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Towards a unified picture for gauge and Higgs fields

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A scheme for a geometrical unification of gauge and Higgs fields, previously given for SU(2), is generalized to include arbitrary semisimple gauge groups. Gauge and physical Higgs fields appear as different components of the same tensor in a high-dimensional manifold, the higher dimensions being comprised by the group coordinates. Their respective inhomogeneous transformation behavior is derived from the same principle. The number of Higgs fields is restricted. The form of the Higgs potential is fixed, and the mass of the Higgs particle is predicted in terms of the vector-boson mass.

I. INTRODUCTION

In the construction of unified[1,2] and grand unified[3,4] models the necessity to introduce Higgs fields appears to be aesthetically disturbing. In fact, the price for the unification of some interactions may be seen as having to introduce a new one, the one produced by the Higgs field.

The objective of the present paper may be seen as the attempt to give, at least for a certain class of Higgs fields, a model that displays a formal unification of gauge and Higgs fields, much in the spirit of the Kaluza[5]-Klein[6] model that gives a formal unification for gravity and gauge fields.[7] In doing so one hopes to get restrictions on available theories. These restrictions may concern the parameters of the theory as well as the form of the Lagrangian. In the Lagrangians of this paper, in particular, the form of the Higgs potential will be fixed and the Higgs mass predicted in terms of the vector-boson mass.

The starting observation is the following. In a Kaluza[5]-Klein[6] formulation of gauge theories (compare Ref. 7 and references therein) it is natural to consider gauge fields $A_\alpha{}^t$ as components of a tensor in a $(4+k)$-dimensional space, k being the order of the gauge group. (Greek indices are Minkowski indices, late Latin indices are group indices.) Further components of the same tensor are in particular $A_s{}^t$, which in the following will be considered to be candidates for the Higgs field.

This identification introduces a formal $SO(1,3+k)$ symmetry that relates gauge and Higgs fields. The symmetry is formal since the dependence of $A_\alpha{}^t$ and $A_s{}^t$ on the group coordinates is fixed.[8] The dependence of $A_\alpha{}^t$ on the group coordinates is characterized by the differential equation

$$L_t A_\alpha{}^s = f^s{}_{vt} A_\alpha{}^t\,, \tag{1.1}$$

where $f^s{}_{vt}$ are the structure constants of the group and L_t are the group generators written as differential operators. The metric in the group space is the Killing-Cartan metric of the group $g_{st} = -\delta_{st}$. The differential equation (1.1) is compatible with gauge transformations in the sense that the gauge-transformed gauge field A_α^{gs} obeys the same differential equation. In fact,[9]

$$A_\alpha^{gs} = U^s{}_t A_\alpha{}^t - \frac{1}{2g} f^{str}(d_\alpha U_t{}^w)\, U^{-1}{}_{wr} \tag{1.2}$$

and the assertion is true provided $[L_t, d_\alpha] = 0$ and U and U^{-1} obey the differential equation for a rank-2 tensor operator under group transformation, viz.[10]

$$L_t U_s{}^v = f_{st}{}^w U_w{}^v + f_s{}^v{}_w U_s{}^w. \tag{1.3}$$

For the would-be Higgs field $A_s{}^t$, two basic observations are the following. One is that, if the formal $SO(1,3+k)$ symmetry is to be preserved under gauge transformations one should attribute an inhomogeneous transformation behaviour to $A_s{}^t$. The $A_s{}^t$ are therefore to be identified with the components of the physical Higgs field in the context of a theory with hidden gauge symmetry. The other observation is that since $A_s{}^t$ is a rank-2 tensor under group transformations, the homogeneous part of the gauge transformation of $A_s{}^t$ should be $U_s{}^v U^t{}_w A_v{}^w$. Such a transformation cannot be reproduced from an expression like UA_sU^{-1}, $A_s = A_s{}^t f_t$. From this point of view the concept of treating both $A_\alpha{}^t$ and $A_s{}^t$ as components of algebra-valued one-forms does not seem to be suitable.[11]

I therefore propose to introduce the two-forms $A_{\alpha s}\,\theta^\alpha \wedge \theta^s$ and $A_{st}\,\theta^s \wedge \theta^t$ as the basic objects of the theory.[12] The algebraic structure of the gauge algebra is hidden in the basic one-forms θ^s. In fact, their duals are the differential operators L_t which satisfy the commutation relations of the gauge algebra.

It will be convenient to formally introduce the two-form $A_{KL}\,\theta^K \wedge \theta^L$, where capital middle indices run from 1 to $3+k$. A contains components A_{st}, $A_{\alpha s}$, and $A_{\alpha\beta}$; the latter will be assumed to vanish in this paper.[13] In order to preserve as much of the formal SO$(1,3+k)$ symmetry as possible, the Langrangian will be required to be a sum of terms

$$F_{\alpha\beta s}F^{\alpha\beta s},\ F_{\alpha st}F^{\alpha st},\ F_{stv}F^{stv},$$

where $F_{\alpha\beta s}$ is the usual Yang-Mills field strength tensor for $A_\alpha{}^s$ and $F_{\alpha st}$, F_{stv} differ $F_{\alpha\beta s}$ in the replacement of differentiations with respect to Minkowski coordinates by differentiations with respect to group coordinates. Terms bilinear in fields will be determined by the requirement of gauge invariance. The Lagrangian itself will be independent of the group coordinates $L_t\mathfrak{L}=0$ as a consequence of the gauge invariance of the theory.

In the following section, a detailed description of the Lagrangian will be given in a concise form. The differentiation with respect to group coordinates will be specified. It can then be seen that the Lagrangian is one for a Yang-Mills Higgs system with hidden gauge symmetry. The maximal number of components for the Higgs field is $k(k-1)/2$, k being the order of the gauge group. The inhomogeneous transformation behavior of the physical Higgs field is derived by analogy with (1.2) and, up to first order in the group parameters, is shown to be identical to the usual one. Gauge invariance is achieved modulo a constraint on a constant matrix. By comparison with the usual formalism, this matrix is identified with the vacuum expectation value of the Higgs field. A first solution of the constraint is given; the construction of further solutions is currently being attempted.

In Ref. 14, preliminary results were given applicable to the case of SU(2) as gauge group. The current paper is formulated for arbitrary semisimple gauge groups.[15] Furthermore, some of the concepts more or less explicitly inherent in Ref. 14 are here being clarified.

II. THE LAGRANGIAN

We introduce the basic two-form

$$A=A_{KL}\,\theta^K\wedge\theta^L\,. \tag{2.1}$$

Adopting the convention $A_{\alpha\beta}=0$, it may be expanded into

$$A=2A_{\alpha s}\,\theta^\alpha\wedge\theta^s+A_{st}\,\theta^s\wedge\theta^t\,. \tag{2.2}$$

It is convenient for the following to introduce a symbol f_{KLM} which is nonvanishing only if all its indices are group indices. Further, a matrix notation will be used such that $(f_K)_{LM}=f_{LKM}$ and $(Af_KA)_{ML}=A_M{}^Nf_{NKN'}A_L^{N'}$. We then introduce a derivative D_M acting on the A_{KL} by

$$D_MA_{KL}=d_MA_{KL}+\kappa(Af_KA)_{ML}\,,\ \kappa=\text{const.} \tag{2.3}$$

In (2.3), for $M=\alpha$, d_α is the usual partial derivative acting on Minkowski coordinates, and for $M=s$, d_s is a derivative acting on group coordinates; the latter will be specified below. A field strength tensor is now introduced by

$$F=DA=(D_MA_{KL})\theta^M\wedge\theta^K\wedge\theta^L+A_{KL}[d(\theta^K\wedge\theta^L)]\,. \tag{2.4}$$

The last term in (2.4) emerges since on the grounds of Cartan's first identity,[16] $d\theta^K$ does not vanish in general.

As a Lagrangian we will choose up to an overall normalization factor

$$\mathfrak{L}\sim(F;F)\,, \tag{2.5}$$

where (;) is a scalar product induced by $(\theta^M,\theta^N)=g^{MN}$, $g_{\alpha\beta}=\text{diag}(+\cdots)$, $g_{st}=-\delta_{st}$, $g_{\alpha s}=g_{s\alpha}=0$. Explicitly,

$$\mathfrak{L}=\tfrac{1}{4}\,(F_{\alpha\beta s}F^{\alpha\beta s}+2F_{\alpha st}F^{\alpha st}+F_{stv}F^{stv})\,, \tag{2.6}$$

where the tensor components F_{KLM} are introduced by

$$F=F_{KLM}\,\theta^K\wedge\theta^L\wedge\theta^M\,. \tag{2.7}$$

Working out (2.4) one finds

$$F_{\alpha\beta s}=[d_\alpha A_{\beta s}-d_\beta A_{\alpha s}-\kappa(Af_sA)_{\alpha\beta}]\,, \tag{2.8}$$

$$F_{\beta vs}=\{d_\beta A_{vs}-d_vA_{\beta s}+d_sA_{\beta v}+\kappa[(Af_vA)_{\beta s}-(Af_sA)_{\beta v}]+[\omega_{vs}{}^t-\omega_{sv}{}^t]A_{\beta t}\}\,, \tag{2.9}$$

$$F_{svt}=\tfrac{1}{3}\{d_sA_{vt}+d_tA_{sv}+d_vA_{ts}-2[\omega_{vs}{}^wA_{wt}+\omega_{tv}{}^wA_{ws}+\omega_{st}{}^wA_{wv}]+\kappa[(Af_tA)_{sv}+(Af_vA)_{ts}+(Af_sA)_{vt}]\}\,, \tag{2.10}$$

where the matrix convention for A and f_K has to be remembered. The Lagrangian displays a manifest formal rotational symmetry under SO$(1,3+k)$. Note that it is by employment of this formal symmetry that the relative factors in (2.6) are being fixed.

Before specifying the derivative with respect to group coordinates let us introduce the analog of (1.2) for Higgs fields. It reads

$$A^g_{sv}=U_s{}^tU_v{}^{t'}A_{tt'}+\frac{1}{4g}\,[f_s{}^{tr}(d_vU_t{}^{t'})U^{-1}_{t'-r}-f_v{}^{tr}(d_sU_t{}^{t'})U^{-1}_{t'r}]\,. \tag{2.11}$$

Guided by our principle of formal symmetry we will identify the differentiations in (2.11) with those appearing in (2.9) and (2.10). It would be suggestive to choose $d_t = L_t$. In fact, in the group space, L_t is the exact geometric analog of d_α. We will see now, however, that such a simple identification will not work.

In fact, using an infinitesimal parametrization for U,

$$U_{tv} = g_{tv} + \epsilon_{vt} = g_{vt} + \epsilon^w f_{vwt}, \tag{2.12}$$

the inhomogeneous term in (2.11) is for $d_s = L_s$ given by

$$\frac{C_2^A}{2g} f_{sv}{}^w \epsilon_w, \tag{2.13}$$

where C_2^A is the value of the quadratic Casimir operator of the group in the adjoint representation.[17] In view of (1.1) and the identification[16] $2\omega_{vst} = f_{vst}$ the transformation behavior of F_{KLM} can now be discussed. In order that $\mathfrak{L}$ be invariant under gauge transformations we require homogeneous transformation laws for F,

$$\begin{aligned} \delta F_{\alpha\beta s} &= \epsilon_s{}^{s'} F_{\alpha\beta s'} \\ \delta F_{\alpha v s} &= \epsilon_v{}^{v'} F_{\alpha v' s} + \epsilon_s{}^{s'} F_{\alpha v s'} \\ \delta F_{vst} &= \epsilon_v{}^{v'} F_{v'st} + \epsilon_s{}^{s'} F_{vs't} + \epsilon_t{}^{t'} F_{vst'}. \end{aligned} \tag{2.14}$$

Using (2.13) one finds however that a homogeneous transformation law for $F_{\beta v s}$ cannot be derived. Therefore, $F_{\beta v s} F^{\beta v s}$ will not be gauge invariant.

A second best choice for d_s would be

$$d_s = N_s{}^{s'} L_{s'}, \tag{2.15}$$

which means geometrically that d_s is a k-tuplet of directional derivatives. In order to preserve Cartan's first identity we have to make the further identification

$$2\omega_{vst} = N_s{}^{s'} f_{vs't}. \tag{2.16}$$

Again $F_{\beta v s}$ and the transformation behavior of $A_{\alpha s}$ and A_{vs} are fully specified. In order to derive a homogeneous transformation law for $F_{\beta v s}$ under (1.2) and (2.11), one finds upon explicit calculation that N_{sv} has to be antisymmetric. In particular the simple choice $d_s = L_s$ is not possible, as has been remarked already above.

In order that A_{sv}^ϵ satisfies the same differential equation as A_{sv} we will require

$$d_t N_{sv} = f_{st}{}^w N_{wv} + f_{vt}{}^w N_{sw}. \tag{2.17}$$

The matrix N itself will be assumed not to transform under gauge transformations and not to display explicit dependence on Minkowski coordinates; i.e., $d_\alpha N_{sv} = 0$.

(2.11) is now well defined. For completeness we write its infinitesimal version as well as that of (1.2):

$$\delta A_{sv} = [\epsilon, A]_{sv} - \frac{C_2^A}{4g}[\epsilon, N]_{sv}, \tag{2.18}$$

$$\delta A_{\alpha s} = \epsilon_{st} A_\alpha{}^t + \frac{C_2^A}{2g} d_\alpha \epsilon_s. \tag{2.19}$$

(2.18) coincides with the transformation behavior of the physical Higgs field in the conventional hidden gauge symmetry formalism.[18] This aspect will be discussed in more detail below.

We are now in position to carry out the differentiations with respect to group coordinates in (2.9) and (2.10). One finds

$$\begin{aligned} F_{\beta v s} = \{ & d_\beta A_{vs} - \tfrac{1}{2}[(Nf_s A)_{v\beta} - (Nf_v A)_{s\beta}] \\ & + \kappa[(Af_s A)_{v\beta} - (Af_v A)_{s\beta}]\} \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} F_{stv} = \tfrac{1}{3}\{ & \tfrac{1}{2}[(Nf_t A)_{sv} + (Nf_v A)_{ts} + (Nf_s A)_{vt} \\ & + (Af_t N)_{sv} + (Af_v N)_{ts} + (Af_s N)_{vt}] \\ & - \kappa[(Af_t A)_{sv} + (Af_v A)_{ts} + (Af_s A)_{vt}]\}. \end{aligned} \tag{2.21}$$

The requirement that under gauge transformations (2.19)

$$\delta F_{\alpha\beta s} = \epsilon_s{}^t F_{\alpha\beta t} \tag{2.22}$$

fixes g in terms of the gauge coupling κ:

$$2g = -C_2^A \kappa. \tag{2.23}$$

Therefore, instead of (2.18) and (2.19),

$$\delta A_{\alpha s} = \epsilon_{st} A_\alpha{}^t - \frac{1}{\kappa} d_\alpha \epsilon_s, \tag{2.24}$$

$$\delta A_{sv} = [\epsilon, A]_{sv} + \frac{1}{2\kappa}[\epsilon, N]_{sv}. \tag{2.25}$$

Under (2.24) and (2.25), $F_{\beta v s}$ transforms homogeneously as specified in (2.14). For F_{stv} we find

$$\begin{aligned} \delta F_{stv} = {} & \epsilon_s{}^{s'} F_{s'vt} + \epsilon_v{}^{s'} F_{ss't} + \epsilon_t{}^{s'} F_{svs'} \\ & + \frac{1}{12\kappa}\{([\epsilon, N]f_t N)_{sv} + (Nf_t[\epsilon, N])_{sv} \\ & + ([\epsilon, N]f_v N)_{ts} + (Nf_v[\epsilon, N])_{ts} \\ & + ([\epsilon, N]f_s N)_{vt} + (Nf_s[\epsilon, N])_{vt}\}. \end{aligned} \tag{2.26}$$

In order that F_{stv} transforms homogeneously as is necessary to get an invariant Lagrangian, we have to require

$$\begin{aligned} \{ & ([\epsilon, N]f_t N)_{sv} + (Nf_t[\epsilon, N])_{sv} \\ & + ([\epsilon, N]f_v N)_{ts} + (Nf_v[\epsilon, N])_{ts} \\ & + ([\epsilon, N]f_s N)_{vt} + (Nf_s[\epsilon, N])_{vt}\} = 0, \end{aligned} \tag{2.27}$$

which means that

$$\{(Nf_t N)_{sv} + (Nf_v N)_{ts} + (Nf_s N)_{vt}\} \tag{2.28}$$

are formally to be the components of an invariant tensor under group transformations. Therefore

$$h_{svt}=0, \tag{2.29}$$

or equivalently

$$h_{svt}h^{svt}=0, \tag{2.30}$$

since the gauge group is compact, with q = constant and

$$h_{svt}=\{(Nf_t N)_{sv}+(Nf_v N)_{ts}+(Nf_s N)_{vt}+qf_{svt}\}. \tag{2.31}$$

First solutions of the constraint (2.29) will be given below.

Using (2.20) and (2.21) it can be seen that the Lagrangian displays a Goldstone-Higgs-Kibble phenomenon. In fact, setting

$$A_{sv}=B_{sv}+\frac{1}{2\kappa}N_{sv}, \tag{2.32}$$

one finds from (2.20) and (2.21), respectively, that

$$F_{\beta vs}=\{d_\beta B_{vs}+\kappa[(Bf_s A)_{v\beta}-(Bf_s A)_{s\beta}]\} \tag{2.33}$$

and

$$3F_{stv}=\left\{\frac{1}{4\kappa}[(Nf_t N)_{sv}+(Nf_v N)_{ts}+(Nf_s N)_{vt}] -\kappa[(Bf_t B)_{sv}+(Bf_v B)_{ts}+(Bf_s B)_{vt}]\right\}. \tag{2.34}$$

$F_{\beta vs}$ and F_{stv} transform as above, if N does not change under gauge transformations, if the constraint (2.29) is fulfilled, if $A_{\alpha s}$ transforms according to (2.24), and if B_{vs} transforms homogeneously,

$$\delta B_{sv}=[\epsilon,B]_{sv}. \tag{2.35}$$

Therefore B is to be identified with that Higgs field that the normal hidden gauge symmetry formalism starts off with and N corresponds to the vacuum expectation value of B.

In fact, this identification is compatible with the observation that $F_{svt}F^{svt}$, the would-be Higgs potential, displays a minimum at

$$B_{vs}=\pm\frac{1}{2\kappa}N_{vs} \tag{2.36}$$

or

$$A_{vs}=0,\quad A_{vs}=\frac{1}{\kappa}N_{vs}. \tag{2.37}$$

Note that the latter choice in (2.37) leads to a Lagrangian equivalent to the one following from (2.21).

The shift (2.32) has been done after the differentiations with respect to group coordinates have been carried out. Because of the condition (2.17), however, we would have arrived at the same results if the shift had been carried out first.

I conclude this section with first a discussion of the constraint (2.29). For SU_2 as a gauge group, (2.29) is an identity for all N, provided q is normalized to $q=-\mathrm{tr}(NN)$. (Compare Ref. 14.) For higher groups the situation is much more difficult. There exist special though degenerate solutions (2.29), however. For definiteness consider the gauge group SU_3. Introducing

$$(T_{st})^{vw}=(g_s^v g_t^w-g_t^v g_s^w), \tag{2.38}$$

they may be written (note that late Latin indices run from 5 to 4+8=12 in this case)

$$N=a_1T_{5,12}+a_2T_{6,12}+a_3T_{7,12}, \tag{2.39}$$

where $a_{1,2,3}$ are constants. N may be characterized by

$$\mathrm{tr}(f_v N)=0,\quad v=5\ldots 12. \tag{2.40}$$

The mass term for the vector bosons following from (2.20) is proportional to

$$\mathrm{tr}([N,A_\alpha])^2, \tag{2.41}$$

where A_α is defined as $A_\alpha=A_\alpha{}^s f_s$. Working out the commutators $[f_v,N]$, one finds by counting the number of linear dependencies among them that the little group of N (equation (2.39)) is two dimensional. We thus would have two massless vector bosons. Unfortunately, (2.39) corresponds to $q=0$ [cf. (2.31)] and thus leads to a potential with a degenerate ground state. I am presently trying to find further solutions of the constraint (2.29).

III. DISCUSSION

The reader might wonder how the field strength tensors (2.20) and (2.21) have been found in the first place. In short, the answer is the following. For once one requires a transformation behavior for gauge and Higgs fields as given by (1.2) and (2.11), respectively. One then desires to write all field strength tensors F_{KLM} in the form

$$F_{KLM}=d_k A_{LM}-d_L A_{KM}+G_{KLM}, \tag{3.1}$$

where G_{KLM} is to be determined such that F_{KLM} transforms homogeneously under (2.11) and (1.2). For geometrical reasons, d_s should be essentially given by L_s; for reasons given in the preceding section, d_s has been set $d_s=N_s{}^tL_t$. The form (3.1) is motivated by the fact that $F_{\alpha\beta s}$, the usual Yang-Mills field strength tensor, is already of the form (3.1).

Choosing the A_{st} to be antisymmetric follows from the constraints on the construction of F_{vst}. In fact, in terms of the homogeneously transforming Higgs field B_{vs}, F_{vst} cannot contain terms linear in B [which would have to be of the form $(Nf_vB)_{st}$, etc.] since such terms could not give rise to a homogeneous transformation law for F_{vst} if N does not transform under gauge transformations. Since further F_{vst} as a function of A_{vs} by construction does not contain a term independent of A_{vs}, F_{vst} as a function of B must have a term bilinear in N. Following the arguments of the preceding section, this term must be proportional to an invariant tensor of rank 3 under formal group transformations $\delta N = [\epsilon, N]$. Using the input (3.1), for a symmetric A_{vs} this tensor would have mixed symmetry, i.e., would be partially symmetric, partially antisymmetric. Such an invariant tensor for rank 3 does not exist, however.

For symmetric A_{vs}, a totally symmetric F_{vst} may however be found using much the same arguments as lead to the construction of the totally antisymmetric F_{vst} for antisymmetric A_{vs}. Having however different symmetry properties for components of F_{KLM} with different values of K, L, M is not compatible with the requirement that a formal symmetry exists that transforms the various components of F_{KLM} into each other. Considering the possibility of applications I summarize here what the structure described in the preceeding section at the best can do.

Higgs fields would be restricted to be in the reducible $k(k-1)/2$-dimensional antisymmetric tensor representation of the gauge group, k being the order of the group. This representation is bigger than the adjoint representation [except for SU(2)]. For example, for SU(3) it is 28-dimensional containing irreducible representations $\underline{8}$, $\underline{10}$, and $\underline{10}^*$. The possible symmetry-breaking patterns are however restricted by the constraint to be fulfilled by N.

For given N, the Higgs potential is fixed and, due to the assumption that the tensor F_{KLM} derives from the concise form (2.4), all relative normalizations in (2.6) are fixed. The Lagrangian of (2.6) contains essentially two free parameters: the self coupling of the gauge field and the mass of the vector bosons (or, equivalently, the radius of the group space). The mass of the Higgs bosons is then determined in terms of the vector-boson mass and the self coupling of the Higgs field in terms of the self coupling of the gauge field.

With regard to applications, clearly the biggest obstacle of the present construction is the constraint (2.29), whose possible solutions will be the subject of some future work.

To conclude, I would like to mention two papers whose objectives are rather similar to the present one. The first is by Manton[19] who gets a formal unification of gauge and Higgs fields by employing high dimensional Yang-Mills theories. The second is by Macrae.[20] The starting point of the latter paper is not unlike that of the present one. However, Macrae's Lagrangian is quite different from the one constructed here.

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APPENDIX

As has been remarked in the Introduction, a formally gauge-invariant theory containing gauge fields and would-be Higgs fields based on the use of algebra-valued one-forms may be constructed. This will be demonstrated in this appendix. As will also be seen, however, the theory does not display a Goldstone-Higgs-Kibble mechanism unlike the theory based on two-forms described in the previous sections.

Introduce

$$A_\alpha = A_\alpha{}^s f_s, \quad A_s = A_s{}^t f_t, \tag{A1}$$

and

$$A = A_L \theta^L, \tag{A2}$$

where the range of indices and other conventions are as in the previous sections. In generalization of the usual field strength tensor $F_{\alpha\beta}$ of a Yang-Mills theory, we introduce now

$$F_{\alpha\beta} = d_\alpha A_\beta - d_\beta A_\alpha + \kappa[A_\alpha, A_\beta], \tag{A3}$$

$$F_{\alpha s} = d_\alpha A_s - d_s A_\alpha + \kappa[A_\alpha, A_s], \tag{A4}$$

$$F_{st} = d_s A_t - d_t A_s + \kappa[A_s, A_t] + 2\omega^v{}_{st} A_v. \tag{A5}$$

Equations (A3)–(A5) may be obtained as components of the two-form

$$F = d(A_L \theta^L) + \kappa A \wedge A, \tag{A6}$$

where

$$A \wedge A = A_K A_L \theta^K \wedge \theta^L \tag{A7}$$

and Cartan's first identity has been taken into account.

The gauge transformations

$$\begin{aligned} A_\alpha &\to U A_\alpha U^{-1} + \frac{1}{\kappa} U d_\alpha U^{-1}, \\ A_s &\to U A_s U^{-1} + \frac{1}{\kappa} U d_s U^{-1}, \end{aligned} \tag{A8}$$

lead to homogeneous transformation laws for F_{KL},

$$F_{KL} \to UF_{KL}U^{-1}. \tag{A9}$$

and thus to a gauge-invariant Lagrangian $\mathrm{tr} F_{KL}F^{KL}$ if

$$[d_s, d_t] = 2\omega^v{}_{st} d_v. \tag{A10}$$

Note that for $d_s = L_s$ we have the identification

$$2\omega^v{}_{st} = f^v{}_{st}. \tag{A11}$$

In this case (A10) is identically true. If we set $d_s = N_s{}^t L_t$, (A10) will give a constraint on N_{sv} which plays the same role as (2.29). However in this case, unlike the situation in Sec. II, we would have no reason to introduce directional derivatives $N_s{}^t L_t$ except the desire to produce certain mass spectra for the vector bosons.

In order to see whether the Lagrangian displays a Goldstone-Higgs-Kibble phenomenon it is convenient to project components $F_{\alpha s}{}^t$ and $F_{st}{}^v$. One finds

$$F_{\alpha sw} = d_\alpha A_{sw} - (Nf_w A)_{s\alpha} + \kappa (Af_w A)_{\alpha s} \tag{A12}$$

and

$$F_{stw} = (Nf_w A)_{st} - (Nf_t A)_{sw} - (Nf_w A)_{ts} + \kappa (Af_w A)_{st}. \tag{A13}$$

From (A12) onwards, formulas are specified for antisymmetric A_{vs} and N_{vs}. For the symmetric case a similar analysis can be carried out. The case $N_{vs} \sim g_{vs}$ is a special case, since g_{vs} itself is an invariant tensor under group transformations. From (A12), physical Higgs field A_{vs} and homogeneously transforming Higgs field B_{vs} should be related by

$$A_{sv} = B_{sv} - \frac{1}{\kappa} N_{sv} \tag{A14}$$

which gives

$$F_{\alpha sw} = d_\alpha B_{sw} + \kappa (Af_w B)_{\kappa s} \tag{A15}$$

and

$$F_{stw} = \frac{1}{\kappa}(Nf_t N)_{sw} + \frac{1}{\kappa}(Nf_w N)_{ts} - (Nf_w B)_{ts} + \kappa (Bf_w B)_{st}. \tag{A16}$$

(A16) contains a term linear in B and therefore does not transform in the right way, i.e., homogeneously. We therefore conclude that the model given by (A3)–(A5) is not equivalent to a gauge model with hidden gauge symmetry. The latter requirement is essential in the context of renormalizability considerations.

The only exception to the above consideration is the case when for symmetric A_{vs} we choose $N_{vs} \sim g_{vs}$, then the transformation properties of (A13) survive the shift (A14). In this case (as, actually, in all others) the second equation of (A8) leads to the transformation behavior expected for physical Higgs fields. The model (A3)–(A5) might therefore serve as a theoretical laboratory for the discussion of the questions of the geometrical unification of gauge and Higgs fields, if we choose the identification $d_s = L_s$.

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[8]It is natural in Kaluza-Klein theories to introduce gauge fields $A^s_\alpha(x,y)$ that depend explicitly on the group coordinates y. For SU(2) their relation to the gauge fields normally used $[B^s_\alpha(x)]$ is given by $A^s_\alpha(x,y) \sim B^t_\alpha(x) D^1_{st}(y)$, where $D^1_{st}(y)$ are Wigner rotation functions (i.e., matrix elements of finite rotations) and y are the Eulerian angles (Ref. 7).

[9]This formula can be seen to be equivalent to the usual one

$$A^g_\alpha = UA_\alpha U^{-1} + (1/g) U d_\alpha U^{-1}$$

with

$$A_\alpha = A^s_\alpha f_s, \quad (f_w)_{st} = f_{swt}$$

by writing the latter in terms of matrix elements

$$(A_\alpha)^{vw} = A^s_\alpha f^v{}_s{}^w.$$

[10]The structure constants obey $L_t f_{svw} = 0$, which on the grounds of the Jacobi identity is equivalent to

$$L_t f_{svw} = f_{st}{}^{t'} f_{t'vw} + f_{vt}{}^{t'} f_{st'w} + f_{wt}{}^{t'} f_{svt'}.$$

[11]Nevertheless, a formal theory based on this concept that possesses a local gauge symmetry is possible. See the Appendix.

[12]This specifies the group representation given by the Higgs field to be the reducible antisymmetric tensor representation with dimension $k(k-1)/2$.

[13]This introduces a further breaking of the formal SO(1, 3 + k) symmetry.

[14]W. Mecklenburg, J. Phys. G 6, 1049 (1980).

[15]The exclusion of Abelian factors is necessary if one

wants to work with a nonsingular metric g_{st}.

[16]Cartan's first identity is $d\theta^K + \omega^K{}_L \wedge \theta^L = 0$, where $\omega^K{}_L = \omega^K{}_{LM}\theta^M$ are the connection one-forms. For the Minkowski coordinates a flat geometry with holonomic coordinates is being used. The only nonvanishing $\omega^K{}_{LM}$ are $\omega^s{}_{vt}$. If d is introduced through $df = (L_s f)\theta^s$, then $\omega^s{}_{tv} = \frac{1}{2} f^s{}_{tv}$ on grounds of the Maurer-Cartan identity for Lie groups. For details compare Y. Choquet-Bruhat, C. DeWitt-Morette, and M. Dillard-Bleick, *Analysis, Manifolds and Physics* (North-Holland, Amsterdam, 1977).

[17]It is introduced through

$f_s{}^{tv} f_{wtv} = -g_{sw} C_2^A$. For SU_2: $C_2^A = 2$; For SU_3: $C_2^A = 3$.

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