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Summable chains of instantons. II. Explicit integration of quantum fluctuation determinants

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Determinants arising in the computation of one-loop quantum fluctuations are obtained explicitly for a particular class of SU(2) multi-instanton backgrounds. For this class the space-time integral involved is evaluated analytically for arbitrary index. Results are given, for isospin 1/2 and 1, in terms of either the Dirac or the Klein-Gordon operator, these two being simply related. Correction to the dilute-gas approximation is displayed explicitly. The periodic Prasad-Sommerfield background is discussed as a limit of our class. Concluding remarks indicate further possible developments.

I. INTRODUCTION

Quantum fluctuation determinants corresponding to nonzero modes in multi-instanton backgrounds have been studied by a number of authors at different levels of generality.[1-9] Brown and Creamer[1] considered 't Hooft's multi-instanton solutions and expressed their result as a correction to a sum of separate single-instanton contributions obtained in an early work of 't Hooft.[10] This is expressed as a space-time integral. Then they remarked that "Unfortunately the integral $\overline{W}_1$ in (4.8) cannot be evaluated explicitly for arbitrary winding number n." In fact, even for $n=2$ one had to resort to approximations. Only for $n=1$ was the result totally explicit. (A recent work for the $n=2$ case is quoted at the end of Sec. III.) Hence an exact, explicit expression for the correction to the dilute-gas approximation is not available even for the 't Hooft class of multi-instanton background. Other authors[2-8] have studied the general instanton background corresponding to the $(8n-3)$-parameter Atiyah-Drinfeld-Hitchin-Manin (ADHM) solutions. Interesting general results have been obtained. But they naturally involve even more complicated integrals (including the relatively simple 't Hooft background as a particular case) which prove to be intractable so far as explicit analytic evaluation is concerned.

We will show that the relevant space-time integral can be evaluated explicitly, *for arbitrary index*, for a particular class of multi-instanton configurations. We studied in a preceding paper (Ref. 11 quoted hereafter as I) the Green's functions corresponding to this class instanton backgrounds. This class is obtained by imposing constraints on the parameters of the centers and sizes of the 't Hooft SU(2) solution. In the notations of I, the gauge potentials turn out to be

$$A_{\mu a}=\eta^{+}_{\mu\nu a}\partial_\nu[\ln\rho(x)]\,, \tag{1.1}$$

where

$$\eta^{\pm}_{jka}=\epsilon_{jka},\quad \eta^{\pm}_{k4a}=\pm\delta_{ka}\quad (a,j,k=1,2,3)$$

and

$$\rho(x)=1+\sum_{k=1}^{\alpha-1}\frac{\csc^2(k\pi/\alpha)}{[t-\cot(k\pi/\alpha)]^2+r^2}\quad(\alpha=2,3,\ldots) \tag{1.2}$$

for index $(\alpha-1)$.

An equivalent form is given by

$$\rho(x)=\sum_{k=0}^{\alpha-1}\frac{\sec^2(k\pi/\alpha)}{[t-\tan(k\pi/\alpha)]^2+r^2}\,. \tag{1.3}$$

Various properties and significances of (1.2) or (1.3) are discussed in detail in I, for example, their relations to static de Sitter solutions and to single "multicharged" instantons of "size" one in Witten's sense. (Detailed references can be found in I). Here, for a given index, the parameters are fixed. But this is done in such a way that in many applications the results can be made (without restricting the instanton number) as explicit as for the one-instanton case. This is the remarkable property we demonstrated in I concerning the Green's functions and will demonstrate again here for the fluctuation determinants.

II. THE INTEGRAL IN QUESTION

Various cases are considered in Refs. 1 to 9 and even for the same case different authors do not always express the result in an identical form. But one finds that for the 't Hooft SU(2) multi-instanton background the integral that remains to be evaluated can be taken to be

$$I=\int d^4x(\ln\tilde\rho)(\Box\Box\ln\tilde\rho)\,, \tag{2.1}$$

where, for index n,

$$\rho = 1 + \sum_{i=1}^{n} \frac{\lambda_i^2}{(x-y_i)^2} \tag{2.2}$$

and

$$\tilde{\rho} = \rho\left(\prod_{i=1}^{n} (x-y_i)^2\right). \tag{2.3}$$

See, for example, the Eqs. (28a) and (28b) of Ref. 8, which give, for the Dirac operator, the results for the fundamental and the adjoint representation, respectively.

Restricting (2.2) to (1.2) means [with $n=(\alpha-1)$] setting

$$\lambda_k = \frac{1}{\sin(k\pi/\alpha)}, \quad y_k = \left(\cot\frac{k\pi}{\alpha}, \vec{0}\right) \quad (k=1,\ldots,\alpha-1). \tag{2.4}$$

This gives [Eq. (2.22) of I]

$$\rho = \alpha\frac{\cot\alpha\omega - \cot\alpha\bar{\omega}}{\cot\omega - \cot\bar{\omega}}, \tag{2.5}$$

where

$$z \equiv (t+ir) = \cot\omega. \tag{2.6}$$

In the coordinates (τ, R_*) defined by

$$t + ir = \tan\left(\frac{\tau + iR_*}{2}\right) \tag{2.7}$$

or

$$\omega = \frac{\pi}{2} - \tfrac{1}{2}(\tau + iR_*), \tag{2.8}$$

$$\begin{aligned}\rho &= \left(\frac{\alpha\sinh\alpha R_*}{\sinh R_*}\right)\left(\frac{\cosh R_* + \cos\tau}{\cosh\alpha R_* - (-1)^\alpha\cos\alpha\tau}\right)\\ &= \alpha^2 e^{-\chi}\left(\frac{\cosh R_* + \cos\tau}{\cosh\alpha R_* - (-1)^\alpha\cos\alpha\tau}\right),\end{aligned} \tag{2.9}$$

where

$$e^{\chi} = \left(\frac{\alpha\sinh R_*}{\sinh\alpha R_*}\right) \quad (\alpha = 2,3,\ldots). \tag{2.10}$$

It will be noted that while

$$-\infty < t < \infty, \quad 0 \leq r < \infty \tag{2.11}$$

in terms of (τ, R_*) the equivalent domain becomes

$$0 \leq \tau < 2\pi, \quad 0 \leq R_* < \infty. \tag{2.12}$$

(See I for a detailed discussion of the significance of these coordinates.) In the integration to follow, we will *not* ampute the domain $[-\infty, +\infty]$ of the Euclidean time in flat space. The finite domain $(0, 2\pi)$ of τ, achieved by a coordinate transformation, has a significance quite different in the context of flat space from the restriction of the t integration over one chosen period for periodic solutions (see Sec. IV). Except in the limiting case of Sec. IV, we consider solutions which are *not* periodic in t. We emphasize this fact, probably evident to many, to avoid trivial confusions.

Again, now

$$\begin{aligned}\prod_{k=1}^{\alpha-1}(x-y_k)^2 &= \prod_{k=1}^{\alpha-1}\left[\left(t-\cot\frac{k\pi}{\alpha}\right)^2 + r^2\right]\\ &= \frac{1}{\alpha^2}\frac{\sin\alpha\omega\,\sin\alpha\bar{\omega}}{(\sin\omega\,\sin\bar{\omega})^\alpha}.\end{aligned} \tag{2.13}$$

Hence, finally

$$\tilde{\rho} = 2^{(\alpha-1)}e^{-\chi}(\cos\tau + \cosh R_*)^{-(\alpha-1)}. \tag{2.14}$$

The factor

$$\Box\Box\ln\tilde{\rho} = -\mathrm{Tr}FF^* \tag{2.15}$$

such that

$$-\frac{1}{16\pi^2}\int d^4x\,\Box^2\ln\tilde{\rho} = (\alpha-1).$$

The change from ρ to $\tilde{\rho}$ eliminates the δ singularities. This gauge-invariant density is most conveniently derived for our purpose by starting with the potentials in the gauge [see (2.10) of I]

$$\begin{aligned}A_t &= \frac{\partial\chi}{\partial r}\hat{\Phi},\\ A_r &= -\left(\frac{\partial\chi}{\partial t}\right)\hat{\Phi},\\ A_j &= (e^\chi - 1)i[\hat{\Phi}, \partial_j\hat{\Phi}] \quad (j=\theta,\varphi),\end{aligned} \tag{2.16}$$

where

$$\hat{\phi} = \tfrac{1}{2}(\sin\theta\cos\varphi\sigma_1 + \sin\theta\sin\varphi\sigma_2 + \cos\theta\sigma_3). \tag{2.17}$$

The required gauge transformation is given in I. The well-known expression for such an ansatz is

$$\mathrm{Tr}FF^* = \frac{2}{r^2}\left\{\partial_r\left[(e^{2\chi}-1)\frac{\partial\chi}{\partial r}\right] + \partial_t\left[(e^{2\chi}-1)\frac{\partial\chi}{\partial t}\right]\right\} \tag{2.18}$$

$$= \frac{2}{r^2}(\partial_t^2 + \partial_r^2)(\tfrac{1}{2}e^{2\chi} - \chi). \tag{2.19}$$

Hence carrying out the trivial angular integrations

$$I = -8\pi\iint dt\,dr\ln\tilde{\rho}[(\partial_t^2 + \partial_r^2)(\tfrac{1}{2}e^{2\chi} - \chi)]. \tag{2.20}$$

From (2.7) one notes that

$$dt\,dr(\partial_t^2 + \partial_r^2) = d\tau\,dR_*(\partial_\tau^2 + \partial_{R_*}^2). \tag{2.21}$$

Since χ is independent of τ,

$$I = -8\pi\int_0^\infty dR_*\left(\int_0^{2\pi} d\tau\ln\tilde{\rho}\right)\frac{\partial^2}{\partial R_*^2}(\tfrac{1}{2}e^{2\chi} - \chi). \tag{2.22}$$

The τ integration gives, using (2.14) and Ref. 12,

$$\int_0^{2\pi} d\tau \ln\tilde{\rho} = 2\pi[-\chi + (\alpha-1)\ln 2] - 2(\alpha-1)\int_0^{\pi} d\tau \ln(\cosh R_* + \cos\tau) = -2\pi[\chi + (\alpha-1)R_* - 2(\alpha-1)\ln 2]\,. \quad (2.23)$$

Hence

$$I = 16\pi^2 \int_0^\infty dR_* [\chi + (\alpha-1)R_* - 2(\alpha-1)\ln 2] \frac{d^2}{dR_*^2}(\tfrac{1}{2}e^{2\chi} - \chi)\,. \quad (2.24)$$

(Note how the coordinates τ, R_* drastically simplify the integration problem. This is no accident since we started in I from static de Sitter solutions.) This is a finite integral. But in order to do the integration we had to break it up into pieces some of which are individually divergent. So we will take the limits ϵ and $\epsilon' \to 0$, where the domain of R_* is $[\epsilon', 1/\epsilon]$. All the divergent terms will be seen to cancel out at the end.

Partial integrations lead to

$$I = \lim_{\epsilon,\epsilon'\to 0}(-16\pi^2)\left[\left([\chi + (\alpha-1)R_* - 2(\alpha-1)\ln 2](e^{2\chi}-1)\frac{d\chi}{dR_*}\right)_{\epsilon'}^{1/\epsilon} - (\alpha-1)(\tfrac{1}{2}e^{2\chi} - \chi)_{\epsilon'}^{1/\epsilon}\right] + \lim_{\epsilon,\epsilon'\to 0}(16\pi^2)\int_{\epsilon'}^{1/\epsilon} dR_* (e^{2\chi}-1)\left(\frac{d\chi}{dR_*}\right)^2. \quad (2.25)$$

Using (2.10) one has (neglecting terms tending to zero)

$$\frac{I}{(-16\pi^2)} = 2(\alpha-1)^2\ln 2 - 2(\alpha-1)\ln\alpha - \tfrac{1}{2}(\alpha-1) + \lim_{\epsilon,\epsilon'\to 0}\left[(\alpha-1)^2\frac{1}{\epsilon} + \int_{\epsilon'}^{1/\epsilon}(e^{2\chi}-1)\left(\frac{d\chi}{dR_*}\right)^2 dR_*\right]. \quad (2.26)$$

The term in the square brackets is finite. Thus finally it remains to evaluate the integral [using (2.10)]

$$\lim_{\epsilon,\epsilon'\to 0}\int_{\epsilon'}^{1/\epsilon} dR_* \left(\frac{\alpha^2\sinh^2 R_*}{\sinh^2\alpha R_*} - 1\right)(\coth R_* - \alpha\coth\alpha R_*)^2\,. \quad (2.27)$$

This is done in Appendix A. Putting all the terms together, one obtains (for integer $\alpha > 1$)

$$I = (-16\pi^2)\left[\frac{\alpha^3}{6} + 2(\alpha-1)^2\ln 2 - 2(\alpha-1)\ln\alpha - \frac{1}{2} - \frac{\alpha}{6}(\alpha^2-1)\left(\frac{\pi}{\alpha}\cot\frac{\pi}{\alpha}\right) - 2\left(\sum_{k=1}^{\alpha-1}\frac{k\pi}{\alpha}\cot\frac{k\pi}{\alpha}\right)\right]. \quad (2.28)$$

For the simplest case $\alpha = 2$, this gives

$$I = (-16\pi^2)\tfrac{5}{6}\,. \quad (2.29)$$

This can be verified by a direct integration of (2.1) setting

$$\rho = 1 + \frac{1}{t^2 + r^2} = 1 + \frac{1}{x^2}\,. \quad (2.30)$$

Namely, by evaluating directly,

$$I = (-16\pi^2)6\int_0^\infty dy \ln(1+y)\frac{y}{(1+y)^4} \quad (y = x^2). \quad (2.31)$$

After a partial integration, this leads to (2.29). For a scale factor λ, i.e., for

$$\rho = 1 + \frac{\lambda^2}{x^2}\,, \quad (2.32)$$

one has by rescaling

$$I = (-16\pi^2)(\ln\lambda^2 + \tfrac{5}{6})\,. \quad (2.33)$$

III. EXPLICIT MULTI-INSTANTON DETERMINANTS AND CORRECTIONS TO "DILUTE-GAS" APPROXIMATIONS

In Ref. 8 the logarithms of the multi-instanton determinants (for the Dirac operator and the 't Hooft background) are conveniently displayed for both isospin $\frac{1}{2}$ and isospin 1. For our class of configurations we can now use (2.28) and the results of Appendix B to obtain the explicit forms of Γ and $\tilde{\Gamma}$ [Eqs. (28a) and (28b) of Ref. 8]. For $k = \alpha - 1$ one obtains

$$\Gamma = -(\alpha-1)[\tfrac{2}{3}\ln\mu + 4\zeta'(-1) + \tfrac{2}{3}\ln 2] + \tfrac{2}{3}\sum_{i=1}^{\alpha-1}\ln\lambda_i + \tfrac{1}{3}\sum_{r>s}\ln(y_r - y_s)^2 + \frac{1}{96\pi^2}\int d^4x \ln\tilde{\rho}\,\Box^2\ln\tilde{\rho} \quad (3.1)$$

$$= -(\alpha-1)[\tfrac{2}{3}\ln\mu + 4\zeta'(-1) + \tfrac{1}{3}\ln 2] - \tfrac{1}{3}\ln\alpha - \frac{1}{36}\left[(\alpha^3 - 3) - \alpha(\alpha^2-1)\left(\frac{\pi}{2}\cot\frac{\pi}{\alpha}\right)\right] + \frac{1}{3}\left(\sum_{k=1}^{\alpha-1}\frac{k\pi}{\alpha}\cot\frac{k\pi}{\alpha}\right). \quad (3.2)$$

(Here $\mu = -\sum ei \ln M_i$ represents the Pauli-Villars regularization terms.) For $\alpha = 2$, this reduces to

$$\Gamma = -[\tfrac{2}{3}\ln\mu + 4\zeta'(-1) + \tfrac{2}{3}\ln 2 + \tfrac{5}{36}] \tag{3.3}$$

for

$$\rho = 1 + \frac{1}{x^2}. \tag{3.4}$$

Similarly,

$$\tilde{\Gamma} = -4(\alpha-1)[\tfrac{2}{3}\ln\mu + 4\zeta'(-1) + \tfrac{1}{6}\ln 2] + \tfrac{20}{3}\sum_{i=1}^{\alpha-1}\ln\lambda_i + \tfrac{16}{3}\sum_{r>s}\ln(y_r - y_s)^2 + \frac{1}{6\pi^2}\int d^4x \ln\tilde{\rho}\,\Box^2\ln\tilde{\rho} + 2\ln R(\alpha-1) \tag{3.5}$$

$$= -4(\alpha-1)[\tfrac{2}{3}\ln\mu + 4\zeta'(-1) + \tfrac{1}{3}\ln 2] + \tfrac{2}{3}\ln\alpha - \tfrac{4}{9}(\alpha^3 - 3) + 4\ln[\Gamma(\alpha)] + \tfrac{4}{9}\alpha(\alpha^2-1)\frac{\pi}{\alpha}\cot\frac{\pi}{\alpha} + \frac{16}{3}\left(\sum_{k=1}^{\alpha-1}\frac{k\pi}{\alpha}\cot\frac{k\pi}{\alpha}\right) \quad [\Gamma(\alpha) = (\alpha-1)!\,]. \tag{3.6}$$

For $\alpha = 2$ this reduces to

$$\tilde{\Gamma} = -\tfrac{8}{3}\ln\mu - 16\zeta'(-1) - \tfrac{2}{3}\ln 2 - \tfrac{20}{9}. \tag{3.7}$$

This is seen to agree with the result (16) of Ref. 8, with $\ln\lambda = 0$ [since (3.4) holds for our case]. Again, comparing (3.3) and (3.7), one has (*for* $\alpha = 2$, i.e., for one instanton)

$$\tilde{\Gamma} = 4\Gamma + 2\ln 2 - \tfrac{5}{3}. \tag{3.8}$$

In view of (2.29) and (B20) this agrees with (20b) of Ref. 8. For a general scale factor λ_i, for $\alpha = 2$, i.e.,

$$\rho = 1 + \frac{\lambda_i^2}{(x - y_i)^2} \tag{3.9}$$

one obtains, setting in the formulas of Ref. 8,

$$\det M^{-1} = \det M_s^{-1} = (2\lambda_i^2), \quad \det M_A^{-1} = 1 \tag{3.10}$$

for single instantons

$$\Gamma_{(1,i)} = -[\tfrac{2}{3}\ln\mu + 4\zeta'(-1) + \tfrac{2}{3}\ln 2 + \tfrac{5}{36}] + \tfrac{1}{6}\ln\lambda_i^2 \tag{3.11}$$

and

$$\tilde{\Gamma}_{(1,i)} = -[\tfrac{8}{3}\ln\mu + 16\zeta'(-1) + \tfrac{2}{3}\ln 2 + \tfrac{20}{9}] + \tfrac{2}{3}\ln\lambda_i^2. \tag{3.12}$$

By a sensible choice of regularization method one can use the same μ throughout. Hence subtracting from (3.2) and (3.6) the sum of the respective *single*-instanton contributions with parameters $(y_i, \lambda_i; i = 1, \ldots, \alpha-1)$ given by (B2), we obtain [using (B14) to evaluate $\sum_i \ln\lambda_i^2$]

$$\Delta\Gamma \equiv \left(\Gamma - \sum_{i=1}^{\alpha-1}\Gamma_{(1,i)}\right) = -\frac{1}{36}\left[(\alpha^3 - 5\alpha + 2) - \alpha(\alpha^2-1)\left(\frac{\pi}{\alpha}\cot\frac{\pi}{\alpha}\right)\right] + \frac{1}{3}\left(\sum_{k=1}^{\alpha-1}\frac{k\pi}{\alpha}\cot\frac{k\pi}{\alpha}\right) \tag{3.13}$$

and

$$\Delta\tilde{\Gamma} \equiv \left(\tilde{\Gamma} - \sum_{i=1}^{\alpha-1}\tilde{\Gamma}_{(1,i)}\right) = 2\ln\alpha - 2(\alpha-1)\ln 2 - \tfrac{4}{9}(\alpha^3 - 5\alpha + 2) + 4\ln[(\alpha-1)!\,] + \tfrac{4}{9}\alpha(\alpha^2-1)\left(\frac{\pi}{\alpha}\cot\frac{\pi}{\alpha}\right) + \frac{16}{3}\left(\sum_{k=1}^{\alpha-1}\frac{k\pi}{\alpha}\cot\frac{k\pi}{\alpha}\right). \tag{3.14}$$

$\Delta\Gamma$ *and* $\Delta\tilde{\Gamma}$ *give the corrections to the "dilute-gas" approximation for our configurations. Their remarkable properties enable us to provide explicit expressions.* One point should however be noted. The dilute-gas approximation, namely additive treatment of single-instanton, or anti-instanton, contributions is associated with a cutoff on the instanton sizes, the centers being sufficiently far apart. This is what can render it consistent, though not necessarily realistic. In our configurations $\lambda_i^2 = 1 + y_i^2$, as emphasized in the remarks in Sec. V of I. Thus the $(\alpha - 1)$ instantons overlap thoroughly (Fig. 1). We hope that the quotation marks on the words dilute gas will serve as a reminder of this aspect. On the other hand, what is really remarkable is that by *not* cutting off sizes but letting them vary in a particular way one can

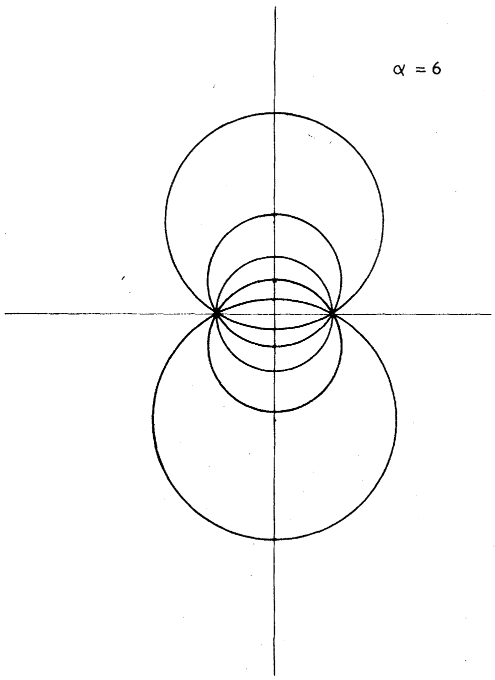

FIG. 1. Schematically, the relation between centers and sizes and the consequent overlapping of the instantons in our class of configurations. The case $\alpha = 6$ or index 5 is presented as a typical example. The definitions of centers and sizes are the usual ones employed in the context of 't Hooft-gauge representations.

extract an explicit, exact expression for the departure from additivity in this context.

So far we have been illustrating the properties of our background with reference to the Dirac operator.[8] The Klein-Gordon (KG) operator is essentially the same for our purposes. Still, for completeness and for direct comparison with the results of Ref. 6, we now display some of ours in terms of the KG operator.

In contrast to Ref. 8, the results of Ref. 6 we will have to use are given in the conformally extended 't Hooft gauge or the Jackiw-Nohl-Rebbi (JNR) gauge. As in Appendix B, we take the usual limit

$$\lambda_0^2 \to \infty, \quad y_0^2 \to \infty \text{ with } \lambda_0^2/y_0^2 \to 1 .$$

The remaining λ_i's and y_i's ($i = 1, 2, \ldots, \alpha - 1$) are all assumed to be finite. Then from the definition of f_ν of Ref. 6 (with $k = \alpha - 1$)

$$\lim(\ln \det f_\nu)$$

$$= -\ln\tilde{\rho} + \lim\left[-\ln\left(\frac{\lambda_0}{x - y_0}\right)^2 + \ln\left(1 + \frac{\sum_1^k \lambda_n^2}{\lambda_0^2}\right)\right]$$

$$= -\ln\tilde{\rho} , \qquad (3.15)$$

where $\tilde{\rho}$ is given by our (2.2) and (2.3). Thus $(-16\pi^2)J$ of Ref. 6 is our I of Sec. II. Similarly, using notations of Ref. 6,

$$\left(\ln \sum_0^k \lambda_n^2 - \ln \prod_0^k \lambda_n^2\right) \to -(k+1) \prod_1^k \lambda_k^2$$

and

$$t_{0n} \to \lambda_n^2 . \qquad (3.16)$$

Thus for $\det M_s^{-1}$, $\det M_A^{-1}$ (and their product $\det M^{-1}$), one gets back, as expected, exactly the 't Hooft gauge expressions of Ref. 8, evaluated for our configuration in Appendix B.

Different regularization conventions are used in Refs. 6 and 8 (Pauli-Villars and ζ-function regularizations, respectively). But for our purpose, it is sufficient to note that, formally, after taking the above-mentioned limit one has simply (comparing the corresponding results of Refs. 6 and 8)

$$(\Gamma + 2\mathfrak{D}) = -(\alpha - 1)(\tfrac{1}{3}\ln\mu^2 + \tfrac{1}{3}\ln 2) \qquad (k = \alpha - 1) \quad (3.17)$$

and

$$(\tilde{\Gamma} + 2\tilde{\mathfrak{D}}) = 4(\Gamma + 2\mathfrak{D}) . \qquad (3.18)$$

[As a useful check we have used the JNR gauge (1.3) for ρ, with *odd* α to have all λ's finite, and calculated directly, using the relevant results of Ref. 6 without any limiting process being involved. One gets back exactly, as one should, the results (3.15) and (3.16).] Moreover, defining $\Delta\mathfrak{D}$ and $\Delta\tilde{\mathfrak{D}}$ analogously to (3.11) and (3.12), it is now evident that

$$\Delta\Gamma + 2\Delta\mathfrak{D} = 0 , \qquad (3.19)$$

$$\Delta\tilde{\Gamma} + 2\Delta\tilde{\mathfrak{D}} = 10 . \qquad (3.20)$$

Hence the necessary expressions for $\mathfrak{D}$ and $\tilde{\mathfrak{D}}$ are obtained immediately from the preceding ones of this section. Using (3.11) and (3.12), we have displayed the numerical values of $\Delta\mathfrak{D}$ and $\Delta\tilde{\mathfrak{D}}$ for the lower values of α (see Table I and Fig. 2). Using the limiting expressions in Appendix A, one can show that as α becomes very large, retaining only the asymptotically leading terms,

$$\Delta\mathfrak{D} \to \tfrac{1}{6}\,\alpha\ln\alpha - 0.22\alpha \qquad (3.21)$$

and

$$\Delta\tilde{\mathfrak{D}} \to 4(\Delta\mathfrak{D}) . \qquad (3.22)$$

Thus it is seen that $\Delta\mathfrak{D}$ and $\Delta\tilde{\mathfrak{D}}$ (also, of course, $\mathfrak{D}$ and $\tilde{\mathfrak{D}}$) diverge faster than α as $\alpha\ln\alpha$.

It should be remembered in this context that for our configurations (and for our chosen scale) the "sizes" as defined in the 't Hooft gauge (through λ_i^2) become linked with the index $(\alpha - 1)$. We have

TABLE I. The values of $\Delta\mathfrak{D}$ and $\Delta\tilde{\mathfrak{D}}$ are obtained from (3.11), (3.12), (3.17), and (3.18). They represent, for isospin $\frac{1}{2}$ and isospin 1, respectively, the corrections to the so-called dilute-gas approximation, i.e., the departure from the sum of the contribution of the component instantons, each taken separately. The numerical evaluations involve small approximations. The exact value for $\alpha=2$ is, of course, zero for $\Delta\mathfrak{D}$ and $\Delta\tilde{\mathfrak{D}}$.

α	$\Delta\mathfrak{D}$	$\Delta\tilde{\mathfrak{D}}$
2	-0.389×10^{-9}	-0.433×10^{-8}
3	0.936×10^{-1}	0.400
4	0.246	0.104×10^{1}
5	0.441	0.187×10^{1}
6	0.671	0.284×10^{1}
7	0.929	0.392×10^{1}
8	0.121×10^{1}	0.509×10^{1}
9	0.151×10^{1}	0.636×10^{1}
10	0.183×10^{1}	0.769×10^{1}
11	0.217×10^{1}	0.910×10^{1}
12	0.252×10^{1}	0.105×10^{2}
13	0.289×10^{1}	0.120×10^{2}
14	0.327×10^{1}	0.136×10^{2}
15	0.366×10^{1}	0.152×10^{2}
16	0.407×10^{1}	0.169×10^{2}

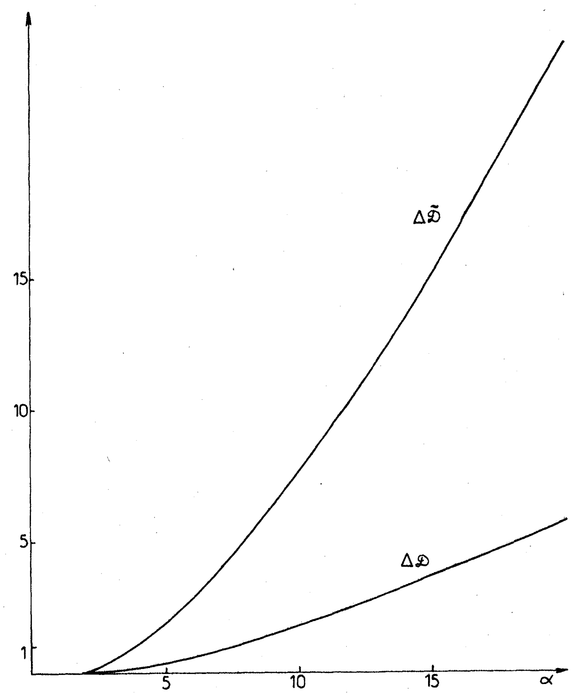

FIG. 2. Here the corrections to the dilute-gas approximations $\Delta\mathfrak{D}$ and $\Delta\tilde{\mathfrak{D}}$ are compared graphically. The values are taken from Table I. Only the integer values of α are significant in the present context. The curves serve merely to guide the eye.

$$\lambda_k^{\,2}=\frac{1}{\sin^2(k\pi/\alpha)}\quad(k=1,\ldots,\alpha-1). \tag{3.23}$$

For $k=1$, for example, as $\alpha\to\infty$,

$$\lambda_1^{\,2}\to\alpha^2. \tag{3.24}$$

Hence to limit sizes, for our class of configurations, we have to limit α. The divergence associated with sizes takes, for our case, the special form indicated by (3.21). The strict connection (3.23) can however be relaxed by a suitable rescaling of our initial ansatz. These aspects will be studied elsewhere, comparing different possible modifications, in connection with the construction of zero modes corresponding to our configurations.

We close this section by comparing an interesting ansatz for the 't Hooft case presented by Osborn and Moody[6] with our exact results. Their approximate form [Eqs. (24) of Ref. 6] gives for J

$$J^0=\ln\det M^{-1}+(\alpha-1)(\tfrac{5}{6}-\ln 2), \tag{3.25}$$

where

$$J=-\frac{1}{16\pi^2}\int d^4x\,\ln\tilde{\rho}\,\Box\Box\ln\tilde{\rho} \tag{3.26}$$

(taking the usual limit $\lambda_0^{\,2}\to\infty$, $y_0^{\,2}\to\infty$). Even J^0 presents some problems.

Rules can be given for construction of the determinant in J^0 for unconstrained centers and sizes.[2,8] But it seems that they cannot be used in practice to obtain a compact, explicit expression for arbitrary index. For our configurations we have exact, explicit expressions for the integral and also for $(\ln\det M^{-1})$. Using (2.28) and (B15), one obtains

$$\begin{aligned}\Delta J=J-J^0=\tfrac{1}{6}\,\alpha\left[\alpha^2-(\alpha^2-1)\frac{\pi}{\alpha}\cot\frac{\pi}{\alpha}\right]+(\alpha-1)\ln 2\\ -2\ln(\alpha-1)!-\tfrac{5}{6}\,\alpha-\ln\alpha+\tfrac{1}{3}\\ -2\left(\sum_{k=1}^{\alpha-1}\frac{k\pi}{\alpha}\cot\frac{k\pi}{\alpha}\right).\end{aligned} \tag{3.27}$$

The right-hand side looks complicated but the numerical values of ΔJ (see Table II) turn out to be quite small and the ansatz is seen to work remarkably well. The error propagated in $\mathfrak{D}$ is $\frac{1}{12}\,\Delta J$. For large α, asymptotically (using the limits obtained in Appendix A)

$$\begin{aligned}\Delta J\to\alpha\left[\frac{\pi^2}{18}+\frac{4}{3}-\ln 2\pi^2+2\gamma\right]\\ \approx 0.05\alpha.\end{aligned} \tag{3.28}$$

This should be compared with the asymptotically leading term $\frac{1}{6}\,\alpha\ln\alpha$ in $\mathfrak{D}$. Thus the ansatz is seen to work very well for all values of α. Our

TABLE II. Some numerical values of ΔJ given by (3.25). The Osborn-Moody ansatz evidently works remarkably well. The exact value for $\alpha=2$ is again zero, as in Table I.

α	$J-J_0$
2	-0.277×10^{-8}
3	0.255×10^{-1}
4	0.639×10^{-1}
5	0.108
6	0.155
7	0.204
8	0.254
9	0.305
10	0.356
11	0.408
12	0.460
13	0.512
14	0.564
15	0.617
16	0.669
17	0.722
18	0.775

exact results confirm the usefulness of the conformal and limiting properties incorporated in J^0.[6]

We add finally, that we have just received a report by G. P. Moody,[13] where the integral is calculated for the case of *two* instantons in 't Hooft's gauge. For $\alpha=3$, our result should emerge as a particular case of his. But this does not seem easy to verify as the results become quite complicated (even for index 2) as our particular type of constraints on center and sizes are relaxed.

IV. THE PERIODIC LIMIT

The periodic solution, gauge equivalent to the static Prasad-Sommerfield monopole, is given by setting in (1.1)

$$\rho=\sum_{m=-\infty}^{\infty}\frac{1}{[(t-2m\pi)^2+r^2]}=\frac{\cot(\bar z/2)-\cot(z/2)}{(z-\bar z)/2}=\frac{\sinh r}{2r(\cosh r-\cos t)}\quad(z\equiv t+ir)\,, \tag{4.1}$$

where we have normalized a scale parameter to 1. We showed in I how this case can be studied very conveniently as a limit ($\alpha\to\infty$) of our class given by (1.2) or (1.3). [In fact, (1.3) is better suited to the limiting process.] The gauge potentials giving the background and the corresponding Green's functions were obtained effortlessly as limits. In I, as in the present paper, our main topic is the nonperiodic, finite-action case of the preceding sections. The periodic limit served to show that the results of various lengthy studies (see the relevant references quoted in I) can be obtained immediately as byproducts of our technique. Here we want to examine briefly what an analogous limiting procedure gives for the determinants of Sec. III.

For the whole domain $[-\infty,\infty]$ of the Euclidean time t, (4.1) of course leads to infinite action. In the context of periodic backgrounds, one however integrates over one period 2π (for our chosen scale). This should not be confused with the domain 2π of τ in Secs. II and III [see the remarks following (2.12)]. It is known that for $0\leq t<2\pi$. (4.1) leads to an action $8\pi^2$ or index 1. For (1.2) or (1.3) one has the index $(\alpha-1)$. Here the limiting procedure to obtain the result for the periodic case is indicated in the simplest possible fashion, namely

$$\lim_{\alpha\to\infty}\frac{1}{\alpha}(\alpha-1)=1\,. \tag{4.2}$$

The factor $1/\alpha$ represents the effect of restriction to a single period $(0,2\pi)$ [or $(-\pi,+\pi)$] of t. Let us now note the effect of applying the same simple procedure on Γ, given by (3.2), noted here as $\Gamma_{(\alpha)}$. (The limits of $\tilde\Gamma$, $\mathfrak{D}$, $\tilde{\mathfrak{D}}$ can be treated analogously.) One has

$$\lim_{\alpha\to\infty}\frac{1}{\alpha}\Gamma_{(\alpha)}=-[\tfrac{2}{3}\ln\mu+4\zeta'(-1)+\tfrac{1}{3}\ln 2]-\tfrac{1}{36}\lim_{\alpha\to\infty}\left[\alpha^2-(\alpha^2-1)\frac{\pi}{\alpha}\cot\frac{\pi}{\alpha}\right]+\tfrac{1}{3}\lim_{\alpha\to\infty}\left(\frac{1}{\alpha}\sum_{k=1}^{\alpha-1}\frac{k\pi}{\alpha}\cot\frac{k\pi}{\alpha}\right)\,. \tag{4.3}$$

Using (A15) and (A20), one has

$$\lim_{\alpha\to\infty}\frac{1}{\alpha}\Gamma_{(\alpha)}=-[\tfrac{2}{3}\ln\mu+4\zeta'(-1)]-\frac{1}{36}\left(\frac{\pi^2}{3}+1\right)+\tfrac{1}{3}(\ln\pi+\gamma)-\tfrac{1}{3}\ln\alpha\,. \tag{4.4}$$

The source of the divergent term ($\approx\ln\alpha$) on the right-hand side can be better understood as follows. Rescaling the coordinates as $t\to t/2\alpha$, $r\to r/2\alpha$ [and similarly τ and R_* of (2.7)], one obtains from (2.9), as $\alpha\to\infty$,

$$\rho\to(4\alpha^2)\frac{\sinh r}{2r[\cosh r-(-1)^\alpha\cos t]}\,. \tag{4.5}$$

[In comparing with (4.1), one can suppose for simplicity that $\alpha\to\infty$ through even values. The relevant integrals, anyhow, do not depend on the sign preceding $\cos t$.] As mentioned in I, in deriving (4.1) as a limit one drops the divergent constant factor ($4\alpha^2$) in (4.5) since it does not contribute to A_μ [(1.1) involving logarithmic derivatives]. But in constructing determinants we started with a form with all the derivatives on one factor as in (2.1). The factor α^2 of ρ in (4.5) contributes to Γ a term

$$-\frac{(2\ln\alpha)}{(96\pi^2)}\int d^4x(\mathrm{Tr}F^*F) \qquad (4.6)$$

and in the limit precisely the term $-\frac{1}{3}\ln\alpha$ in (4.4). An exactly similar situation arises also for $\tilde{\Gamma}$, $\mathfrak{D}$, and $\tilde{\mathfrak{D}}$. In each case we have a term of the type $c\ln\alpha$, the constant c depending on the case considered. The origin in each case is the α^2 factor in (4.5) which does not contribute to the limiting form of the background potential A_μ. Hence through a suitable interpretation one should be able to regularize or simply discard such terms in a consistent fashion. Here we simply display the consequences of our limiting process.

Periodic backgrounds have been studied by several authors. Apart from the references quoted in I, the statistical mechanics for periodic backgrounds has been discussed in Refs. 14 and 15. Our (4.1) is a particular case of the periodic potentials considered (along with finite-temperature effects) in Refs. 14 and 15. It may be noted in this connection that the asymptotically leading-logarithmic term found in Ref. 15 [Eq. (6.13) of Ref. 15] corresponds formally to our ($\ln\alpha$) term. A close comparison makes this soon evident. In our approach and for our particular case [without an additional term +1 in (4.1)], we have to insist on the limit $\alpha\to\infty$ and reinterpret. But we are also able to calculate the nonleading terms exactly. As indicated in the following section, the next logical step in our program should be to formulate completely the functional integral for our finite-action nonperiodic configuration and some of its natural generalizations. So far as this turns out to be possible, results for the corresponding limiting periodic cases should again be obtainable in a relatively simple fashion.

V. REMARKS

For our class of configurations we have shown that for arbitrary index the Green's functions and the nonzero-mode determinants can be constructed as explicitly as for a single instanton. We started from the fact that the de Sitter space is not only conformally flat (a fact much exploited by various authors in studying flat-space instantons), but at the same time has a static spherical symmetry. In a series of papers we have shown how to exploit this situation to obtain various interesting results.[16] The paths thus opened up can lead to other results. In particular, we hope to continue the work started in I and in this paper in the following directions. We will study elsewhere the zero modes for our class of backgrounds and the corresponding measures. The complete functional integral should be obtained explicitly. This would be the logical next step. But one can try to go further. One can try to construct a hierarchy of multi-instanton configurations with specially simple and interesting properties, of which ours is the simplest. They can have, for example, as limits the multicharged monopole states recently constructed,[17-19] just as our present class has the singly charged Prasad-Sommerfield monopole as an infinite action limit. A passage via de Sitter space should again prove fruitful in this context. We hope to explore these aspects elsewhere.

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APPENDIX A: USEFUL INTEGRALS AND LIMITS

Our aim is to evaluate the integral (2.27). We start with the corresponding indefinite integral and break it up into relatively simple pieces. Partial integrations are used to extract the would-be divergent terms for the limits to be imposed. The remaining integrals are finite and easily available for the domain $[\infty, 0]$. This permits an easy and careful treatment of the limits. A good check is provided by the fact that the divergences of the separate terms must all cancel out at the limit zero and the only surviving divergent term at the limit of infinity must cancel exactly the term $(\alpha-1)^2/\epsilon$ in (2.26). Let

$$\tilde{I} = \int dx\left(\frac{\alpha^2\sinh^2 x}{\sinh^2\alpha x}-1\right)(\coth x-\alpha\coth\alpha x)^2 \qquad (A1)$$

$$=\alpha^2(I_1+\alpha^2 I_2-2\alpha I_3)-(I_4+\alpha^2 I_5-2\alpha I_6)\ (\alpha=2,3,\ldots)\,, \qquad (A2)$$

where

$$I_1=\int dx\frac{\sinh^2 x}{\sinh^2\alpha x}\coth^2 x=-\frac{1}{\alpha}\coth\alpha x+\int dx\frac{\sinh^2 x}{\sinh^2\alpha x}\,, \qquad (A3)$$

$$I_2 = \int dx \frac{\sinh^2 x}{\sinh^2 \alpha x} \coth^2 \alpha x$$
$$= -\frac{1}{3\alpha}\left[\sinh^2 x \coth^3 \alpha x + \frac{1}{2\alpha}\left(\frac{\sinh 2x}{\sinh^2 \alpha x} + \frac{2}{\alpha}\coth \alpha x\right) - \tfrac{1}{2}\sinh 2x - \frac{2}{\alpha}\int dx \frac{\sinh^2 x}{\sinh^2 \alpha x} + \int dx \frac{\sinh(\alpha-2)x}{\sinh \alpha x}\right], \quad \text{(A4)}$$

$$I_3 = \int dx \frac{\sinh^2 x}{\sinh^2 \alpha x} \coth x \coth \alpha x$$
$$= -\frac{1}{4\alpha}\left[\frac{\sinh 2x}{\sinh^2 \alpha x} + \frac{2}{\alpha}\coth \alpha x - 4\int dx \frac{\sinh^2 x}{\sinh^2 \alpha x}\right], \quad \text{(A5)}$$

$$I_4 = \int dx \coth^2 x = x - \coth x\,, \quad \text{(A6)}$$

$$I_5 = \int dx \coth^2 \alpha x = x - \frac{1}{\alpha}\coth \alpha x\,, \quad \text{(A7)}$$

and

$$I_6 = \int dx \coth x \coth \alpha x$$
$$= \int dx \left[1 + \frac{2}{e^{2x}-1} + \frac{2}{e^{2\alpha x}-1} + \frac{4}{(e^{2x}-1)(e^{2\alpha x}-1)}\right]$$
$$= x + \ln(1-e^{-2x}) + \frac{1}{\alpha}\ln(1-e^{-2\alpha x}) + 2\int \frac{du}{(u^\alpha - 1)}\left[\frac{1}{u-1} - \frac{1}{u}\right], \quad \text{(A8)}$$

where $u = e^{2x}$. Using (for $\alpha = 2, 3, \ldots$)

$$\frac{1}{u^\alpha - 1} = \frac{1}{\alpha}\sum_{k=0}^{\alpha-1} \frac{\omega_k}{(u-\omega_k)}\,, \quad \omega_k = e^{i2\pi k/\alpha}\,, \quad \text{(A9)}$$

$$I_6 = x + \ln\left(\frac{e^{2x}-1}{e^{2x}}\right) + \frac{1}{\alpha}\ln\left(\frac{e^{2\alpha x}-1}{e^{2\alpha x}}\right) + 2\left[-\frac{1}{\alpha(e^{2x}-1)} + 2x - \left(\frac{\alpha+1}{2\alpha}\right)\ln(e^{2x}-1) - \frac{1}{\alpha}\left(\sum_{k=1}^{\alpha-1}\frac{1}{(1-\omega_k)}\ln(e^{2x}-\omega_k)\right)\right]. \quad \text{(A10)}$$

Now we put in the limits. One has for the values of α considered,[20]

$$\int_0^\infty dx \frac{\sinh^2 x}{\sinh^2 \alpha x} = \frac{1}{2\alpha}\left(1 - \frac{\pi}{\alpha}\cot\frac{\pi}{\alpha}\right) \quad \text{(A11)}$$

and

$$\int_0^\infty dx \frac{\sinh(\alpha-2)x}{\sinh \alpha x} = \frac{\pi}{2\alpha}\cot\frac{\pi}{\alpha}\,. \quad \text{(A12)}$$

To evaluate the limits of (A10) one must use special properties of ω_k such as

$$\sum_{k=1}^{\alpha-1}\frac{\omega_k}{1-\omega_k} = -\left(\frac{\alpha-1}{2}\right), \quad \sum_{k=1}^{\alpha-1}\frac{1}{1-\omega_k} = \left(\frac{\alpha-1}{2}\right)$$

and (A13)

$$\sum_{k=1}^{\alpha-1}\frac{\ln(1-\omega_k)}{(1-\omega_k)} = \frac{1}{2}\left[\ln\alpha - \sum_{k=1}^{\alpha-1}\frac{k\pi}{\alpha}\cot\frac{k\pi}{\alpha}\right].$$

Finally one obtains

$$\lim_{\epsilon,\epsilon'\to 0}\int_{\epsilon'}^{1/\epsilon} dx\left(\frac{\alpha^2\sinh^2 x}{\sinh^2\alpha x} - 1\right)(\coth x - \alpha\coth\alpha x)^2\,,$$
$$= -(\alpha-1)^2\frac{1}{\epsilon} + \frac{1}{6}\alpha\left[\alpha^2 - (\alpha^2-1)\frac{\pi}{\alpha}\cot\frac{\pi}{\alpha}\right] - 2\left(\sum_{k=1}^{\alpha-1}\frac{k\pi}{\alpha}\cot\frac{k\pi}{\alpha}\right) + \tfrac{1}{2}(\alpha-2) \quad (\alpha = 2, 3, \ldots)\,. \quad \text{(A14)}$$

This leads to (2.28).

Let us now note the following useful asymptotic expressions for the terms involving cotangents,

$$\lim_{\alpha\to\infty}\left[\alpha^2 - (\alpha^2-1)\frac{\pi}{\alpha}\cot\frac{\pi}{\alpha}\right] = \left[\alpha^2 - (\alpha^2-1)\left(1 - \frac{1}{3}\frac{\pi^2}{\alpha^2}\right)\right] = \frac{\pi^2}{3} + 1\,. \quad \text{(A15)}$$

For the sum involving cotangents we proceed as follows. As $\alpha\to\infty$,

$$\frac{1}{\alpha}\sum_{k=1}^{\alpha-1}\left(\frac{k\pi}{\alpha}\cot\frac{k\pi}{\alpha} + \frac{k\pi/\alpha}{\pi - k\pi/\alpha}\right) \to \frac{1}{\alpha}\int_1^{\alpha-1}\left[\frac{\pi z}{\alpha}\cot\frac{\pi z}{\alpha} + \frac{\pi z/\alpha}{\pi - \pi z/\alpha}\right]dz\,, \quad \text{(A16)}$$

the remainder term tending to zero.

We note that

$$\int_0^\pi\left(x\cot x + \frac{x}{\pi - x}\right)dx = [x\ln\sin x - x - \pi\ln(\pi - x)]_0^\pi - \int_0^\pi \ln\sin x\, dx = [\pi\ln\pi - \pi] + \pi\ln 2 = \pi(\ln 2\pi - 1)\,. \quad \text{(A17)}$$

Also

$$\frac{1}{\alpha}\sum_{k=1}^{\alpha-1}\frac{k}{\alpha-k} = -\left(\frac{\alpha-1}{\alpha}\right) + \sum_{k=1}^{\alpha-1}\frac{1}{\alpha-k} = -1 + \left(1 + \tfrac{1}{2} + \cdots + \frac{1}{\alpha}\right) \quad \text{(A18)}$$

and

$$\lim_{\alpha\to\infty}\left(1 + \tfrac{1}{2} + \cdots + \frac{1}{\alpha}\right) = \ln\alpha + \gamma\,, \quad \text{(A19)}$$

where γ is the Euler's constant.[21] Combining the results (A16) to (A19) and neglecting remainder terms vanishing in the limit, one obtains

$$\lim_{\alpha\to\infty}\frac{1}{\alpha}\left(\sum_{k=1}^{\alpha-1}\frac{k\pi}{\alpha}\cot\frac{k\pi}{\alpha}\right)=-\ln\alpha-\gamma+\ln 2\pi\,. \tag{A20}$$

Hence the leading term for the sum

$$\sum_{k=1}^{\alpha-1}(k\pi/\alpha)\cot(k\pi/\alpha)$$

is seen to be $-\alpha\ln\alpha$ which is the same as that for $-\ln[(\alpha-1)!]$.

APPENDIX B: PRODUCTS AND DETERMINANTS

In Sec. III, we need the determinant $\det M^{-1}$, where [see Eq. (22) of Ref. 8]

$$M^{-1}_{il,jm}=\lambda_i\lambda_j\delta_{lm}+\lambda_l\lambda_m\delta_{ij}+(y^i-y^l)^2\delta_{ij}\delta_{lm}\quad(i,j,l,m=1,\ldots,\alpha-1) \tag{B1}$$

with in our case

$$\lambda_k=\left(\sin\frac{k\pi}{\alpha}\right)^{-1},\quad y_k=\cot\frac{k\pi}{\alpha}\quad(k=1,\ldots,\alpha-1,\ \alpha=2,3,\ldots)\,. \tag{B2}$$

This determinant can be factorized[8] as

$$\det M^{-1}=\det M_S^{-1}\det M_A^{-1}\,, \tag{B3}$$

where

$$\det M_S^{-1}=2^{(\alpha-1)}\prod_{i=1}^{\alpha-1}\lambda_i^2\prod_{r>s}(y^r-y^s)^2 \tag{B4}$$

and a more complicated expression in Ref. 8 for M_A^{-1}. We will evaluate directly $\det M^{-1}$ for our configurations using the results of Ref. 2.

Adapting the results of Sec. V of Ref. 2 to our case, namely making $(\lambda_0,y_0)\to\infty$ such that $\lambda_0^2/y_0^2\to 1$ and dividing each element by λ_0^4, one recovers (B1) from their (5.3). For the remaining λ's and y's we then use the special values (B12). Following through this technique, the results of Ref. 2 give

$$\det M^{-1}=2^{(\alpha-1)}(\det p)\left(\prod_{j=1}^{\alpha-1}\sin\frac{j\pi}{\alpha}\right)^4\left(\prod_{i\neq j}^{\alpha-1}\sin^2(i-j)\frac{\pi}{\alpha}\right), \tag{B5}$$

where the elements of the matrix p turn out upon examination to be

$$p_{ij}=\left(\frac{\alpha^2-1}{3}\right)\delta_{ij}-\left(\sin(i-j)\frac{\pi}{\alpha}\right)^{-2}(1-\delta_{ij})\quad(i,j=1,\ldots,\alpha-1)\,. \tag{B6}$$

(The last term is to be taken as strictly zero for $i=j$.)

This is exactly the matrix considered in Appendix C of I, where we inverted it. So, without going through the elaborate process indicated in Ref. 2, we can obtain $\det p$ from the results of Appendix C of I. One easily obtains

$$\det p=\frac{\det\Lambda}{\det Y}\,, \tag{B7}$$

where

$$\Lambda=\mathrm{diag}(\Lambda_1,\Lambda_2,\ldots,\Lambda_{\alpha-1}) \tag{B8}$$

with

$$\Lambda_k=2k(\alpha-k) \tag{B9}$$

and

$$Y_{ij}=1+\delta_{ij}\quad(i,j=1,\ldots,\alpha-1)\,. \tag{B10}$$

It can be shown that

$$\det Y=\alpha\,. \tag{B11}$$

Hence

$$\det p=\frac{2^{(\alpha-1)}[1\times(\alpha-1)][2\times(\alpha-2)]\cdots[(\alpha-1)\times 1]}{\alpha} \tag{B12}$$

$$=\frac{1}{\alpha}2^{(\alpha-1)}[\Gamma(\alpha)]^2\,. \tag{B13}$$

One can also show that

$$\prod_{j=1}^{\alpha-1}\sin\frac{j\pi}{\alpha}=\left(\frac{\alpha}{2^{\alpha-1}}\right) \tag{B14}$$

and

$$\prod_{i\neq j}^{\alpha-1}\sin^2(i-j)\frac{\pi}{\alpha}=\left(\frac{\alpha}{2^{(\alpha-1)}}\right)^{2(\alpha-2)}\,.$$

(Such and related, more elaborate results have been much used in I.)

Thus, finally,

$$\det M^{-1}=\alpha^{-(2\alpha-3)}2^{2(\alpha-1)^2}[\Gamma(\alpha)]^2\,. \tag{B15}$$

From (B4) and (B14) one also obtains readily

$$\det M_S^{-1}=\alpha^{-\alpha}2^{(\alpha^2-1)}\,. \tag{B16}$$

Hence from (B3),

$$\det M_A^{-1}=\left(\frac{2^{(\alpha-1)}}{\alpha}\right)^{(\alpha-3)}[\Gamma(\alpha)]^2\,. \tag{B17}$$

If one defines[8]

$$\det M_A^{-1}=R_{(\alpha-1)}\left[\prod_{r>s}(y_r-y_s)^2\right], \tag{B18}$$

one gets

$$R_{(\alpha-1)}=\left(\frac{\alpha}{2^{(\alpha-1)}}\right)[\Gamma(\alpha)]^2 . \qquad \text{(B19)}$$

For $\alpha=2$, the direct comparison of (B1) and (B15) is trivial giving

$$\det M^{-1}=2 \quad (\alpha=2) . \qquad \text{(B20)}$$

(Remember that for our class $\lambda_1=1$ for $\alpha=2$.)

For $\alpha=3$ one gets directly from (B1) and (B2)

$$\det M^{-1}=\det\begin{vmatrix} 2\lambda_1^2 & \lambda_1\lambda_2 & \lambda_1\lambda_2 & 0 \\ \lambda_2\lambda_1 & [\lambda_1^2+\lambda_2^2+(y_1-y_2)^2] & 0 & \lambda_1\lambda_2 \\ \lambda_2\lambda_1 & 0 & [\lambda_1^2+\lambda_2^2+(y_1-y_2)^2] & \lambda_1\lambda_2 \\ 0 & \lambda_2\lambda_1 & \lambda_2\lambda_1 & 2\lambda_2^2 \end{vmatrix} , \qquad \text{(B21)}$$

where

$$\lambda_1=\frac{2}{\sqrt{3}}=\lambda_2, \quad y_1=-y_2=\frac{1}{\sqrt{3}} . \qquad \text{(B22)}$$

This gives

$$\det M^{-1}=2^{10}3^{-3} , \qquad \text{(B23)}$$

which confirms (B15).

Also using (27b) of Ref. 8, we have checked $R_{(\alpha-1)}$ for $\alpha=2$, 3, and 4. The result (B19) is verified.

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