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Summation of coupling-constant logarithms in three-dimensional QED

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The leading and first subleading coupling-constant logarithms are explicitly calculated and summed to all orders for the self-energy graphs of massless three-dimensional fermionic QED.

I. INTRODUCTION

Finite-temperature field theories have been increasingly studied since it was realized that spontaneously broken symmetry is restored at high temperature.[1] Since then, it has been argued that in quantum chromodynamics (QCD) there is a phase transition to unconfined quarks above a critical temperature.[2] Above a critical point, imaginary-time degrees of freedom decouple if only the long-distance (infrared) behavior of the theory is required and hence it is governed by an effective theory in one dimension smaller. At infinite temperature, the decoupling is complete so physical four-dimensional theories become equivalent to super-renormalizable three-dimensional ones.[3] Thus, the latter have gained considerable interest in their own right.

The super-renormalizable interactions of massless field theories lead to severe perturbative infrared divergences which have been investigated by various methods.[4,5] In particular, Jackiw and the author considered the gauge theories of massless fermionic three-dimensional QED and QCD in which a simple resummation of perturbation theory removes the divergences and leads to logarithms in the coupling constant.[5] At high orders, high powers of logarithms arise, the leading terms of which have been calculated and summed by the author.[6] This paper explains in more detail and extends some of the results first presented in Ref. 6.

Section II of this paper repeats the explanation in Ref. 5 of the origin of the coupling-constant logarithm, modifying and extending it for use in calculating higher-order logarithms.

In Section III, I derive rules for calculating the dominant logarithms and use them to sum the terms proportional to $(e^4 \ln e^2)^n$ (leading logarithms) and $e^2(e^4 \ln e^2)^n$ (first subleading logarithms) for all n in the exact fermion propagator.

Section IV performs the same calculation for the vacuum polarization and explains why certain classes of diagrams neglected in the calculations do not contribute any further logarithms to the orders considered.

Section V discusses how nonperturbative terms appear to arise in the summation of the logarithms and Sec. VI summarizes the results.

II. COUPLING-CONSTANT LOGARITHMS

In three-dimensional massless fermionic quantum electrodynamics (QED_3), the coupling constant e has dimensions of $(\text{mass})^{1/2}$. This super-renormalizable interaction makes the ultraviolet divergences trivial but leads to infrared divergences in the loop integrations of perturbation theory. A self-energy graph of high order behaves like a high power of e^2/p on dimensional grounds, where p is the external momentum of the graph. When this is inserted as a subgraph into a loop integration of a higher-order diagram, the inverse powers of p lead to a divergence at small p.

Reference 5 gave a simple resolution of this problem. Whenever a self-energy occurs in a propagator, replace it by the dressed propagator in which an arbitrary number of self-energy insertions are summed. Then the small-p divergence occurs in the denominator instead of the numerator, so it is innocuous. This resummation of perturbation theory has interesting consequences—it leads to analytic terms which are not perturbatively calculable and to coupling-constant logarithms.

To proceed, the Euclidean Feynman rules of Fig. 1 for QED_3 in a covariant gauge are used. The complete photon and fermion propagators are shown in Fig. 2. The crosshatched subgraphs are the one-particle irreducible self-energies which to lowest order are shown in Fig. 3. The evaluation of the graphs of Fig. 3 gives

$$D_{\mu\nu}(p)=\frac{P_{\mu\nu}(p)}{p^2-\Pi(p^2)}+\alpha\frac{p_\mu p_\nu}{p^4}\,,\quad \Pi(p^2)=-\frac{e^2p}{16}+O(e^4) \tag{2.1}$$

and

$$S(p)=\frac{1}{\not p-\Sigma(p)}\,,\quad \Sigma(p)=-\frac{\alpha e^2\not p}{16p}+O(e^4)\,, \tag{2.2}$$

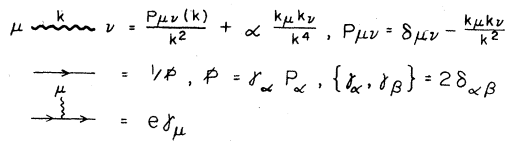

FIG. 1. Euclidean Feynman rules for QED_3.

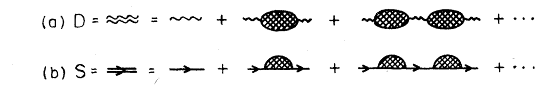

FIG. 2. Dressed propagators for (a) photon and (b) fermion.

where $p=(p^2)^{1/2}$ and by gauge invariance $\Pi_{\mu\nu}(p) = \Pi(p^2)P_{\mu\nu}(p)$ for some scalar Π. In the calculation of the fermion self-energy (2.2), the ultraviolet divergence vanishes by the angular integration. For the photon self-energy (vacuum polarization) in (2.1), the divergence is zero with any gauge-invariant regularization.

The perturbative infrared divergences first occur at $O(e^4)$ when the vacuum polarization is inserted into the photon propagator of the lowest-order fermion self-energy of Fig. 3(b). So I will first consider the consequences of the resummation in this self-energy using the exact photon propagator $D_{\mu\nu}$ of (2.1),

$$\Sigma_\alpha(p) = e^2 \int \frac{d^3k}{(2\pi)^3}\gamma_\mu \frac{1}{\not p + \not k}\gamma_\nu\left(\frac{P_{\mu\nu}(k)}{k^2 - \Pi(k^2)} + \alpha\frac{k_\mu k_\nu}{k^4}\right). \tag{2.3}$$

The gauge-dependent α term gives the lowest-order result of (2.2). To calculate the remainder, write the unknown part of the propagator as a dimensionless function

$$f\left(\frac{e^2}{k}\right) = \frac{k^2}{k^2 - \Pi(k^2)}, \tag{2.4}$$

where Π is given by (2.1). After the angular integration is done for the α-independent ($\alpha = 0$) part, the result is

$$\Sigma_0(p) = \int_0^\infty dk(k^2/p)C(k/p)f(e^2/k)/k^2, \tag{2.5}$$

where

$$C(k/p) = p\frac{e^2}{(2\pi)^3}\int d\Omega_k \gamma_\mu \frac{1}{\not p + \not k}\gamma_\nu P_{\mu\nu}(k)$$
$$= \frac{e^2 \not p}{4\pi^2 p}\frac{1-x^2}{x^2}\left[1 - \frac{1+x^2}{4x}\ln\left(\frac{1+x}{1-x}\right)^2\right] \tag{2.6}$$

and $x = k/p$. For the moment, I will not be concerned about the precise form of $C(k/p)$ or $f(e^2/k)$, but will analyze (2.5) more generally.

Using the dimensionless variable $x=k/p$, the required integral is

$$\Sigma_0 = \int_0^\infty dx\, C(x) f(\lambda/x),\quad \lambda = e^2/p, \tag{2.7}$$

where C is some known function analytic at $x=0$ and vanishing at $x=\infty$ and the first few derivatives of f are known at the origin (i.e., its small-coupling expansion). A direct expansion of f leads to divergences in the small-x integration corresponding to the previously mentioned infrared divergences. However, using the known form (2.4) of f, one can assume that it and its derivatives vanish at small x (large coupling). Then (2.7) can be analyzed more carefully and the following result is derived in the Appendix (the order-λ terms were derived in Ref. 5):

$$\int_0^\infty dx\, C(x) f(\lambda/x) = f(0)\int_0^\infty dx\, C(x) + \sum_{n=0}^\infty \frac{\lambda^{n+1}}{n!\,(n+1)!}\Bigg[-\ln\lambda\, C^{(n)}(0) f^{(n+1)}(0) - C^{(n)}(0)\int_0^\infty dy \ln y\, f^{(n+2)}(y)$$
$$- f^{(n+1)}(0)\int_0^\infty dx \ln x\, C^{(n+1)}(x) + C^{(n)}(0) f^{(n+1)}(0)\left(\frac{1}{n+1} + 2\sum_{i=1}^n \frac{1}{i}\right)\Bigg]. \tag{2.8}$$

In addition to the vanishing of f and its derivatives at infinite argument, the derivation of (2.8) assumes that C is analytic at the origin and its derivatives vanish at infinity, which is true for our case (2.6). Also f must be analytic at the origin. However, this is not true as we shall see that eventually logarithms of the coupling λ appear in the vacuum polarization and feed back into f causing its higher derivatives to diverge. Hence, the analysis leading to (2.8) breaks down beyond some order in λ. Nevertheless, to the orders discussed here, there is no problem. Further, the logarithmic singularity at $x=1$ in $C(x)$ means the logarithmic integrals of the derivatives of C in (2.8) diverge at $x=1$ for large enough n—they have to be regulated. See the Appendix for details.

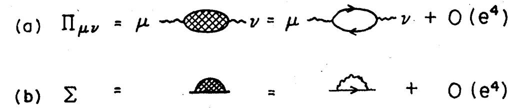

FIG. 3. Lowest-order self-energies for (a) photon and (b) fermion.

In the expression (2.8), the first term is the lowest-order result. Logarithms of the coupling $\lambda = e^2/p$ appear in the first term of the sum in (2.8) and occur as $(e^2)^n \ln e^2$ for all $n \geq 1$ and their coefficients are perturbatively calculable. The second term in the sum involves integrals of the derivatives of f over the entire range of its argument. Perturbation theory only gives the derivatives of f at the origin, hence can never give us the values of these integrals. So, in addition to the logarithms at order e^2 and above, at the same orders there are terms analytic in the coupling which are not perturbatively calculable. The remainder of the terms in (2.8) are perturbatively calculable since $C(x)$ is known exactly, and they give just a power series in λ.

Returning to the explicit form of C in (2.6), we see that it is an even function of x so only the even derivatives are nonzero at the origin. At $x = 0$, its value is

$$C(0) = \frac{e^2 \not{p}}{4\pi^2 p}\left(-\frac{4}{3}\right) \tag{2.9}$$

and from (2.4) with (2.1)

$$f^{(1)}(0) = -1/16\,. \tag{2.10}$$

So up to order λ, (2.8) gives

$$\Sigma_\alpha = -\frac{\alpha e^2 \not{p}}{16 p} - \frac{e^2 \not{p}}{48\pi^2 p}\frac{e^2}{p}\ln\frac{e^2}{p} + \frac{e^2\not{p}}{3\pi^2 p}\frac{e^2}{p}\int_0^\infty dy\, \ln y\, f^{(2)}(y)$$

$$+\text{analytic perturbatively calculable } O(e^4) \text{ terms}$$

$$+O(e^6)\,. \tag{2.11}$$

Because of the symmetry of $C(x)$, there are no logarithms or incalculable terms at $O(e^6)$ in Σ and they next appear at $O(e^8)$.

When the dressed fermion propagator of Fig. 2(b) is inserted into the self-energies of Fig. 3, a similar analysis applies. In these cases

$$f = \not{p} S(p) = \frac{1}{1 + \alpha e^2/16p + O(e^4)} \tag{2.12}$$

and for Fig. 3(b)

$$C\left(\frac{k}{p}\right) = \frac{e^2 p}{(2\pi)^3}\int d\Omega_k \gamma_\mu \not{k} \gamma_\nu \left[\frac{P_{\mu\nu}(k+p)}{(k+p)^2} - \alpha\frac{(k+p)_\mu (k+p)_\nu}{(k+p)^4}\right]$$

$$= \frac{\alpha e^2 \not{p}}{4\pi^2 p^2}\left[\frac{1+x^2}{4x}\ln\left(\frac{1+x}{1-x}\right)^2 - 1\right]. \tag{2.13}$$

Here $C(0) = 0$ due to the $\not{k}$ factor from the fermion propagator, and again $C(x)$ is even, so the first nonzero derivative is the second. Hence the first logarithms from the fermion propagator arise at order e^8. With the dressed fermion inserted into the vacuum polarization of Fig. 3(a), $C(x)$ is proportional to

$$1 - \frac{1-x^2}{4x}\ln\left(\frac{1+x}{1-x}\right)^2,$$

which is also even and vanishes at the origin so the same conclusions apply.

Previously, the analysis was done using a spectral representation of the exact photon propagator.[5,6] Though this was sufficient for obtaining the lowest-order logarithm and nonperturbative analytic term, it does not generalize as well to the higher orders considered here. Otherwise, the two methods are equivalent.

In general, for diagrams with more than one dressed propagator of Fig. 2, logarithms arise independently from each one, so powers of logarithms occur. The next section explains how in a diagram with n dressed photons with n independent loop momenta flowing through them, each photon will contribute an $e^4 \ln e^2$ leading to an overall $(e^4 \ln e^2)^n$ term [$e^2 \ln e^2$ comes from the lowest-order logarithm in (2.8) and an e comes from each of the vertices at the photon ends]. These I call the leading logarithms (LL), since for a given power of logarithm they are the terms with the lowest power of e^2. The e^2 term from only one of the photons gives terms like $e^2(e^4 \ln e^2)^{n-1}$—the first subleading logarithms (SL). Further subleading logarithms $(e^2)^m (e^4 \ln e^2)^n$ with $m \geq 2$ include diagrams with a nonperturbative e^4 or higher term occurring in at least one of the photon propagators. Hence, in general, the coefficients of these subleading logarithms are not perturbatively calculable.

I perturbatively calculate the leading and first subleading logarithms and sum them for the complete fermion propagator (and hence self-energy) and the vacuum polarization. The techniques apply equally well to other amplitudes and, in general, all further subleading logarithms are nonperturbative.

III. SUMMED LOGARITHMS OF THE FERMION PROPAGATOR

First I calculate the leading and first subleading logarithms for the simplest case—the complete fermion propagator—and then apply the methods developed to the photon vacuum polarization in the next section. I claim that the leading logarithms are given by the sum of all diagrams in which an arbitrary number of dressed photons form arches on a bare fermion line in all possible ways—a typical such graph is shown in Fig. 4. The first subleading logarithms are given by the same diagrams except one of the lines gives only its bare contribution instead of a logarithm. A generalization of the results of the previous section shows

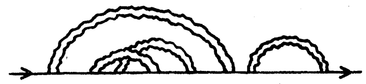

FIG. 4. A typical contribution to the leading logarithms of the fermion propagator.

that each photon line contributes a leading $e^4 \ln e^2$ logarithm, but no subleading $e^6 \ln e^2$ logarithm except through the bare e^2 term which contributes a gauge (α) dependent part. In the analysis of Sec. II, it was assumed that there were no logarithms in the perturbative vacuum polarization [the derivatives of $f(x)$ were finite at zero]. In Sec. IV, this will be shown to be true, at least for leading and first subleading order which might feed back to give extra logarithms in the complete fermion propagator.

Other diagrams such as Fig. 5, with fermion loops not already included in the vacuum polarization, also give no additional logarithms at the leading and first subleading orders. This is also shown in Sec. IV.

Again the results of the previous section, when generalized, show that any insertion of dressed fermion propagators also does not lead to new logarithms at the orders considered.

I now consider the complete contribution of diagrams such as Fig. 4. The required angular integrations corresponding to $C(x)$ of (2.6) are greatly simplified when one picks out only the leading and first subleading loagrithm coefficients, which enables one to calculate them exactly to all orders. The contribution of a diagram such as Fig. 4 to the fermion propagator with n complete photon propagators can, analogously to (2.7), be written as

$$S_n(p) = \prod_{i=1}^{n} \left[\int_0^\infty dx_i f(\lambda/x_i) \right] C_n(x_1, \ldots, x_n)\,, \quad (3.1)$$

where $x_i = k_i/p$ are the scaled magnitudes of the momenta k_i flowing through each of the photon lines and p is the external momentum. In C, all the angular integrations of the k_i have been performed. Repeating the analysis of the Appendix for each x_i integration gives an $(e^2/p)\ln(e^2/p)$ from each to obtain the leading logarithms with coefficient $C_n(0, \ldots, 0)$ times $[-f^{(1)}(0)]^n$. $C_n(0)$ is proportional to $(e^2)^n$ because of the $2n$ vertices at the ends of the photon lines.

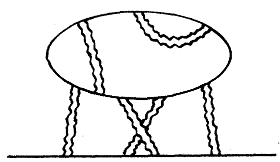

FIG. 5. A typical diagram that does not contribute leading or first subleading logarithms.

The first subleading logarithms arise when one x_i integration—say x_j—gives its $e^4 \ln e^2$ contribution [$n=1$ in (2.8)] with coefficient proportional to $(\partial/\partial x_j) C_n(0, \ldots, 0)$ (which I show to vanish) or when one photon line contributes only its e^2 bare α-dependent part and the remaining $n-1$ lines contribute leading logarithms with coefficient $C_{n-1}(0)$.

In general,

$$C_n(x_1, \ldots, x_n) = \prod_{i=1}^{n} \left(\int \frac{p\, d\Omega_i}{(2\pi)^3} \right) \frac{1}{\not{p}}\, e\gamma_{\alpha_1} \frac{1}{\not{p} + \sum_{j\in V_1} \not{k}_j}\, e\gamma_{\alpha_2} \cdots e\gamma_{\alpha_{2n}} \frac{1}{\not{p}} \times \mathcal{P}\left(P_{\alpha_1\alpha_{n+1}}(k_1) \cdots P_{\alpha_n\alpha_{2n}}(k_n)\right), \quad (3.2)$$

where $\mathcal{P}$ is an arbitrary permutation of the summed indices $\alpha_1, \ldots, \alpha_{2n}$ and $\sum_{j\in V_N} k_j$ is the net sum of the incoming photon momenta at the first N vertices ($N = 1, \ldots, 2n-1$), with V_N being the set of labels j for the photons at these vertices. Writing $k_j = p x_j z_j$, where z_j is a unit vector in the k_j direction, one sees that x_j only occur in the bare fermion propagators $1/(\not{p} + p\sum_{j\in V_N} x_j \not{z}_j)$, since the $P_{\alpha\beta}$ depend only on the directions z_j. Taking the limit of all $x_j \to 0$ in C_n simplifies the fermion propagators so that the only remaining angular integrations are

$$\int d\Omega_i P_{\alpha\beta}(z_i) = \tfrac{2}{3}\, \delta_{\alpha\beta} 4\pi\,. \quad (3.3)$$

Hence one obtains

$$C_n(0, \ldots, 0) = \left(\frac{e^2 \not{p}}{(2\pi)^3} \frac{2}{3}\, 4\pi \right)^n \frac{1}{\not{p}}\, \gamma_{\alpha_1} \frac{1}{\not{p}}\, \gamma_{\alpha_2} \cdots \gamma_{\alpha_{2n}} \frac{1}{\not{p}} \times \mathcal{P}(\delta_{\alpha_1\alpha_{n+1}} \cdots \delta_{\alpha_n\alpha_{2n}})\,. \quad (3.4)$$

Combining (3.4) with each of the $-f^{(1)}(0)(e^2/p) \times \ln(e^2/p)$ of (2.8) coming from each x_j integration gives the leading-logarithm contribution of

$$S_n^{\rm LL}(p) = \left(\frac{e^4}{48\pi^2} \ln \frac{e^2}{p} \right)^n \frac{1}{\not{p}}\, \gamma_{\alpha_1} \frac{1}{\not{p}}\, \gamma_{\alpha_2} \cdots \gamma_{\alpha_{2n}} \frac{1}{\not{p}} \times \mathcal{P}(\delta_{\alpha_1\alpha_{n+1}} \cdots \delta_{\alpha_n\alpha_{2n}})\,. \quad (3.5)$$

Instead of attempting to calculate and sum (3.5) directly, the following trick is used. One observes that (3.5) is exactly the result of calculating the same diagram of Fig. 4, but with the modified Feynman rules of Fig. 6. In these rules, the photon carries no momentum and there are no loop integrations. Also, so far, there are no fermion loops since the diagrams of Fig. 4 contain none. The leading-logarithm contribution to the

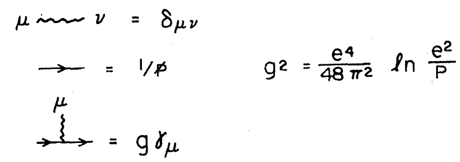

FIG. 6. Modified Feynman rules for the leading logarithms.

complete fermion propagator is given by the sum of all terms like (3.5), which is the same as the exact fermion propagator generated by the rules of Fig. 6, but with all fermion loops suppressed. Hence the sum is

$$S^{\mathrm{LL}}(\not p)=\frac{1}{N}\int d^3A\,e^{-A^2/2}\frac{1}{\not p+g\not A} \tag{3.6}$$

with normalization

$$N=\int d^3A\,e^{-A^2/2}=(2\pi)^{3/2}. \tag{3.7}$$

When (3.6) is expanded in powers of g and the A integrations performed, it reproduces term by term the expressions of (3.5). If, instead of the complete propagator, I tried to calculate the fermion self-energy, the same rules of Fig. 6 would apply but the summation corresponding to (3.6) would be much more difficult—hence this choice.

One expects the leading logarithms to be important for small coupling $e^2 \ll p$, so g^2 given in Fig. 6 is negative and g pure imaginary. Calculating (3.6) with imaginary g gives extra terms which do not arise in the perturbative expansion of (3.5).[6] Instead, I will evaluate (3.6) assuming g is real (which leads to some simplifications) and only at the end recognize the fact that g^2 is negative. The consequences of having g complex initially are discussed in Sec. V.

Equation (3.6) is evaluated by shifting the A integral by an amount $-p/g$,

$$\begin{aligned}S^{\mathrm{LL}}(p)&=\frac{1}{N}\int d^3A\,e^{-(A-p/g)^2/2}\frac{1}{g\not A}\\&=\frac{1}{N}\frac{\not p}{p^2}\int d^3A\frac{p\cdot A}{gA^2}\\&\qquad\times\exp[-\tfrac{1}{2}(A^2-2p\cdot A/g+p^2/g^2)].\end{aligned} \tag{3.8}$$

The angular integral gives

$$\begin{aligned}S^{\mathrm{LL}}(p)&=\frac{\not p}{Np^2}2\pi e^{-p^2/2g^2}\\&\quad\times\int_0^\infty dA\,e^{-A^2/2}\bigg[e^{Ap/g}+e^{-Ap/g}\\&\qquad-\frac{g}{Ap}(e^{Ap/g}-e^{-Ap/g})\bigg].\end{aligned} \tag{3.9}$$

The scale integrals are standard and with the normalization (3.7) the result is

$$S^{\mathrm{LL}}(p)=\frac{\not p}{p^2}e^{-p^2/2g^2}\left[e^{p^2/2g^2}+\frac{ig}{p}\frac{\pi}{(2\pi)^{1/2}}\operatorname{erf}\left(\frac{ip}{\sqrt{2}g}\right)\right]. \tag{3.10}$$

Equation (3.10) is the exact integration of (3.6) for real g. For small g it has an asymptotic expansion in powers of g^2,

$$S^{\mathrm{LL}}(p)\sim\frac{1}{\not p}\sum_{n=0}^{\infty}\frac{\Gamma(n-\frac{1}{2})}{\Gamma(-\frac{1}{2})}\left(\frac{2g^2}{p^2}\right)^n, \tag{3.11}$$

the coefficients of which are those given by the perturbation theory of (3.5). For negative g^2, the series (3.11) has a Borel sum. Writing $\Gamma(n-\frac{1}{2})=\int_0^\infty dt\,e^{-t}t^{n-3/2}$, $n=1,2\ldots$, we obtain

$$\begin{aligned}S^{\mathrm{LL}}(p)&=\frac{1}{\not p}\left[1+\int_0^\infty dt\,e^{-t}\sum_{n=1}^{\infty}\frac{t^{n-3/2}}{-2\sqrt{\pi}}(-2|g^2|/p^2)^n\right]\\&=\frac{1}{\not p}\left[1+\frac{|g^2|}{\sqrt{\pi}p^2}\int_0^\infty dt\,e^{-t}t^{-1/2}\Big/\left(1+\frac{2|g^2|}{p^2}t\right)\right].\end{aligned} \tag{3.12}$$

A change of variables to $z=t^{1/2}$ gives

$$\begin{aligned}S^{\mathrm{LL}}(p)&=\frac{1}{\not p}\left[1+\frac{2|g^2|}{\sqrt{\pi}p^2}\int_0^\infty dz\,e^{-z^2}\Big/\left(1+\frac{2|g^2|}{p^2}z^2\right)\right]\\&=\frac{1}{\not p}\left[1+\left(\frac{\pi}{2}\right)^{1/2}\frac{|g|}{p}e^{p^2/2|g^2|}\operatorname{erfc}\frac{p}{\sqrt{2}|g|}\right].\end{aligned} \tag{3.13}$$

Note that this is slightly different from a simple replacement of $g=i|g|$ in (3.10)—for a discussion see Sec. V.

As stated earlier, a diagram such as Fig. 4 with n dressed photon lines can give a first subleading logarithm if one of the first derivatives of $C_n(x_1,\ldots,x_n)$ at the origin is nonzero or when one of the photons is bare. They also occur when a fermion line has been replaced by the exact fermion propagator of Fig. 2(b) and the corresponding $C(x)$ function has a nonvanishing first derivative for zero momentum going through the fermion line. I show that the multivariable C's have the same symmetry as the lowest-order single-variable C's of (2.6) and (2.13); hence all the first derivatives vanish at the origin.

In C of (3.2), x_i only occurs in the bare fermion propagator $1/(\not p+p\sum_{j\in V_N}x_j\not z_j)$, so replacing x_i with $-x_i$ for some i can be compensated for by replacing z_i with $-z_i$. The latter can be done since z_i is integrated over all directions and $P_{\alpha\beta}(z_i)$ is independent of the sign of z_i. Hence C is unchanged by the change of sign in x_i and so is an even function, and the odd derivatives vanish at the origin.

A similar consideration applies to the generalization of the C of (2.13) for the dressed fermion lines. By the angular integration any $x_i \to -x_i$ leaves C unchanged and it vanishes when all $x_i \to 0$, because of the $\sum_{i\in V_N} \not{k}_i$ factors in the numerators of the fermion propagators. This confirms that the dressed fermions give no leading or first subleading logarithms.

The only remaining first subleading contribution to Fig. 4 is when one of the photon lines is bare and an arbitrary number of dressed photons cross in all possible ways. The same kind of analysis applies and the first subleading terms arise when each of the dressed photons gives its leading $e^2 \ln e^2$ contribution and the coefficient is determined by the corresponding $C(x)$ when each of the momenta of the dressed photons vanish. The $C(x)$ is defined as before with the angular integration over the dressed photon momenta performed, but in addition the complete integration over the bare photon momentum is done. As a result the same modified Feynman rules of Fig. 6 apply to the dressed photons and their vertices but the original rules apply to the bare photon and its vertices. After an analogous summation of the modified rules as for the leading-logarithm case, the result is

$$S_{\rm SL}(p)=\frac{1}{N}\int d^3A\, e^{-A^2/2}\frac{1}{\not{p}+g\not{A}} \times \int\frac{d^3k}{(2\pi)^3}\, e\gamma_\mu \frac{1}{\not{p}+\not{k}+g\not{A}} \times e\gamma_\nu\left(\frac{P_{\mu\nu}(k)}{k^2}-\alpha\frac{k_\mu k_\nu}{k^4}\right)\frac{1}{\not{p}+g\not{A}}\,. \qquad (3.14)$$

To find the summed contribution, I again evaluate (3.14) exactly assuming real g, pick out the powers of g^2 in the asymptotic expansion for small g (i.e., only those terms which arise from the modified perturbation theory) and then sum taking g^2 negative. The k integral in (3.14) is exactly that for the lowest-order fermion self-energy with external momentum $\not{p}+g\not{A}$ and the result is given in (2.2). Hence

$$S_{\rm SL}(p)=\frac{1}{N}\int d^3A\, e^{-A^2/2}\left(-\frac{\alpha e^2}{16}\right)\frac{\not{p}+g\not{A}}{|p+gA|^3}, \qquad (3.15)$$

which is again evaluated by shifting the A integration by $-p/g$:

$$S_{\rm SL}(p)=\frac{\not{p}}{p^2}\left(-\frac{\alpha e^2}{16N}\right)e^{-p^2/2g^2}\int d^3A\, e^{-A^2/2}e^{p\cdot A/g}\frac{p\cdot A}{g^2A^3}\,. \qquad (3.16)$$

The angular integration followed by an integration by parts on the scale integral leads to

$$S_{\rm SL}(p)=-\left(\frac{\alpha e^2}{16N}\right)\frac{\not{p}}{p^2}\,2\pi e^{-p^2/2g^2}\int_0^\infty dA\, e^{-A^2/2}\left[\frac{1}{gA}\left(e^{Ap/g}+e^{-Ap/g}\right)-\frac{1}{pA^2}\left(e^{Ap/g}-e^{-Ap/g}\right)\right]$$
$$=-\frac{\alpha e^2}{16N}\frac{\not{p}}{p^2}\,2\pi e^{-p^2/2g^2}\left[-\frac{2}{g}+\frac{1}{p}\int_0^\infty dA\, e^{-A^2/2}\left(e^{Ap/g}-e^{-Ap/g}\right)\right]. \qquad (3.17)$$

The scale integral is again standard and results in

$$S_{\rm SL}(p)=-\frac{\alpha e^2}{16}\frac{\not{p}}{p^2}\left[-\left(\frac{2}{\pi}\right)^{1/2}\frac{1}{g}\,e^{-p/2g^2}+\frac{1}{p}\,\mathrm{erf}(p/\sqrt{2}g)\right]. \qquad (3.18)$$

An asymptotic expansion of $\mathrm{erfc}=1-\mathrm{erf}$ for small g gives

$$S_{\rm SL}(p)\sim-\frac{\alpha e^2}{16}\frac{\not{p}}{p^3}+\frac{\alpha e^2}{16}\frac{\not{p}}{p^2}\left(\frac{2}{\pi}\right)^{1/2}\frac{1}{g}\,e^{-p^2/2g^2}\left[1+\frac{g^2}{p^2}+\frac{g^2}{p^2}\sum_{n=1}^{\infty}1\times3\times\cdots\times(2n-1)\left(-\frac{g^2}{p^2}\right)^n\right]. \qquad (3.19)$$

The only term which has even powers of g is the first. The remainder is not only odd in g but the exponential has no expansion in positive powers of g. What this tells us is that an expansion of (3.14) or (3.15) in powers of g is zero term by term apart from the lowest order. The formal expansion of $1/(\not{p}+g\not{A})$ is only valid when $gA \lesssim p$, so it is not surprising that extra terms arise when A is integrated over all space becoming large.

To check the above conclusion, (3.15) is expanded to reproduce the modified perturbative terms,

$$S_{\rm SL}(p)=-\frac{\alpha e^2}{16N}\int d^3A\, e^{-A^2/2}\sum_{n=0}^{\infty}\frac{g^n}{n!}\frac{\partial^n}{\partial g^n}\frac{\not{p}+g\not{A}}{|p+gA|^3}\bigg|_{g=0}$$
$$=-\frac{\alpha e^2}{16N}\int d^3A\, e^{-A^2/2}\sum_{n=0}^{\infty}\frac{g^n}{n!}A_{\alpha_1}A_{\alpha_2}\cdots A_{\alpha_n}\times\frac{\partial^n}{\partial p_{\alpha_1}\partial p_{\alpha_2}\cdots\partial p_{\alpha_n}}\frac{\not{p}}{p^3}. \qquad (3.20)$$

By the angular integration of A all the odd n terms vanish, which always happens since there are an even number of g vertices in the modified perturbative graphs. For the even-n terms the angular integration gives the totally symmetric tensor in $\alpha_1, \ldots, \alpha_n$. Also the p derivative is totally symmetric in $\alpha_1, \ldots, \alpha_n$, so when contracted the result is proportional to

$$\frac{\partial^n}{\partial p_{\beta_1}\partial p_{\beta_1}\cdots\partial p_{\beta_{n/2}}\partial p_{\beta_{n/2}}}\frac{\not p}{p^3}=0\ . \tag{3.21}$$

The vanishing of (3.21) is due to $(\partial^2/\partial p_\alpha \partial p_\alpha)\not p/p^3$ being identically zero $(p\neq 0)$, hence all the even terms $(n>0)$ of (3.20) are zero, confirming the previous argument. The final summed perturbative result is just the lowest order

$$S_{\rm SL}(p) = -\frac{\alpha e^2}{16}\frac{\not p}{p^3}\ . \tag{3.22}$$

IV. THE VACUUM POLARIZATION AND OTHER FERMION LOOPS

To complete the assertions of the previous section it remains to check that fermion loops do not contribute additional logarithms. First, it is convenient to consider the leading and first subleading logarithms in the vacuum polarization.

The leading logarithms are the $(e^4 \ln e^2)^n$ terms for all n and the first subleading are e^2 times these. This is a different definition from that given in Ref. 6 since it turns out that the current one is more consistent. These two orders are perturbatively calculable but the next order of subleading logarithms is not. For example, insert a dressed fermion propagator of Fig. 2(b) in place of one of the bare fermions in the lowest-order vacuum polarization of Fig. 3(a). From the analysis of Sec. II the coefficient of the fermion's leading $e^2\ln e^2$ contribution vanishes and the first nonzero contribution is $e^6\ln e^2$ coming from the second derivative of C and $f^{(3)}(0)$. However, the latter depends on the perturbatively incalculable e^4 and e^6 fermion self-energy terms. Hence the $e^2\times e^6 \ln e^2$ contribution to the vacuum polarization is incalculable. Note that the perturbatively calculable $e^4\ln e^2$ fermion self-energy term arising from the lowest-order graph of Fig. 3(b) with a single dressed photon should not be included in the f function for the dressed fermion in the vacuum polarization. It is included in the first subleading logarithm calculation of the vacuum polarization next, in which an arbitrary number of dressed photons are added to the bare fermion propagator.

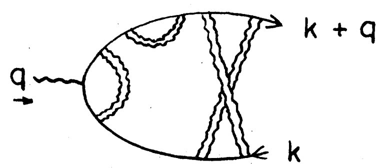

FIG. 7. Part of a typical graph contributing first subleading logarithms to the vacuum polarization.

In Fig. 7 there is an independent loop momentum for each of the dressed photons. When each momentum becomes small it gives a leading $e^2\ln e^2$ contribution in the same way as before and the same modified Feynman rules apply. When the expansion in the modified perturbation series is summed the lower fermion leg becomes $1/(\not k + g\not A)$ and the upper becomes $1/(\not k + \not q + g\not A)$. The fermion loop can then be closed and the remaining k integration performed to give the first subleading logarithm vacuum polarization,

$$\Pi^{\rm SL}_{\alpha\beta}(q) = \frac{1}{N}\int d^3A\, e^{-A^2/2}(-e^2)\times\int\frac{d^3k}{(2\pi)^3}\,{\rm tr}\left(\gamma_\alpha\frac{1}{\not k+\not q+g\not A}\gamma_\beta\frac{1}{\not k+g\not A}\right)\ . \tag{4.1}$$

Shifting the k integral by $-gA$ shows that it is A independent and equals the lowest-order bare result of (2.1). Then the A integral is trivial and cancels the normalization N giving

$$\Pi^{\rm SL}_{\alpha\beta}(q) = -\frac{e^2 q}{16}P_{\alpha\beta}(q)\ . \tag{4.2}$$

Again one can check that the perturbation series in g vanishes order by order. An extension of the gauge invariance arguments of Ref. 5 can be used. At a given order the dressed photons are removed but the paired ends still have equal and opposite momenta going in. Then gauge invariance tells us that the amplitude (the sum of such diagrams at the given order) must vanish when these momenta go to zero. Hence when the dressed photon is reinserted its leading-logarithm coefficient corresponding to $C(0)$ vanishes. Alternatively, an argument of Jackiw can be used.[7] The original modified perturbation theory corresponds to an expansion of (4.1) in powers of g,

$$\begin{aligned}\Pi^{\rm SL}_{\alpha\beta}(q) &= -\frac{e^2}{N}\int d^3A\, e^{-A^2/2}\int\frac{d^3k}{(2\pi)^3}\sum_{n=0}^{\infty}\frac{g^n}{n!}\frac{\partial^n}{\partial g^n}\,{\rm tr}\gamma_\alpha\frac{1}{\not k+\not q+g\not A}\gamma_\beta\frac{1}{\not k+g\not A}\bigg|_{g=0}\\ &= -\frac{e^2}{N}\sum_{n=0}^{\infty}\frac{g^n}{n!}\int d^3A\, e^{-A^2/2}A_{\alpha_1}\cdots A_{\alpha_n}\int\frac{d^3k}{(2\pi)^3}\frac{\partial^n}{\partial k_{\alpha_1}\cdots\partial k_{\alpha_n}}\,{\rm tr}\gamma_\alpha\frac{1}{\not k+\not q}\gamma_\beta\frac{1}{\not k}\ .\end{aligned} \tag{4.3}$$

For $n \geq 2$ the k integral is now infrared divergent as $k \to 0$ or $-q$. This however is an artifact of the fact that the k integration is being done after the momenta going through the dressed photons are sent to zero. If the k integration had been done initially in defining the appropriate $C(x)$ function, then no such divergences would have arisen [and also $C(0)$ would be seen to vanish] However, except for the lowest-order logarithm which was examined in Ref. 5, this procedure is difficult. Instead we can ignore the infrared divergences in (4.3) (or remove them by dimensional regularization) and integrate by parts giving

$$\Pi^{SL}_{\alpha\beta}(q) = -\frac{e^2}{N} \sum_{n=0}^{\infty} \frac{g^n}{n!} \int d^3A e^{-A^2/2} A_{\alpha_1} \cdots A_{\alpha_n} \times \int \frac{dS_{\alpha_1}}{(2\pi)^3} \frac{\partial^{n-1}}{\partial k_{\alpha_2} \cdots \partial k_{\alpha_n}} \times \mathrm{tr}\gamma_\alpha \frac{1}{\not{k}+\not{q}} \gamma_\beta \frac{1}{\not{k}}, \qquad (4.4)$$

which clearly vanishes for $n \geq 1$ as the sphere of integration grows to infinity. Hence all except the first term vanish.

The above results that there are no leading or first subleading logarithms in the vacuum polarization (i.e., it is analytic up to order e^6) show that the analysis of Sec. II and the Appendix works to higher order than one might have expected It requires $f^{(n)}(0)$ be finite where f depends on the exact vacuum polarization through (2.4). One sees that it is finite through $n=3$ so there is no feedback of further logarithms until higher subleading order.

Using the insight gained with the vacuum polarization, it is easy to see why fermion loops in diagrams such as Fig. 5 also give no additional logarithms to the complete fermion propagator. Consider diagrams like Fig. 5 with at least two dressed photons connecting a fermion loop to the main propagator. Detach one of the photons from the fermion propagator so that it carries momentum q into the fermion loop as in Fig. 8. The leading-logarithm contribution of the resulting graph occurs when as many as possible of the dressed photons carry zero momentum and one of the photons connecting the fermion loop to the main propagator carries all the external momentum q. For convenience this distinguished photon can be treated as bare and is depicted as such in Fig. 8, though when the free photon end is reattached to the main fermion line and the q integration performed to give Fig. 5, we must remember that this special photon is also dressed.

In Fig. 8, for each dressed photon there is an independent momentum integration but not for the

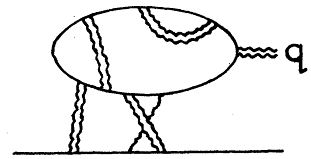

FIG. 8. Analysis of Fig. 5 by breaking one of the dressed photon lines and selecting one of the remaining lines (shown as bare) to carry the resulting momentum q.

distinguished bare photon. When the loop momenta vanish each dressed photon gives a leading logarithm with the modified Feynman rules of Fig. 6 applying. After summation of the rules and a shift in the fermion loop momentum by $-gA$ in the same way as for the vacuum polarization, one sees that the contribution of the fermion loop is independent of A. Hence just the lowest-order term from a simple fermion loop, attached by one photon, remains. However the resulting lowest-order graph is part of a single dressed photon with momentum q (remember that in a dressed photon an arbitrary number of simple fermion loops have been inserted into the bare photon). When the free end is reattached and the q integration done, the resulting graph has already been included in the dressed photon arches calculated in Sec. III so there is no further contribution.

The same arguments apply if the fermion loop is attached to the fermions of the vacuum polarization or if it is attached to the fermion propagator by an additional bare photon so as to try to obtain additional subleading logarithms. Only the lowest-order nonlogarithmic terms could contribute.

V. COMPLEX g AND NONPERTURBATIVE EFFECTS

In this section, I discuss the case when the coupling g of the modified Feynman rules is taken as imaginary. To be more general, I evaluate (3.6) when $g=a+ib$ is complex. For simplicity assume $a,b>0$. The shift in the A integral used previously now causes the A integration contour to cross poles so giving extra pole terms.

Equation (3.6) is equal to $\not{p}/Np^2$ times the following integral:

$$I(p^2) = \int d^3A e^{-A^2/2} \frac{p\cdot(p+gA)}{(p+gA)^2} . \qquad (5.1)$$

$I(p^2)$ is rotationally invariant so one can choose p_α to lie along say the three direction $p_\alpha=(0,0,p)$. Denote by $\underline{A}$, the 1,2 vector of A_α, then

$$I(p^2)=\int d^2\underline{A}e^{-\underline{A}^2/2}\int_{-\infty}^{\infty}dA_3e^{-A_3{}^2/2}\frac{p^2+gpA_3}{g^2[A_3+p/g+i(\underline{A}^2)^{1/2}][A_3+p/g-i(\underline{A}^2)^{1/2}]}. \quad (5.2)$$

Shifting the A_3 contour down by $\mathrm{Im}p/g=-pb/(a^2+b^2)$ from the real axis [i.e., up by $pb/(a^2+b^2)$], it crosses a pole at $A_3=-p/g-i(\underline{A}^2)^{1/2}$ provided $(\underline{A}^2)^{1/2}<pb/(a^2+b^2)$. After a further real shift so that the net shift is $-p/g$, the A_3 integral becomes

$$\int_{-\infty}^{\infty}dA_3e^{-(A_3-p/g)^2/2}\frac{pA_3/g}{[A_3+i(\underline{A}^2)^{1/2}][A_3-i(\underline{A}^2)^{1/2}]}+2\pi i\exp\{-[-p/g-i(\underline{A}^2)^{1/2}]/2\}\frac{-igp(\underline{A}^2)^{1/2}}{-2i(\underline{A}^2)^{1/2}g^2}\theta\left(\frac{pb}{a^2+b^2}-(\underline{A}^2)^{1/2}\right).$$

Inserting this back into (5.2) gives

$$I(p^2)=\int d^3Ae^{-(A-p/g)^2/2}\frac{p\cdot A}{gA^2}+\frac{i\pi p}{g}e^{-p^2/2g^2}\int^{|\underline{A}|=pb/(a^2+b^2)}d^2\underline{A}e^{-i|\underline{A}|p/g}. \quad (5.3)$$

The first integral, which has been rewritten in a rotationally invariant form, is exactly that which arose after a shift by real g in (3.8) and the result (3.10) still applies for complex g. The second two-dimensional integral in (5.3) evaluates to

$$\int^{|A|=pb/|g|^2}d^2\underline{A}e^{-i|\underline{A}|p/g}=2\pi\left[\left(\frac{g^2}{p^2}+\frac{igb}{|g|^2}\right)e^{-ip^2b/g|g|^2}-\frac{g^2}{p^2}\right], \quad (5.4)$$

giving the final result for complex g,

$$S_{\mathrm{LL}}(p)=\frac{\not p}{p^2}\left[1-\frac{ig}{p}\left(\frac{\pi}{2}\right)^{1/2}e^{-p^2/2g^2}\mathrm{erfc}\left(\frac{ip}{\sqrt{2}g}\right)+\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{ig}{p}-\frac{p\,\mathrm{Im}g}{|g|^2}\right)e^{-p^2/2|g|^2}\right]. \quad (5.5)$$

For real g [i.e., $b=\mathrm{Im}(g)=0$, $|g|^2=g^2$], (5.5) reproduces the previous result (3.10) and for pure imaginary $g=ib$ it gives

$$S_{\mathrm{LL}}(p)=\frac{\not p}{p^2}\left[1+\left(\frac{\pi}{2}\right)^{1/2}\frac{b}{p}e^{p^2/2b^2}\mathrm{erfc}\left(\frac{p}{\sqrt{2}b}\right)-\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{b}{p}+\frac{p}{b}\right)e^{-p^2/2b^2}\right]. \quad (5.6)$$

Again this is not the simple replacement of g with ib in (3.10). The first two terms of (5.6) agree with the Borel sum (3.13) of the perturbative series. In addition there is a nonperturbative term which has no expansion in positive powers of g. The significance of the extra terms depends on how much faith one has in the representation (3.6) for the sum of the leading logarithms. Since it was derived on a term-by-term basis from the logarithms arising perturbatively, only the summed perturbative series is trustworthy. The latter has been assumed in the results of Secs. III and IV. However one may take the point of view that (3.6) is in some sense an approximation to the exact functional integral for the fermion propagator when the low-momentum limit of the dressed photon is taken. Then the nonperturbative result (5.6) may have some validity. If so then there are further nonperturbative effects.

In Ref. 6 the sum of first subleading logarithms for the vacuum polarization with imaginary g was calculated in a similar way to the derivation of (5.6). The asymptotic expansion for small g, in addition to the perturbative result (for real g) of (4.2), has terms with only odd powers of g which cannot arise perturbatively.

One problem with including nonperturbative terms is that it leads to a breakdown of the arguments of Sec. IV that no additional logarithms arise. In particular, the lowest-order nonperturbative logarithm in the vacuum polarization (proportional to g—see Ref. 6) invalidates the analysis of Sec. II and the Appendix at the orders required. So additional logarithms feed into all orders of the complete fermion propagator and the simple methods of analysis presented here break down.

VI. CONCLUSION

The techniques described in this paper apply equally well to calculating the logarithms in other amplitudes. For those considered here—the complete fermion propagator and vacuum polarization—the results are very simple provided one only considers logarithms that arise "perturbatively." At the leading and first subleading order all the logarithms vanish except for the leading fermion propagator. The results are

$$S(p)=\frac{1}{\not p}\left[1+\left(\frac{\pi}{2}\right)^{1/2}\frac{|g|}{p}e^{p^2/2|g|^2}\mathrm{erfc}\frac{p}{\sqrt{2}|g|}-\frac{\alpha e^2}{16p}\right]$$
$$+\text{ further subleading logarithms} \quad (6.1)$$

with $g^2=(e^2/48\pi^2)\ln(e^2/p)$ and

$$\Pi_{\alpha\beta}(p)=-\frac{e^2p}{16}P_{\alpha\beta}(p)$$
$$+\text{ further subleading logarithms}, \quad (6.2)$$

where the further subleading logarithms are terms proportional $(e^2)^m(g^2)^n$ for all $n\geq 0$ and $m=2,3,\ldots$. They are in general not calculable perturbatively as explained, for example, for the $m=2$, $n=1$ term of the vacuum polarization at the beginning of Sec. IV.

Note that if the fermion has nonzero mass, then the first incalculable $e^8 \ln e^2$ term in the vacuum polarization would be absent. The first nonzero logarithm would be $e^{10}\ln e^2$ from the masslessness of the photon since all the second subleading logarithm ($m=2$) contributions of the dressed photon can be shown to vanish by similar arguments to those in Sec. IV. However a fermion mass would induce the photon to gain a gauge-invariant mass as explained in Ref. 5, removing all perturbative infrared divergences and coupling-constant logarithms.

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APPENDIX

To evaluate (2.7) split up the range of integration at an arbitrary point r into small-x and a large-x integrals. Naturally the final result should be independent of r:

$$\int_0^\infty dx C(x) f(\lambda/x) = \int_0^r dx C(x) f(\lambda/x) + \int_r^\infty dx C(x) f(\lambda/x)\,. \tag{A1}$$

Assuming $C(x)$ is analytic about $x=0$ and r is smaller than its radius of convergence [$r<1$ for $C(x)$ of (2.6)], the first term of (A1) can be expanded as

$$\int_0^r dx C(x) f(\lambda/x) = \sum_{n=0}^\infty C^{(n)}(0)\frac{1}{n!}\int_0^r dx\, x^n f(\lambda/x)\,. \tag{A2}$$

Changing variables to $y=\lambda/x$ in the latter integral gives

$$\int_0^r dx C(x) f(\lambda/x) = \sum_{n=0}^\infty C^{(n)}(0)\frac{\lambda^{n+1}!}{n!}\int_{\lambda/r}^\infty \frac{dy}{y^{n+2}} f(y)\,. \tag{A3}$$

Integrating the y integral by parts r 2 times results in

$$\int_z^\infty \frac{dy}{y^{n+2}} f(y) = \sum_{i=0}^n \frac{(n-i)!}{(n+1)!} z^{i-n-1} f^{(i)}(z) - \frac{1}{(n+1)!}(\ln z) f^{(n+1)}(z) - \frac{1}{(n+1)!}\int_z^\infty dy (\ln y) f^{(n+2)}(y)\,, \tag{A4}$$

where it is assumed that the derivatives of f vanish at infinite argument. Writing the last integral of (A4) as

$$\int_z^\infty dy (\ln y) f^{(n+2)}(y) = \int_0^\infty dy (\ln y) f^{(n+2)}(y) - \int_0^z dy (\ln y) f^{(n+2)}(y)\,, \tag{A5}$$

one can expand the derivatives of f about the origin to express (A4) entirely in terms of perturbative information apart from the first integral on the right of (A5),

$$\begin{aligned}\int_z^\infty \frac{dy}{y^{(n+2)}} f(y) &= \sum_{i=0}^n \frac{(n-i)!}{(n+1)!} z^{i-n-1}\sum_{j=0}^\infty f^{(i+j)}(0)\frac{z^j}{j!} - \frac{1}{(n+1)!}(\ln z)\sum_{j=0}^\infty f^{(n+1+j)}(0)\frac{z^j}{j!} \\ &\quad + \frac{1}{(n+1)!}\sum_{j=0}^\infty f^{(n+2+j)}(0)\frac{1}{j!}\frac{z^{j+1}}{j+1}\left(\ln z - \frac{1}{j+1}\right) - \frac{1}{(n+1)!}\int_0^\infty dy (\ln y) f^{(n+2)}(y) \\ &= \frac{1}{(n+1)!}\Bigg[-\int_0^\infty dy (\ln y) f^{(n+2)}(y) - (\ln z) f^{(n+1)}(0) \\ &\quad + \sum_{m=0}^\infty f^{(m)}(0) z^{m-n-1}\sum_{i=0}^n \frac{(n-i)!}{(m-i)!} - \sum_{m=n+2}^\infty \frac{f^{(m)}(0) z^{m-n-1}}{(m-n-1)!\,(m-n-1)}\Bigg]\,. \end{aligned} \tag{A6}$$

The $\sum_{i=0}^n (n-i)!/(m-i)!$ in this expression is well defined for $n \le m$. For $n>m$, the correct factor is $\sum_{i=0}^m (n-i)!/(m-i)!$, which is identical to the sum up to n if $(m-i)!$ is interpreted as $\Gamma(m-i+1)$ for negative argument. When (A6) is substituted back into (A3) with $z=\lambda/r$, this completes the analysis of the first term of (A1).

In the second term on the right of (A1), f can be expanded about the origin of its argument giving

$$\int_r^\infty dx C(x) f(\lambda/x) = \sum_{m=0}^\infty f^{(m)}(0)\frac{\lambda^m}{m!}\int_r^\infty \frac{dx}{x^m} C(x)\,. \tag{A7}$$

As it stands (A7) is already perturbatively calculable since $C(x)$ is known exactly and the x integrals are convergent hence calculable. Also the required logarithms and perturbatively noncalculable terms are already present in (A6). However, in order to present the elegant result of (2.8) and show the independence of r, I proceed to analyze (A7) further. The integral on the right-hand side of (A7) is entirely analogous to the left side of (A6) so

$$\int_r^\infty dx\,\frac{C(x)}{x^m}=\frac{1}{(m-1)!}\Bigg[-\int_0^\infty dx(\ln x)C^{(m)}(x)-(\ln r)C^{(m-1)}(0)+\sum_{n=0}^{\infty}C^{(n)}(0)r^{n-m+1}\sum_{i=0}^{m-2}\frac{(m-2-i)!}{(n-i)!}-\sum_{n=m}^{\infty}\frac{C^{(n)}(0)r^{n-m+1}}{(n-m+1)!(n-m+1)}\Bigg],\quad m\geq 1. \tag{A8}$$

For $m=0$, (A8) does not apply but instead one has

$$\int_r^\infty dxC(x)=\int_0^\infty dxC(x)-\sum_{n=0}^{\infty}C^{(n)}(0)\frac{r^{n+1}}{(n+1)!}. \tag{A9}$$

As for (A6), the same kind of interpretation is given to $\sum_{i=0}^{m-2}(m-2-i)!/(n-i)!$ in (A8) when $m-2>n$. Also when $m-2<0$, (i.e., $m=1$) the sum is taken as zero to give the correct results. Because of the logarithmic singularity at $x=1$ in $C(x)$ of (2.6), the integrations by parts leading to (A8) are in fact invalid since I have already required $r<1$ for (A2). As a consequence the logarithmic integrals of the derivatives of C in the first term of (A8) are divergent. They should be regulated in some fashion which is done at the end of the Appendix. For now I will proceed formally assuming (A8) is all right. The analysis of the second term of (A1) is completed by the substitution of (A8) into (A7).

Before collecting the results to show the r independence of (A1), I make some simplification of (A6) and (A8) using the following results which are easily proved by induction on n:

$$\sum_{j=0}^{n}\frac{(k+j)!}{j!}=\frac{(k+n+1)!}{(k+1)n!},\quad k\geq 0 \tag{A10}$$

$$\sum_{j=0}^{n}\frac{j!}{(k+j)!}=\frac{1}{(k-1)!(k-1)}-\frac{(n+1)!}{(n+k)!(k-1)},\quad k\geq 2. \tag{A11}$$

Equation (A10) corresponds to the sum referred to after (A6) when $n\geq m$ and (A11) when $n<m$. With these (A6) becomes

$$\int_{\lambda/r}^\infty dy\,\frac{f(y)}{y^{n+2}}=\frac{1}{(n+1)!}\Bigg[-\int_0^\infty dy(\ln y)f^{(n+2)}(y)-\ln(\lambda/r)f^{(n+1)}(0)+f^{(n+1)}(0)\sum_{i=0}^{n}\frac{1}{(n+1-i)}+\sum_{m=0,\neq n+1}^{\infty}f^{(m)}(0)\frac{(\lambda/r)^{m-n-1}(n+1)!}{(n-m+1)m!}\Bigg] \tag{A6$'$}$$

and (A8) becomes

$$\int_r^\infty dx\frac{C(x)}{x^m}=\frac{1}{(m-1)!}\Bigg[-\int_0^\infty dx(\ln x)C^{(m)}(x)-(\ln r)C^{(m-1)}(0)+C^{(m-1)}(0)\sum_{i=0}^{m-2}\frac{1}{(m-1-i)}+\sum_{n=0,\neq m-1}^{\infty}C^{(n)}(0)\frac{r^{n-m+1}(m-1)!}{(m-n-1)n!}\Bigg],\quad m\geq 1. \tag{A8$'$}$$

Equation (A8$'$) is substituted into (A7) and the double sum reordered, then collecting terms gives

$$\begin{aligned}\int_0^\infty dxC(x)f\Big(\frac{\lambda}{x}\Big)=&\sum_{n=0}^{\infty}\frac{\lambda^{n+1}}{n!(n+1)!}\Bigg[-C^{(n)}(0)\int_0^\infty dy(\ln y)f^{(n+2)}(y)-\ln(\lambda/r)C^{(n)}(0)f^{(n+1)}(0)\\&-f^{(n+1)}(0)\int_0^\infty dx(\ln x)C^{(n+1)}(x)-(\ln r)f^{(n+1)}(0)C^{(n)}(0)\Bigg]\\&+\sum_{n=0}^{\infty}\frac{C^{(n)}(0)\lambda^{n+1}}{n!(n+1)!}\Bigg[-\sum_{m=0,\neq n+1}^{\infty}f^{(m)}(0)\frac{(\lambda/r)^{m-n-1}(n+1)!}{(m-n-1)m!}+f^{(n+1)}(0)\sum_{i=1}^{n+1}\frac{1}{i}\\&+\frac{n!(n+1)!}{\lambda^{n+1}}\sum_{m=1,\neq n+1}^{\infty}\frac{f^{(m)}(0)\lambda^m}{(m-1)!m!}\frac{r^{n-m+1}(m-1)!}{(m-n-1)n!}\Bigg]\\&+\sum_{m=1}^{\infty}f^{(m)}(0)\frac{\lambda^m}{m!}\frac{1}{(m-1)!}C^{(m-1)}(0)\sum_{i=1}^{m-1}\frac{1}{i}+f(0)\Bigg[\int_0^\infty dxC(x)-\sum_{n=0}^{\infty}C^{(n)}(0)\frac{r^{n+1}}{(n+1)!}\Bigg].\end{aligned} \tag{A12}$$

The logarithmic r dependence in the first term of (A12) clearly cancels to give the first three terms of the sum in (2.8). The remaining sums over n of (A12) cancel for $m \geq 1$ leaving the $m = 0$ term to give

$$-\sum_{n=0}^{\infty} \frac{C^{(n)}(0)\lambda^{n+1}}{n!(n+1)!} \frac{f(0)(\lambda/r)^{-n-1}(n+1)!}{(-n-1)0!},$$

which cancels the very last term in (A12). The remaining terms are r independent and combine to give the result (2.8).

To regularize the divergent logarithmic C integral, assuming C has a logarithmic singularity at $x = s$, I repeat the analysis, but in (A1) the upper limit of the first term is $r < s$ and the lower limit of the second is $r' > s$. The correction, which is the integral from r to r', is regular and vanishes as r, $r' \to s$. In order to derive the same result (A8) as r, $r' \to s$, one needs to make the following replacement in the formulas:

$$\int_0^\infty dx(\ln x)C^{(m)}(x) \to \lim_{\substack{r\to s_- \\ r'\to s_+}} \Bigg[\int_0^r dx(\ln x)C^{(m)}(x) + \int_{r'}^\infty dx(\ln x)C^{(m)}(x) + (\ln s)[C^{(m-1)}(r') - C^{(m-1)}(r)] - \sum_{i=0}^{m-2} (m-2-i)!\, s^{(i-m+1)}[C^{(i)}(r') - C^{(i)}(r)]\Bigg], \quad \text{(A13)}$$

the right side of which is convergent. Integrating the right-hand side of (A13) by parts m times also gives the simpler expression

$$\int_0^\infty dx(\ln x)C^{(m)}(x) \to \lim_{\epsilon\to 0} \left[-\int_\epsilon^\infty dx \frac{C(x)}{x^m} - (\ln\epsilon)C^{(m-1)}(0) + \sum_{i=0}^{m-2} (m-2-i)!\, \epsilon^{(i-m+1)} C^{(i)}(0)\right], \quad \text{(A14)}$$

where the $C(x)$ integral is now regular at $x = s$.

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[1] D. Kirzhnits and A. Linde, Phys. Lett. 47B, 471 (1972); L. Dolan and R. Jackiw, Phys. Rev. D 9, 3320 (1974); S. Weinberg, Phys. Rev. D 9, 3357 (1974).

[2] J. Collins and M. Perry, Phys. Rev. Lett. 34, 135 (1975); A. M. Polyakov, Phys. Lett. 72B, 477 (1978); L. Susskind, Phys. Rev. D 20, 2610 (1979).

[3] For reviews see S. Weinberg, in *Understanding the Fundamental Constituents of Matter*, edited by A. Zichichi (Plenum, New York, 1978); A. Linde, Rep. Prog. Phys. 42, 389 (1979); D. Gross, R. Pisarski, and L. Yaffe, Rev. Mod. Phys. 53, 43 (1981).

[4] R. Jackiw, Ph.D. thesis, Cornell University, 1966 (unpublished). K. Symanzik, Lett. Nuovo Cimento 8, 771 (1973); A. Linde, Phys. Lett. 96B, 289 (1980); 96B, 293 (1980); G. 't Hooft, in *Field and Strong Interactions*, edited by Paul Urban (Springer, New York, 1980); O. K. Kalashnikov and V. V. Klimov, Yad. Fiz. 33, 595 (1981); O. K. Kalashnikov, Pis'ma Zh. Eksp. Teor. Fiz. 33, 173 (1981) [JETP Lett. 33, 165 (1981)]; T. Appelquist and R. Pisarski, Phys. Rev. D 23, 2305 (1981).

[5] R. Jackiw and S. Templeton, Phys. Rev. D 23, 2291 (1981).

[6] S. Templeton, Phys. Lett. 103B, 134 (1981).

[7] R. Jackiw, private communication.