $-\infty<t<\infty$. $\theta=\pi$ is specifically excluded from the transformation at the north pole because this refers to the portion of the axis immediately below the pole where we know that the geometry is singular. In this coordinate system, the t, r, θ, ϕ components of the null tetrad are

$$k^\mu = \left(\frac{[1+8r\cos^2(\frac{1}{2}\theta)]^{1/2}-1}{[1+8r\cos^2(\frac{1}{2}\theta)]^{1/2}+1}\right)^{\gamma}\left(1,\ -\frac{[1+8r\cos^2(\frac{1}{2}\theta)]^{1/2}}{4m\cos^{\gamma^2-1}(\frac{1}{2}\theta)},\ \frac{\tan(\frac{1}{2}\theta)[1+8r\cos^2(\frac{1}{2}\theta)]^{1/2}}{4mr\cos^{\gamma^2-1}(\frac{1}{2}\theta)},\ 0\right),$$

$$l^\mu = \left(\frac{1}{2}\left(\frac{[1+8r\cos^2(\frac{1}{2}\theta)]^{1/2}+1}{[1+8r\cos^2(\frac{1}{2}\theta)]^{1/2}-1}\right)^{\gamma},\ \frac{[1+8r\cos^2(\frac{1}{2}\theta)]^{1/2}}{8m\cos^{\gamma^2-1}(\frac{1}{2}\theta)},\ -\frac{\tan(\frac{1}{2}\theta)[1+8r\cos^2(\frac{1}{2}\theta)]^{1/2}}{8mr\cos^{\gamma^2-1}(\frac{1}{2}\theta)},\ 0\right), \tag{8}$$

$$m^\mu = \left(\frac{[1+8r\cos^2(\frac{1}{2}\theta)]^{1/2}-1}{[1+8r\cos^2(\frac{1}{2}\theta)]^{1/2}+1}\right)^{\gamma}\left(0,\ \frac{i\sin(\frac{1}{2}\theta)[1-8r\sin^2(\frac{1}{2}\theta)]^{1/2}}{4\sqrt{2}m\cos^{\gamma^2}(\frac{1}{2}\theta)},\ \frac{i\cos(\frac{1}{2}\theta)[1-8r\sin^2(\frac{1}{2}\theta)]^{1/2}}{4\sqrt{2}\,mr\cos^{\gamma^2}(\frac{1}{2}\theta)},\ \frac{1}{4\sqrt{2}r\sin\theta}\right).$$

In this coordinate system the nonvanishing tetrad components of the conformal tensor may be written

$$\psi_0 = -\frac{\gamma(\gamma^2-1)\sin^2(\frac{1}{2}\theta)[1+8r\cos^2(\frac{1}{2}\theta)]^{1/2}}{16m^2(2r)^2\cos^{2\gamma^2+2}(\frac{1}{2}\theta)}\left(\frac{[1+8r\cos^2(\frac{1}{2}\theta)]^{1/2}-1}{[1+8r\cos^2(\frac{1}{2}\theta)]^{1/2}+1}\right)^{2\gamma},$$

$$\psi_2 = \frac{\gamma}{32m^2}\,\frac{[1+8r\cos(\frac{1}{2}\theta)]^{1/2}[\gamma^2\sin^2(\frac{1}{2}\theta)+\cos^2(\frac{1}{2}\theta)+1]-2\gamma}{(2r)^2\cos^{2\gamma^2+2}(\frac{1}{2}\theta)}\left(\frac{[1+8r\cos^2(\frac{1}{2}\theta)]^{1/2}-1}{[1+8r\cos^2(\frac{1}{2}\theta)]^{1/2}+1}\right)^{\gamma}, \tag{9}$$

$$\psi_4 = \frac{\gamma(\gamma^2-1)}{64m^2}\,\frac{\sin^2(\frac{1}{2}\theta)}{(2r)^2\cos^{2\gamma^2+2}(\frac{1}{2}\theta)}\,[1+8r\cos^2(\tfrac{1}{2}\theta)]^{1/2}.$$

One can readily establish that as $r\to 0$; $\psi_0 \to 0$ $(\gamma \geq 2)$, $\psi_2 \to 0$ $(\gamma > 2)$, ψ_2 has a finite limit $(\gamma = 2)$. The behavior of ψ_4 is that it becomes infinite as $r\to 0$ unless $\theta = 0$ in which case it is always 0. So the surface $r=0$ has curvature singularities everywhere except at the pole itself. There is no directional singularity in this coordinate system.

IV. THE γ METRIC APPROXIMATED AT THE POLE

The exact metric (7) has a complicated form and we are interested in its behavior chiefly in the region $r\to 0$. It is a temptation to write a simpler metric which approximates this metric in the region r close to zero. Ernst[3] and Economou[4] have studied the behavior of an approximate metric for the $\delta = 2$, TS solution. As we have remarked, in the case of no rotation, this is the same as the γ solution for $\gamma = 2$. After finding an approximate metric following the path of Ernst and Economou we should have a way of determining whether or not the approximation is good. We shall give a covariant approach to testing the approximation. It will turn out that the approximation is qualitatively good except for the very case, $\gamma = 2$, considered by Ernst and Economou. The Ernst-Economou approximate metric is a poor approximation in the limit of vanishing rotation.

If one replaces the components of the metric tensor by a power series in the coordinate r, the question naturally arises how the series should be truncated to obtain a good approximation of the actual geometry near $r = 0$. We propose replacing r by λr and t by $\lambda^{1-\gamma} t$ after which the limit $\lambda \to 0$ is taken, terms of the order $\lambda^{2-\gamma}$ are kept, and λ is dropped from the ensuing expression. This agrees with the procedure of Ernst when $\gamma = 2$. The resulting approximate line element is

$$ds^2 = 16m^2 \cos^{2\gamma^2-2\gamma}(\tfrac{1}{2}\theta)\frac{dr^2}{(2r)^\gamma} + \frac{4m^2}{(2r)^{\gamma-2}}\left[\cos^{2\gamma^2-2\gamma}(\tfrac{1}{2}\theta)d\theta^2 + \frac{\sin^2\theta}{\cos^{2\gamma}(\tfrac{1}{2}\theta)}d\phi^2\right] - (2r)^\gamma \cos^{2\gamma}(\tfrac{1}{2}\theta)dt^2 . \tag{10}$$

It turns out that this metric is an exact solution of the Einstein vacuum field equations for any γ and for all r, θ; not only in the region $r\to 0$. The Riemann tensor of the approximate metric is given in the Appendix. When $\gamma = 2$ the metric agrees with the approximate metric used by Ernst[3] and Economou. For constant t and $r = r_0$, a constant, the induced line element on the (θ, ϕ) hypersurface is

$$ds^2 = \frac{4m^2}{(2r_0)^{\gamma-2}}\left[\cos^{2\gamma^2-2\gamma}(\tfrac{1}{2}\theta)d\theta^2 + \frac{\sin^2\theta}{\cos^{2\gamma}(\tfrac{1}{2}\theta)}d\phi^2\right] , \tag{11}$$

$0 \leq \theta < \pi$, $0 \leq \phi \leq 2\pi$. The Gaussian curvature K of this two-dimensional hypersurface is

$$K = -\frac{(2r_0)^{\gamma-2}}{4m^2 \cos^{2\gamma^2-2\gamma+2}(\tfrac{1}{2}\theta)} \times [(\gamma-2)(\gamma^2-2)\cos^2(\tfrac{1}{2}\theta) + \gamma^2(\gamma-1)] . \tag{12}$$

For $0 \leq \theta < \pi$ the Gaussian curvature is negative and finite. However, as $r_0 \to 0$, if $\gamma = 2$ the curvature is nonzero, if $\gamma > 2$ the curvature is zero.

The area of the portion of the hypersurface from $\theta = 0$ to $\theta = \theta_0$ is

$$A = 8\pi m^2 \frac{[1-\cos^{\gamma^2-2\gamma+2}(\tfrac{1}{2}\theta_0)]}{(2r_0)^{\gamma-2}(\gamma^2 - 2\gamma + 2)} . \tag{13}$$

For $\gamma = 2$, the area is finite as $r_0 \to 0$; for $\gamma > 2$ it is infinite as $r_0 \to 0$ (no matter how small θ_0 is as long as $\theta_0 > 0$).

We shall try to establish whether the approximate metric (10) is really a good approximation to the exact metric (7). In order to examine this we will choose a null tetrad for the approximate metric reasonably similar to the null tetrad used for the real metric and compare the resulting tetrad components of the conformal tensor. In the remainder of this section, vectors and other quantities associated with the metric (10) will be written with a label AP; those associated with the metric (7) will be written without subscript. The (t, r, θ, ϕ) components of the null tetrad appropriate to (10) are

$$k^\mu_{\text{AP}} = \left[1, -\frac{2^\gamma r^\gamma \cos^{-\gamma^2+2\gamma}(\tfrac{1}{2}\theta)}{4m}, 0, 0\right] ,$$
$$l^\mu_{\text{AP}} = \left[\frac{\cos^{-2\gamma}(\tfrac{1}{2}\theta)}{2(2r)^\gamma}, \frac{\cos^{-\gamma^2}(\tfrac{1}{2}\theta)}{8m}, 0, 0\right], \tag{14}$$
$$m^\mu_{\text{AP}} = \frac{(2r)^{(\gamma-2)/2}\cos^\gamma(\tfrac{1}{2}\theta)}{2\sqrt{2}\,m}[0, 0, i\cos^{-\gamma^2}(\tfrac{1}{2}\theta), \sin^{-1}\theta] .$$

A calculation of the tetrad components of the Weyl conformal tensor yields the results

$$\psi_0^{\text{AP}} = \frac{\gamma(\gamma-1)(\gamma-2)\sin^2(\tfrac{1}{2}\theta)}{16m^2\cos^{2\gamma^2-4\gamma+2}(\tfrac{1}{2}\theta)}(2r)^{2\gamma-2} ,$$
$$\psi_1^{\text{AP}} = \psi_3^{\text{AP}} = 0 ,$$
$$\psi_2^{\text{AP}} = -\frac{\gamma(\gamma-1)[\gamma\sin^2(\tfrac{1}{2}\theta) + 2\cos^2(\tfrac{1}{2}\theta)]}{32m^2\cos^{2\gamma^2-2\gamma+2}(\tfrac{1}{2}\theta)}(2r)^{\gamma-2} , \tag{15}$$
$$\psi_4^{\text{AP}} = -\frac{\gamma(\gamma-1)(\gamma-2)\sin^2(\tfrac{1}{2}\theta)}{64m^2\cos^{2\gamma^2+2}(\tfrac{1}{2}\theta)}(2r)^{-2} .$$

The approximate metric is Petrov type D along the axis (since $\psi_0 = \psi_4 = 0$ along axis where $\theta = 0$) and type I elsewhere, in this sense it resembles the exact metric which has the same classification.

As $r\to 0$; $\psi_0^{AP}\to 0$ $(\gamma\geq 2)$, $\psi_2^{AP}\to 0$ $(\gamma>2)$, ψ_2^{AP} has a finite limit $(\gamma=2)$. The behavior of ψ_4^{AP} is that it becomes infinite as $r\to 0$ $(\gamma>2)$ unless $\theta=0$ in which case it is always 0; it is also 0 if $\gamma=2$. Hence the approximate metric may be a good approximation to the real metric if $\gamma>2$; however, in the very case used by Ernst and Economou $(\gamma=2)$ it cannot be a good approximation. The real metric is singular at $r=0$ everywhere except along the axis and the approximate metric has no singularities in the neighborhood of $r=0$. Although the approximate metric was designed for $\gamma\geq 2$, we might look at it for the Schwarzschild case, $\gamma=1$. All the reasoning previously made would seem to apply although there is no directional singularity at the pole. If $\gamma=1$, all the components of the Riemann tensor are equal to zero, the space-time is everywhere flat; so again the approximate metric does not approximate the real metric.

To compare further the approximate metric to the real metric we look first at the ratios of the tetrad components of the Weyl tensor in the limit $r\to 0$:

$$\frac{\psi_0^{AP}}{\psi_0}=-\frac{\gamma-2}{\gamma+1},\quad \frac{\psi_2^{AP}}{\psi_2}=1,\quad \frac{\psi_4^{AP}}{\psi_4}=-\frac{\gamma-2}{\gamma+1}. \tag{16}$$

To see how meaningful these comparisons are we look at the ratios of the components of the null tetrad vectors as $r\to 0$,

$$\begin{aligned}&\frac{k^t_{AP}}{k^t}=1,\quad \frac{k^r_{AP}}{k^r}=[\cos(\tfrac{1}{2}\theta)]^{-1},\\&\frac{l^t_{AP}}{l^t}=1,\quad \frac{l^r_{AP}}{l^r}=[\cos(\tfrac{1}{2}\theta)]^{-1},\\&\frac{m^\theta_{AP}}{m^\theta}=[\cos(\tfrac{1}{2}\theta)]^{-1},\quad \frac{m^\phi_{AP}}{m^\phi}=1.\end{aligned} \tag{17}$$

One sees that near the axis, the null tetrads become nearly equal to each other and the curvature invariants all approach some constant value. It would seem that near the axis the approximate metric resembles the γ metric reasonably closely as far as qualitative features are concerned; however for $\gamma=2$, the approximation fails because the singularity structure at $r=0$ is so much different. Of course, as we remarked, the γ and the approximate metrics have the same Petrov classification.

V. EXTENSION OF THE SYMMETRY AXIS AND THE CONFORMAL STRUCTURES

It has been remarked that in the (t,r,θ,ϕ) coordinate system there is no directional singularity. Further at the pole, $r=0$, $\theta=0$, there is no curvature singularity. We shall hereafter usually consider $\gamma\geq 2$. If a timelike geodesic reaches the pole in a finite proper time (or finite value of an affine parameter) the space-time is geodesically incomplete at this point and should be extended. It is straightforward to show that the proper time is indeed finite and an extension of the space-time is necessary. We shall describe an extension of the two dimensional space-time subspace spanned by the time coordinate and the rotation axis; this is of course not a complete extension of the four-dimensional manifold. In the (t,x,y,ϕ) coordinate system we shall extend the submanifold along the axis $y=1$, $d\phi=0$. The fact that this extension was necessary and possible is stated in the 1972 paper of Voorhees (Ref. 1)

$$ds^2=-\left(\frac{x-m}{x+m}\right)^\gamma dt^2+\left(\frac{x+m}{x-m}\right)^\gamma dx^2. \tag{18}$$

Introduce the null coordinates

$$du\equiv dt+\left(\frac{x-m}{x+m}\right)^{-\gamma} dx, \tag{19}$$

$$w\equiv u-G(x),\quad G(x)\equiv 2\int_{x_0}^{x}\left(\frac{x'+m}{x'-m}\right)^\gamma dx'. \tag{20}$$

$G(x)$ is monotonic in the regions

I: $x>m$,

II: $-m<x<m$.

The mapping is well defined only if we specify to which region it is being applied. In the u,w coordinate system the line element is

$$ds^2=\left(\frac{x-m}{x+m}\right)^\gamma du\,d\omega. \tag{21}$$

We now introduce the coordinate mappings

$$\begin{aligned}&u=\int_{s_0}^{s}\sec^\gamma s'ds',\quad w=-\int_{r_0}^{r}\csc^\gamma r'dr',\\&s=\tfrac{1}{2}(\psi+\xi),\quad r=\tfrac{1}{2}(\psi-\xi),\\&-\infty<\xi<\infty,\quad -\infty<\psi<\infty.\end{aligned} \tag{22}$$

The line element in the two space becomes

$$\begin{aligned}&ds^2=Q(d\xi^2-d\psi^2),\\&Q\equiv\frac{1}{4}\left(\frac{x-m}{x+m}\right)^\gamma \sec^\gamma\tfrac{1}{2}(\psi+\xi)\csc^\gamma\tfrac{1}{2}(\psi-\xi).\end{aligned} \tag{23}$$

A rather lengthy algebraic calculation can show that Q is finite, nonzero, and a continuous function of ψ and ξ at $x=m$.

A better understanding of the line element (18) and its extension (23) is obtained by the behavior at infinity as described by its conformal structure. The technique for construction of conformal diagrams of geodesic completions of metrics of this type (18) has been known for sometime; a brief review might be beneficial. The following remarks are based on the paper by Walker[6] which should be consulted for details. We discuss a timelike two surface with a metric which can be put in the form

$$ds^2 = -F(r)dt^2 + F(r)^{-1}dr^2 . \qquad (24)$$

The only curvature invariant of this metric is the intrinsic Gaussian curvature $K \equiv \frac{1}{2} d^2 F/dr^2$. If K and F are finite for all r, then every geodesic can be extended indefinitely. If K or F becomes infinite for some $r = r_0$ then there are inextendible geodesics and the space-time is singular.

Let the zeros of F by given by $r = a_i$, $i = 1, 2, 3, \ldots, n$, with n finite. If F approaches a constant value and $K \to 0$ as $r \to \infty$, the space is asymptotically flat and conformal infinity can be represented on diagrams by a pair of finite null lines.

When F vanishes (horizons), the timelike Killing vector becomes null; we can think of the $r = a_i$ as dividing the space into $n+1$ distant regions. Each region is bounded by the horizons, by one horizon and conformal infinity, or by one horizon and a singularity (say at $r = r_0$ on the figures). The conformal structure of each such region will be referred to as a block (see Fig. 1). The conformal representation of the space-time is found by "gluing" the blocks together according to a well-defined scheme. Those seams along which $F = 0$ and K is finite are nonsingular while those along which either F or K are infinite are singular.

We now turn to the extension of the geodesics by gluing the blocks together. Our convention for the resulting diagram is that the timelike coordinate in each block increases vertically upward. If $F > 0$ in a block, t is the timelike coordinate; if $F < 0$, r is the timelike coordinate. Figure 2 shows the possible arrangements.

In summary, the rules for constructing the conformal representation of the space time are very simple.

(1) The timelike coordinate in each region changes vertically.

(2) The blocks are combined in all possible ways by joining them along nonsingular seams.

(3) K must be smooth across a seam, hence a block cannot be flipped and joined to itself.

The above rules may be applied to the γ metric along the polar axis. Equation (19) represents the line element in an appropriate form with

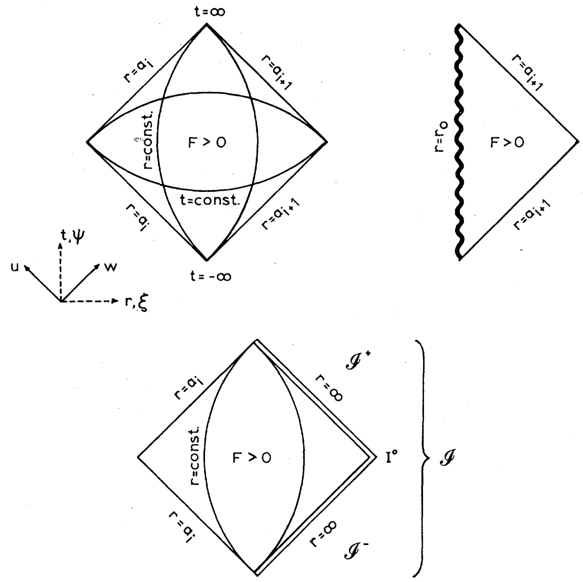

FIG. 1. The conformal structure of blocks. (a) is a nonsingular block. (b) is a singular block. The wavy line $r = r_0$ shows when the intrinsic curvature K is infinite. (c) is a nonsingular block with conformal infinity $\mathscr{I}$ represented by double lines.

$$F = \left(\frac{x-m}{x+m}\right)^{\gamma} . \qquad (25)$$

The results are displayed in Figs. 3 and 4. The only root is a root of order γ at $x = m$. We can only apply the considerations of the preceding paragraphs to integer values of γ ($\gamma = 1, 2, 3, \ldots$). Since there is only one root, there are only two blocks. For γ odd, F changes sign as x crosses m and one obtains a diagram of the two space of "finite" size. For γ even, $F > 0$ in both blocks and an infinitely long chain of blocks results. For values of γ such that $0 < \gamma < 1$ and $1 < \gamma < 2$,

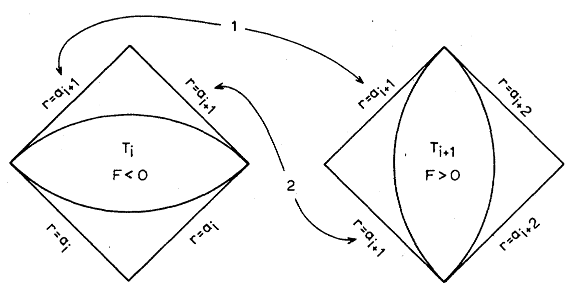

FIG. 2. Gluing blocks together: 1 and 2 indicate the two ways in which neighboring blocks may be glued together. A representative pair of Killing vector orbits r = const is shown.

although γ is not integer, no extension is expected as the space is singular at $x=m$.

The points on the diagrams labeled exceptional points are not covered by a coordinate patch. However they are at infinite affine distance and cause no concern. For $\gamma \neq 1$, the exceptional points occur at $u=w=0$.[7] If $\gamma>2$ and not an integer, an analytic extension is not possible. If $n<\gamma<n+1$ for integer n, an extension is possible which is C^n.

VI. CONCLUSIONS

Upon examination of the γ solution at the "point" $x=m$, $y=\pm 1$, we have found that this "point" is actually a two dimensional hypersurface for $\gamma \geq 2$. The Gaussian curvature of the hypersurface can be calculated (this was done in an $r-\theta$ coordinate system). For $0<\gamma<2$, the Gaussian curvature is infinite at $r=0$ ($x=m, y=\pm 1$) and the "hypersurface" is a point; for $\gamma=2$, the Gaussian curvature is finite at $r=0$; for $\gamma>2$ the Gaussian curvature is zero at $r=0$.

The directional behavior of the curvature invariants in the original space-time is thereby explained as the evaluation of these invariants at

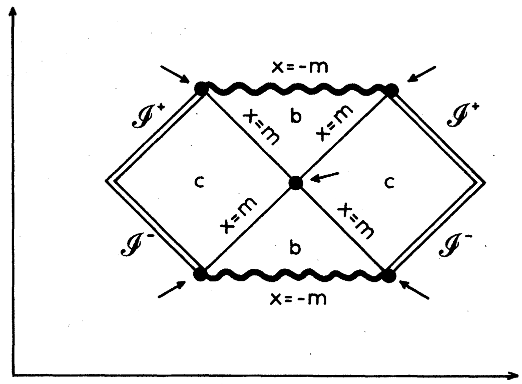

FIG. 4. The conformal representation of the space-time (x, t) plane when $\gamma>2$ and odd, $y=+1$. The wavy line represents the physical singularity. The arrows refer to the exceptional points. These points are not covered by the coordinate patch but are at an infinite affine distance.

different points on the hypersurface. We attempted to understand the behavior of the metric at the pole ($r=0, \theta=0$) of the hypersurface by finding a metric which gave similar behavior to the real metric in the neighborhood of the pole. The attempt was unsuccessful for $\gamma=2$ and only partially successful for other values of γ.

We then showed that the space-time was incomplete at the poles for $\gamma \geq 2$. We looked at the polar axis and found an analytic extension of two-dimensional space-time which includes the axis and time coordinates. We described the conformal structure of the extended space-times.

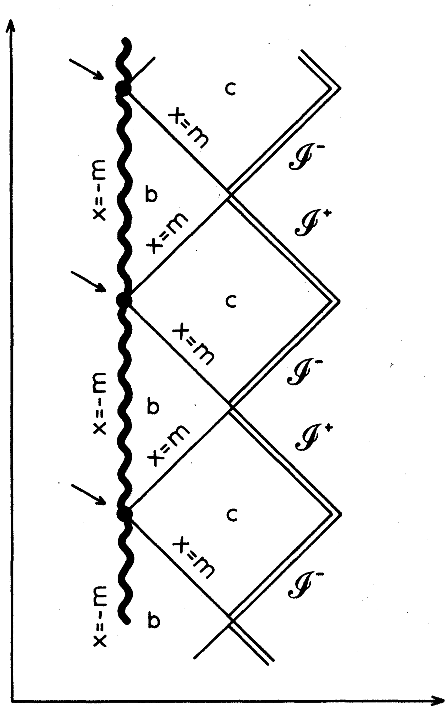

FIG. 3. The conformal representation of the space-time (x, t) plane when $\gamma \geq 2$ and even, $y=+1$. The wavy line represents the physical singularity. The arrows refer to the exceptional points. These points are not covered by the coördinate patch but are at infinite affine distance.

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APPENDIX

The approximate γ metric [Eq. (10)] was intended to approximate the exact γ metric only at the poles ($r=0$) and for $\gamma \geq 2$. It turns out that the approximate metric satisfies the vacuum Einstein field equations for any value of γ and over the whole four-space (r, θ, ϕ, t). The nonvanishing components of the Riemann tensor are given below. $\gamma=0$ and $\gamma=1$ both yield Minkowski space-time:

$$R_{1212}=\frac{4\gamma(\gamma-1)m^2}{(2r)^\gamma}\cos^{2\gamma^2-2\gamma-2}(\tfrac{1}{2}\theta)\,, \tag{A1}$$

$$R_{1313}=\frac{4\gamma\sin^2\theta\, m^2}{(2r)^\gamma\cos^{2\gamma+2}(\frac{1}{2}\theta)}[\gamma(\gamma-1)\sin^2(\tfrac{1}{2}\theta)+(\gamma-1)\cos\theta]\,, \tag{A2}$$

$$R_{1414} = \frac{\gamma(\gamma-1)[\gamma \sin^2(\tfrac{1}{2}\theta) + 2\cos^2(\tfrac{1}{2}\theta)](2r)^\gamma \cos^{2\gamma}(\tfrac{1}{2}\theta)}{4r^2\cos^2(\tfrac{1}{2}\theta)}, \tag{A3}$$

$$R_{1424} = \frac{\gamma(\gamma^2-3\gamma+2)}{2}(2r)^{\gamma-1}\sin(\tfrac{1}{2}\theta)\cos^{2\gamma-1}(\tfrac{1}{2}\theta), \tag{A4}$$

$$R_{2323} = -\frac{4\gamma r^2 \sin^2\theta m^2}{(2r)^\gamma \cos^{2\gamma+2}(\tfrac{1}{2}\theta)} \times [(2-2\gamma) + (\gamma^2-3\gamma+2)\sin^2(\tfrac{1}{2}\theta)], \tag{A5}$$

$$R_{2424} = -\frac{\gamma(\gamma-1)(2r)^\gamma\cos^{2\gamma-2}(\tfrac{1}{2}\theta)}{4} \times [(\gamma-1)\sin^2(\tfrac{1}{2}\theta) + \cos^2(\tfrac{1}{2}\theta)], \tag{A6}$$

$$R_{3434} = -\frac{\gamma(\gamma-1)(2r)^\gamma \sin^2\theta \cos^{2\gamma-2\gamma^2-2}(\tfrac{1}{2}\theta)}{4}, \tag{A7}$$

$$R_{1323} = \frac{4\gamma(\gamma^2-3\gamma+2)r\sin^2\theta\sin(\tfrac{1}{2}\theta)}{(2r)^\gamma\cos^{4\gamma+1}(\tfrac{1}{2}\theta)}. \tag{A8}$$

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[1]R. Bach and H. Weyl, Math. Z. 13, 134 (1922); G. Darmois, *Les Equations de la Gravitation Einsteinienne* (Gauthier-Villars, Paris, 1927), p. 36; G. Erez and N. Rosen, Bull. Res. Council Israel 8F, 47 (1959); D. M. Zipoy, J. Math. Phys. 7, 1137 (1966); R. Gautreau and J. L. Anderson, Phys. Lett. 25A, 291 (1967); J. Winicour, A. Janis, and E. T. Newman, Phys. Rev. 176, 1507 (1968); B. H. Voorhees, Phys. Rev. D 2, 2119 (1970); F. I. Cooperstock and G. J. Junevicus, Int. J. Theor. Phys. 9, 59 (1968), B. H. Voorhees, Phys. Rev. D 2, 2119 (1970); B. H. Voorhees, *Methods of Local and Global Differential Geometry in General Relativity* (Springer, Berlin, 1972), p. 137; F. P. Esposito and L. Witten, Phys. Lett. 58B, 357 (1975). The metric which is called the γ metric in this paper is often referred to as the Voorhees metric or the Zipoy-Voorhees metric. For example, see J. Meinhardt, Nuovo Cimento 36B, 2 (1976); W. Kinnersley and D. M. Chitre, J. Math. Phys. 19, 10 (1978); hardt, Nuovo Cimento 36B, 2 (1976); W. Kinnersley and D. M. Chitre, J. Math. Phys. 19, 10 (1978); C. Hoenselaers, W. Kinnersley, and B. C. Xanthopoulos, *ibid.* 20, 12 (1979); A. Tomimatsu, Prog. Theor. Phys. 63, 3 (1980); C. Hoenselaers, J. Math. Phys. 21, 8 (1980). In many papers, the parameter we call γ is labeled δ and is called the deformation parameter.

[2]A. Tomimatsu and H. Sato, Phys. Rev. Lett. 29, 1344 (1972); Prog. Theor. Phys. 50, 95 (1973).

[3]F. J. Ernst, J. Math. Phys. 17, 1091 (1976).

[4]J. E. Economou, J. Math Phys. 17, 1095 (1976).

[5]D. Kramer, H. Stephani, M. MacCullum, and E. Herlt, *Exact Solutions of Einstein's Equations* (VEB Deutscher Verlag der Wissenschaften, Berlin, 1980). We follow the notation of this book.

[6]M. Walker, J. Math Phys. 11, 2280 (1970).

[7]B. Carter, Phys. Lett. 21, 423 (1966); B. Godfrey, J. Math Phys. 12, 606 (1970).