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Interaction among systems of finite size in predictive relativistic mechanics. I. General framework

Xavier Fustero and Enric Verdaguer

Departament de Teòrica and Grupo Interuniversitario de Física Teòrica, Universitat Autònoma de Barcelona, Bellattera (Barcelona), Spain

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We derive a Hamiltonian formulation for the three-dimensional formalism of predictive relativistic mechanics. This Hamiltonian structure is used to derive a set of dynamical equations describing the interaction among systems in perturbation theory.

I. INTRODUCTION

The objects we find in the natural world, both in the microscopic and macroscopic domains, are characterized by a general property, namely that they possess an internal structure. Usually the internal structure of the physical systems produces only small effects in their dynamical behavior and therefore a theory for structureless particles can be used successfully. However, if accurate measurements are performed, a theory taking into account the structure of the physical systems is needed. This is of course the case of quantum mechanics, where the spin of the particles is introduced, or the case of the theory of gravitation, where the spin precession of gravitating bodies is calculated.

This paper is the first of a series of three devoted to the study of the interaction among bodies of finite size within the framework of predictive relativistic mechanics, hereafter abbreviated as PRM.

This subject has been studied by several authors, especially in the quantum domain, since the pioneering work of Bakamjian and Thomas.[1] Close and Osborn[2] used a quasi-field-theoretical approach to give the form of the relativistic interaction terms for the electromagnetic interaction. The structure of the terms giving the interaction with an external field was corrected and generalized by Krajcik and Foldy,[3-5] Foldy,[6] and Coester and Havas[7] using the Bakamjian and Thomas formalism and the problem of the center-of-mass variables has been studied also by Liou[8] and Tindle.[9]

The approach followed here differs from that of those authors in several aspects. First of all, our treatment of the subject is purely classical, in contrast with their quantum formulations. This precludes a direct comparison of our results with those of the preceding authors; this comparison will only be possible when a quantization of PRM able to describe spinning particles is available. Second, in our approach, based on PRM and classical field theory, the structure of the interacting terms is uniquely defined (up to changes of variables), and therefore the problems we are faced with are different from those found using the Bakamjian and Thomas formalism where the interaction is introduced by means of a unitary transformation.

Given an isolated system of interacting particles, one is usually interested in defining global quantities such as the total energy, the total momentum, the center-of-mass position, and the total angular momentum. This can be achieved easily if a Hamiltonian formalism is at hand.

It turns out that PRM can be put in Hamiltonian form and that this Hamiltonian formalism is unique in perturbation theory, therefore this Hamiltonian formalism provides us with a unique canonical way of defining the dynamical quantities associated with an isolated system.

The first Hamiltonian formalism devised for PRM was that of Bel and Martin,[14] using the four-dimensional formalism. Its projection on the hypersurface $t = \text{const}$ allowed the construction of a three-dimensional Hamiltonian formalism.

In this paper the Hamiltonian formalism is constructed within the three-dimensional framework of PRM from the start and we show its equivalence with the projection of the four-dimensional one. Once the Hamiltonian structure is constructed we can use it to define the ten generating functions H, P^i, J^i, K^i which provide natural definitions for the energy, total linear momentum, and total angular momentum. This covers the first five sections of this paper.

In Sec. VI we give an alternative method to construct these quantities which is simpler and gives the same results although it is not based upon the Hamiltonian formalism. Section VII is a review of the definitions of spin, center of spin, and center of mass of an isolated system to be used in the subsequent sections.

An isolated system as a whole can be characterized by giving its energy H, its intrinsic angular momentum $\vec{S}$, its center-of-mass position $\vec{X}$, and the velocity of the center of mass $\vec{V}$. This char-

acterization has a well-defined meaning in the case in which the constituent particles of this system are closely tied together and behave as a rigid body. When two such systems are under mutual interaction the evolution equations of these dynamical quantities can easily be derived. This is done in Sec. VIII. The resulting system of differential equations is not, however, a closed one. A way to get a closed system is to make a multipolar development, to assume a finite multipolar structure for the bodies in question, and then to provide the evolution equation for the multipolar moments introduced.

We have limited ourselves to one of the simplest cases, assuming spherical symmetry or only small departures from it. Under such assumptions we have been able to write the equations of motion for the spin, the center of mass, and the mass of the systems in terms of $\vec{X}$, $\vec{S}$, $\vec{V}$, the masses, and $\vec{\mu}=\epsilon^{a}e_a\vec{r}_a\times\vec{v}_a$, which in the case of the electromagnetic interaction is the magnetic moment. Assuming then a relation of the form $\vec{S}=\alpha\vec{\mu}$ the system becomes closed to first order.

In this way we have been able to build a system of differential equations that describes the evolution of the global quantities. These equations as we shall see in a forthcoming paper can also be interpreted (to order c^{-2}) as the equations of motion for spinning particles, because to this order the masses are constant, and therefore they provide a dynamical model for the study of the interaction among classical spinning particles.

Finally, in Sec. X we give a general procedure to find the general integral of the dynamical system, to first order in perturbation theory.

II. INTEGRATION OF THE EQUATIONS OF MOTION

In the three-dimensional formalism of PRM the dynamics of an interacting system of structureless particles is described by the following system of second-order differential equations:

$$\frac{d\vec{x}_a}{dt}=\vec{v}_a,\quad \frac{d\vec{v}_a}{dt}=\vec{a}_a(\vec{x}_b,\vec{v}_c)\quad (a,b,c=1,\ldots,n)\,. \tag{2.1}$$

Only those functions a_a^i $(i=1,2,3)$ satisfying the Currie-Hill conditions

$$\begin{aligned}&\mathfrak{L}(\vec{P}_k)a_a^i=0\,,\\ &\mathfrak{L}(\vec{J}_j)a_a^i=\delta_j{}^i{}_k a^k{}_a\,,\\ &\mathfrak{L}(\vec{K}_j)a_b^i=x_{bj}\mathfrak{L}(\vec{H})a_b^i-2v_{bj}a_b^i-v_b^i a_{bj}\,,\end{aligned} \tag{2.2}$$

can be used.[10-12] The symbols $\vec{P}_k$, $\vec{J}_k$, $\vec{K}_j$, $\vec{H}$ stand for the ten vector fields

$$\begin{aligned}&\vec{H}=-v_a^i\frac{\partial}{\partial x_a^i}-a_a^i\frac{\partial}{\partial v_a^i}\,,\\ &\vec{P}_k=-\epsilon_a\frac{\partial}{\partial x_a^k}\quad(\epsilon_a=1,\forall a)\,,\\ &\vec{J}_k=\delta_k{}^i{}_j x_a^j\frac{\partial}{\partial x_a^i}+\delta_k{}^i{}_j v_a^j\frac{\partial}{\partial v_a^i}\,,\\ &\vec{K}_j=-x_{aj}v_a^i\frac{\partial}{\partial x_a^i}+\epsilon_a(\delta_j^i-a_a^i x_{aj}-v_a^i v_{aj})\frac{\partial}{\partial v_a^i}\,.\end{aligned} \tag{2.3}$$

As can easily be verified, they satisfy the commutation relations characteristic of the Lie algebra of the Poincaré group.

We shall now study a general procedure to construct the general integral of (2.1) within the framework of perturbation theory, where the $\vec{a}_a$ of the most significant interactions are known.

If we are able to solve the following system of differential equations,

$$\mathfrak{L}(\vec{H})p_a^i=0\,,\quad \mathfrak{L}(\vec{H})q_a^i=-\frac{p_a^i}{(p_a{}^2+m_a{}^2)^{1/2}}\,, \tag{2.4}$$

then the general integral of (2.1) can be given in implicit form as

$$\begin{aligned}&p_a^i(\vec{x}_b,\vec{v}_c)=p_a^i(\vec{x}_b^{(0)},\vec{v}_c^{(0)})=C_a^i\quad (C_a^i\ \text{constants}),\\ &q_a^i(\vec{x}_b,\vec{v}_c)=q_a^i(\vec{x}_b^{(0)},\vec{v}_c^{(0)})+\frac{C_a^i}{(C_a{}^2+m_a{}^2)}t\end{aligned} \tag{2.5}$$

in the neighborhood of a point $\vec{x}_b$, $\vec{v}_c$ where the p_a^i and q_a^i are C^1 functions satisfying

$$\frac{\partial(q_a^i,p_b^j)}{\partial(x_c^k,v_d^n)}\neq 0\,.$$

The functions q_a^i and p_b^i will be called Hamilton-Jacobi coordinates. For a free particle system this role is played by the functions $\vec{x}_a$, $m_a\gamma_a\vec{v}_a$. Therefore we shall look for solutions of (2.4) that reduce to x_a and $m_a\gamma_a\vec{v}_a$ in the limit $(\vec{x}_a-\vec{x}_{a'})^2\to\infty$. This condition can be expressed in the form

$$\begin{aligned}&\lim_{\lambda\to\pm\infty}R(\lambda)(q_a^i-x_a^i)=0\,,\\ &\lim_{\lambda\to\pm\infty}R(\lambda)(p_a^i-m_a\gamma_a v_a^i)=0\,,\end{aligned} \tag{2.6}$$

where $R(\lambda)$ is the translation operator defined by

$$R(\lambda)f(\vec{x}_a,\vec{v}_b)\equiv f(\vec{x}_a+\lambda\vec{v}_a,\vec{v}_b)\,.$$

We shall now prove the following proposition: A given set of functions q_a^i, p_b^j is a solution of (2.4) satisfying the asymptotic conditions (2.6) if and only if it is a solution of the integral equations

$$p_a^i=m_a\gamma_a v_a^i+\int_0^{-\infty}d\lambda\,R(\lambda)\left(a_b^j\frac{\partial p_a^i}{\partial v_b^j}\right), \tag{2.7}$$

$$q_a^i=x_a^j+\int_0^{-\infty}d\lambda\,R(\lambda)\left(a_b^i\frac{\partial q_a^j}{\partial v_b^i}-\frac{p_a^j}{E_a}+v_a^j\right), \tag{2.8}$$

where

$E_a=(p_a{}^2+m_a{}^2)^{1/2}$.

Proof: Taking the shorthand notation $D=v_b^j(\partial/\partial x_b^j)$ it can be proved that $DR(\lambda)=R(\lambda)D=[dR(\lambda)/d\lambda]$. Therefore, applying $R(\lambda)$ to Eq. (2.4) we get

$$\frac{d}{d\lambda}R(\lambda)p_a^i=-R(\lambda)a_b^j\frac{\partial p_a^i}{\partial v_b^j},$$

$$\frac{d}{d\lambda}R(\lambda)(q_a^i-x_a^i)=-R(\lambda)\left(a_b^j\frac{\partial}{\partial v_b^j}(q_a^i-x_a^i)-\frac{p_a^i}{E_a}+v_a^i\right).$$

Integrating now these equations between $-\infty$ and 0 and taking into account the asymptotic conditions we find (2.7) and (2.8). This completes the first part of the proof.

From the definition of $R(\lambda)$ it is obvious that $R(\lambda)\cdot R(\tilde{x})=R(\lambda+\tilde{\lambda})$. Hence,

$$\lim_{\tilde{\lambda}\to-\infty}R(\tilde{\lambda})p_a^i=m_a\gamma_a v_a^i+\lim_{\tilde{\lambda}\to-\infty}\int_{\tilde{\lambda}}^{-\infty}d\lambda\,R(\lambda)\left(a_b^j\frac{\partial p_a^i}{\partial v_b^j}\right)=m_a\gamma_a v_a^i,$$

$$\lim_{\tilde{\lambda}\to-\infty}R(\tilde{\lambda})(q_a^i-x_a^i)=\lim_{\tilde{\lambda}\to-\infty}\int_{\tilde{\lambda}}^{-\infty}d\lambda\,R(\lambda)\left(a_b\frac{\partial q_a^i}{\partial v_b^j}+v_a^i-\frac{p_a^i}{E_a}\right)=0,$$

which implies that the solutions of (2.7) and (2.8) satisfy the asymptotic conditions (2.6). Applying now the operator D to both sides of (2.7) and (2.8) it is easily verified that Eqs. (2.4) are satisfied, which completes the proof.

In PRM the accelerations $\vec{a}_a$ are known as power series of the coupling constants. This forces us to introduce the expansions

$$p_a^i=\sum_{n=0}^{\infty}g^n p_a^{(n)i},\quad q_a^i=\sum_{n=0}^{\infty}g^n q_a^{(n)i} \tag{2.9}$$

for the solutions of (2.7) and (2.8). If we introduce the expansions (2.9) in (2.7) and (2.8) we obtain the basis for a recurrent algorithm that gives the functions $p_a^{(n)i}$ and $q_a^{(n)i}$ by means of quadratures only,

$$p_a^{(n)i}=\delta^{0n}m_a\gamma_a v_a^i+\int_0^{-\infty}d\lambda\,R(\lambda)\left(\sum_{r+s=n}a_b^{(r)j}\frac{\partial p_a^{(s)i}}{\partial v_b^j}\right), \tag{2.10}$$

$$q_a^{(n)i}=\delta^{0n}x_a^i+\int_0^{-\infty}d\lambda\,R(\lambda)\left[\sum_{r+s=n}\left(a_b^{(r)j}\frac{\partial q_a^{(s)i}}{\partial v_b^j}-\frac{p_a^{(r)i}}{E_a^{(s)}}\right)+\delta^{0n}v_a^i\right]. \tag{2.11}$$

Therefore we have proved the following theorem by explicit construction: Within the framework of perturbation theory there is only one solution of (2.4) satisfying the asymptotic conditions (2.6), which is given by Eqs. (2.10) and (2.11). This solution gives the general integral of (2.1) in implicit form.

III. TRANSFORMATION PROPERTIES OF q_a^i AND p_a^i

Using Eqs. (2.10) and (2.11) and the known transformation properties of the functions x_a^i, v_b^j, and a_c^k under space translations and rotations it is an easy task to prove that the functions q_a^i and p_b^j satisfy the following set of differential equations:

$$\begin{aligned}&\mathfrak{L}(\vec{P}_k)q_a^i=-\epsilon_a\delta_k^i,\quad \mathfrak{L}(\vec{P}_k)p_b^j=0,\\&\mathfrak{L}(\vec{J}_k)q_a^i=\delta_k{}^i{}_j q_a^j,\quad \mathfrak{L}(\vec{J}_k)p_b^i=\delta_k{}^i{}_j p_b^j.\end{aligned} \tag{3.1}$$

Within the framework of perturbation theory, we shall now prove the following result: The functions defined by (2.10) and (2.11) are solutions of the system of differential equations

$$\mathfrak{L}(\vec{K}_j)p_a^i=\delta_j^i E_a, \tag{3.2}$$

$$\mathfrak{L}(\vec{K}_j)q_a^i=-\frac{q_{aj}p_a^i}{E_a}. \tag{3.3}$$

Proof: The function $E_a=(p_a{}^2+m_a{}^2)^{1/2}$ satisfies the differential equation

$$\mathfrak{L}(\vec{H})E_a=0 \tag{3.4}$$

and the asymptotic condition

$$\lim_{\lambda\to\pm\infty}R(\lambda)(E_a-m_a\gamma_a)=0. \tag{3.5}$$

Therefore in perturbation theory there is only one solution of (3.4) satisfying (3.5), given by

$$E_a^{(n)}=\delta^{0n}m_a\gamma_a+\int_0^{-\infty}d\lambda\,R(\lambda)\sum_{r+s=n}a_b^{(r)j}\frac{\partial E_a^{(s)}}{\partial v_b^j}.$$

Applying now the commutation relations of the Lie algebra of the Poincaré group we find

$$\mathfrak{L}(\vec{K}_j)[\mathfrak{L}(\vec{H})p_a^i]=\mathfrak{L}(\vec{H})[\mathfrak{L}(\vec{K}_j)p_a^i]=0.$$

On the other hand from (2.3) and (2.10) it can easily be seen that

$$\lim_{\lambda\to-\infty}R(\lambda)[\mathfrak{L}(\vec{K}_j)p_a^i-\delta_j^i m_a\gamma_a]=0.$$

This gives in perturbation theory the unique solution

$$[\mathfrak{L}(\vec{K}_j)p_a^i]^{(n)}=\delta_j^i\left\{\sum_{n=0}^{\infty}g^n\left[\delta^{0n}m_a\gamma_a+\int_0^{-\infty}d\lambda\,R(\lambda)\sum_{r+s=n}\left(a_b^{(r)j}\frac{\partial E_a^{(s)}}{\partial v_b^j}\right)\right]\right\}=\delta_j^i E_a^{(n)}.$$

This proves (3.2).

To prove (3.3) we consider the function $-q_{aj}p_a^i/E_a$. This function satisfies the equation

$$\mathcal{L}(\vec{H})\left(-\frac{q_{aj}p_a^i}{E_a}\right)=\frac{p_{aj}p_a^i}{E_a^2} \tag{3.6}$$

and the asymptotic condition

$$\lim_{\lambda\to-\infty} R(\lambda)\left(-\frac{q_{aj}p_a^i}{E_a}+x_{aj}v_a^i\right)=0\,, \tag{3.7}$$

and there is only one such function in perturbation theory. As before, using the commutation relations of the Lie algebra of the Poincaré group, (2.3) and (2.11), it can easily be shown that $\mathcal{L}(\vec{K}_j)q_a^i$ satisfies (3.6) and (3.7), which proves (3.3).

IV. HAMILTONIAN FORMULATION

For a given interaction, that is for a given set of functions $\vec{a}_a$ satisfying the Currie-Hill conditions, we shall consider an open set where

$$\frac{\partial(q_a^i,p_b^j)}{\partial(x_c^k,v_d^l)}\neq 0\,,$$

and there we construct the two-form

$$\Omega=dq_a^i\wedge dp_i^a\,.$$

This two-form is therefore closed ($d\Omega=0$) and of maximum rank. As $d\Omega=0$ we have

$$\mathcal{L}(\vec{H})\Omega=d(i(\vec{H})\Omega)\,,$$
$$\mathcal{L}(\vec{P}_k)\Omega=d(i(\vec{P}_k)\Omega)\,,$$
$$\mathcal{L}(\vec{J}_k)\Omega=d(i(\vec{J}_k)\Omega)\,,$$
$$\mathcal{L}(\vec{K}_j)\Omega=d(i(\vec{K}_j)\Omega)\,,$$

and since

$$i(\vec{H})\Omega=-d(\epsilon^a E_a)\,, \tag{4.1}$$
$$i(\vec{P}_k)\Omega=-d(\epsilon^a p_{ak})\,, \tag{4.2}$$
$$i(\vec{J}_k)\Omega=-d(\delta_{kij}\epsilon^a q_a^i p_a^j)\,, \tag{4.3}$$
$$i(\vec{K}_j)\Omega=-d(\epsilon^a E_a q_{aj})\,, \tag{4.4}$$

we have

$$\mathcal{L}(\vec{H})\Omega=\mathcal{L}(\vec{P}_k)\Omega=\mathcal{L}(\vec{J}_k)\Omega=\mathcal{L}(\vec{K}_j)\Omega=0\,.$$

Therefore the symplectic two-form thus constructed is invariant under the Poincaré group, that is, the Poincaré transformations are canonical transformations.

Equations (4.1)–(4.4) define the ten functions

$$H=\epsilon^a E_a\,,\quad P_k=\epsilon^a p_{ak}\,,$$
$$J_k=\epsilon^a\delta_{kij}q_a^i p_a^j\,,\quad K_j=\epsilon^a E_a q_{aj}\,, \tag{4.5}$$

up to additive constants. If these constants are chosen to be zero, then the Poisson bracket relations among these functions are formally identical with the Lie bracket relations among the corresponding vector fields.

The expressions (4.5) are in agreement with the so-called no-interaction theorem[13] owing to Eqs. (3.1)–(3.3).

We have therefore constructed a unique Hamiltonian formalism for the system considered, as long as perturbation theory is used.

V. RELATIONS WITH THE FOUR-DIMENSIONAL FORMALISM

In the four-dimensional Hamiltonian formalism of Bel and Martín[5] use is made of a symplectic two-form ω defined by

$$\omega=d\hat{q}_a^\alpha\wedge d\hat{p}_\alpha^a\,, \tag{5.1}$$

where $\hat{q}_a^\alpha$ and $\hat{p}_\alpha^a$ are solutions of the following system of differential equations:

$$\begin{aligned}
&\mathcal{L}(\vec{P}_\mu)\hat{q}_a^\alpha=-\epsilon_a\delta_\mu^\alpha\,,\quad \mathcal{L}(\vec{P}_\mu)\hat{p}_\beta^b=0\,,\\
&\mathcal{L}(\vec{J}_{\lambda\mu})\hat{q}_a^\alpha=\delta_\lambda^\alpha\hat{q}_{a\mu}-\delta_\mu^\alpha\hat{q}_{a\lambda}\,,\quad \mathcal{L}(\vec{J}_{\lambda\mu})\hat{p}_\beta^b=\eta_{\lambda\beta}\hat{p}_\mu^b-\eta_{\mu\beta}\hat{p}_\lambda^b\,,\\
&\mathcal{L}(\vec{H}_b)\hat{q}_a^\alpha=\delta_{ba}\hat{p}_a^\alpha\,,\quad \mathcal{L}(\vec{H}_b)\hat{p}^a=0\,.
\end{aligned} \tag{5.2}$$

and the asymptotic conditions

$$\lim_{\mu\to\pm\infty}\prod_{a=1}^{n}R_a(\mu\lambda_a)(\hat{p}_\beta^b-\pi_\beta^b) = \lim_{\mu\to\pm\infty}\prod_{a=1}^{n}R_a(\mu\lambda_a)(\hat{q}_b^\beta-x_b^\beta)=0\,. \tag{5.3}$$

The action of the operators $R_a(\lambda_a)$ is defined in (5.6) and the parameters λ_a must be chosen in such a way that the translated points have spacelike separations.[14]

Let us now study the restriction of this symplectic form to the submanifold defined by the equations

$$\begin{aligned}
&x_a^0(\hat{q}_b^\beta,\hat{p}_c^\gamma)=0\,,\\
&\Pi_a^2(\hat{q}_b^\beta,\hat{p}_c^\gamma)=m_a^2\,,
\end{aligned}\qquad a=1,2,\ldots,n\,. \tag{5.4}$$

Taking as independent variables $\hat{q}_a^i$ and $\hat{p}_a^i$, we have

$$d\hat{q}_a^0=\frac{\partial\hat{q}_a^0}{\partial\hat{q}_b^i}d\hat{q}_b^i+\frac{\partial\hat{q}_a^0}{\partial\hat{p}_b^i}d\hat{p}_b^i\,,$$

$$d\hat{p}_a^0=\frac{\partial\hat{p}_a^0}{\partial\hat{q}_b^i}d\hat{q}_b^i+\frac{\partial\hat{p}_a^0}{\partial\hat{p}_b^i}d\hat{p}_b^i\,,$$

and as $\partial\hat{p}_a^0/\partial\hat{q}_b^i=0$ and $\partial\hat{p}_a^0/\partial\hat{p}_{a'}^i=0$ for $a'\neq a$ (see Ref. 14), we have

$$\begin{aligned}
\overline{\omega}&=-\left(d\hat{q}_a^0\wedge\frac{\partial\hat{p}_a^0}{\partial\hat{p}_a^i}d\hat{p}_a^i\right)+d\hat{q}_a^i\wedge d\hat{p}_i^a\\
&=\left(d\hat{q}_a^0-\frac{\partial\hat{p}_a^0}{\partial p_a^i}d\hat{q}_a^0\right)\wedge d\hat{p}_i^a\\
&=d\left(\hat{q}_a^i-\hat{q}_a^0\frac{\hat{p}_a^i}{E_a}\right)\wedge d\hat{p}_i^a\,.
\end{aligned}$$

The functions

$$\tilde{q}_a^i = \hat{q}_a^i - \hat{q}_a^0 \frac{\hat{p}_a^i}{E_a} \quad \text{and} \quad \tilde{p}_a^i = \hat{p}_a^i \tag{5.5}$$

satisfy the same differential equations as the Hamilton-Jacobi coordinates defined in the second paragraph. This can easily be verified using the relations given in the Appendix of Ref. 14 for the action of the ten generators (2.3) on functions projected on the submanifold (5.4).

We shall now prove that they satisfy the same asymptotic conditions. This will prove that they are the same functions as long as perturbation theory is used.

Let f be a function of the variables x_a, u_b. We have

$$\prod_{a=1}^{n} R_a(\lambda_a) f(x_b^\beta, u_c^\gamma) = f(x_b^\beta + \lambda_b u_b^\beta, u_c^\gamma) , \tag{5.6}$$

and under projection into the submanifold

$$\Pi\{f(x_b^\beta, u_c^\gamma)\} = f(0, x_b^i, \gamma_c, \gamma_c v_c^j) . \tag{5.7}$$

If we now apply the operator $R(\lambda)$ on the projected function we get

$$R(\lambda)\{\Pi[f(x_b^\beta, u_c^\gamma)]\} = f(0, x_b^i + \lambda v_b^i, \gamma_c, \gamma_c v_c^j) . \tag{5.8}$$

Therefore if $\lambda_b = \lambda/\gamma_b$ and the x^μ dependence of f arises only through differences of positions,

$$\Pi\left|\prod_{a=1}^{n} R_a(\lambda_a) f\right| = R(\lambda)[\Pi(f)] .$$

The first condition can easily be fulfilled because, as can easily be shown, λ/γ_b belongs to the admissible class of parameters.

Since p_a^μ is translation invariant the only dependence it can have on the x^μ is through differences. Therefore the two conditions are fulfilled and the asymptotic condition on the p_a^μ can be projected giving

$$\lim_{\lambda\to\pm\infty} R(\lambda)\hat{p}_a^i = m_a\gamma_a v_a^i , \quad \lim_{\lambda\to\pm\infty} R(\lambda)\hat{p}_a^0 = m_a\gamma_a .$$

Since $q_a^\mu - x_a^\mu$ is also translation invariant, the same procedure applies and we get

$$\lim_{\lambda\to\pm\infty} R(\lambda)(\hat{q}_a^i - x_a^i) = 0 , \quad \lim_{\lambda\to\pm\infty} R(\lambda)(\hat{q}_a^0) = 0 .$$

Therefore the functions (5.5) satisfy the conditions (2.6). Q.E.D.

VI. CONSTRUCTION OF THE GENERATING FUNCTIONS OF POINCARÉ TRANSFORMATIONS

The ten functions defined in (4.5) can obviously be constructed by computing first the Hamilton-Jacobi coordinates and then applying the definitions (4.5). However, for some kinds of interactions the integrals giving the q_a^i are not defined and this procedure cannot be applied. In this case the asymptotic conditions must be relaxed, and then the q_a^i are not uniquely defined in general. It must be pointed out that in spite of the fact that the q_a^i are not uniquely defined Ω is uniquely defined, and therefore the Hamiltonian formalism is unique also in this case. This is for instance the case for the electromagnetic interaction. For these interactions the ten generating functions are well defined and a direct way of finding them is useful.

The ten vector fields $\vec{H}$, $\vec{P}_k$, $\vec{J}_k$, $\vec{K}_j$ when written in a Hamilton-Jacobi coordinate system take the form

$$\begin{aligned}
&\vec{H} = -\frac{p_a^i}{E_a}\frac{\partial}{\partial q_a^i} , \quad \vec{P}_k = -\epsilon_a \frac{\partial}{\partial q_a^k} , \\
&\vec{J}_k = \delta_k{}^i{}_j\, p_a^j \frac{\partial}{\partial p_a^i} + \delta_k{}^i{}_j\, q_a^j \frac{\partial}{\partial q_a^i} , \\
&\vec{K}_j = E_a \frac{\partial}{\partial p_a^j} - \frac{q_{aj} p_a^i}{E_a}\frac{\partial}{\partial q_a^i} .
\end{aligned} \tag{6.1}$$

With these expressions it is easy to show that the ten functions (4.5) satisfy the following system of differential equations:

$$\begin{aligned}
&\pounds(\vec{H})H = 0 , \quad \pounds(\vec{P}_k)H = 0 , \quad \pounds(\vec{J}_k)H = 0 , \\
&\pounds(\vec{H})P_k = 0 , \quad \pounds(\vec{P}_k)P_j = 0 , \quad \pounds(\vec{J}_j)P_k = \delta_{jk}^n P_n , \\
&\pounds(\vec{H})J_k = 0 , \quad \pounds(\vec{P}_k)J_j = -\delta_{kj}^n P_n , \quad \pounds(\vec{J}_j)J_k = -\delta_{jk}^n J_n , \\
&\pounds(\vec{H})K_j = -P_k , \quad \pounds(\vec{P}_k)K_j = -\delta_{kj} H , \quad \pounds(\vec{J}_j)K_k = \delta_{jk}^n K_n , \\
&\pounds(\vec{K}_j)H = P_j , \quad \pounds(\vec{K}_j)P_k = \delta_{jk} H , \quad \pounds(\vec{K}_j)J_k = -\delta_{jk}^n K_n , \\
&\pounds(\vec{K}_j)K_k = \delta_{jk}^n J_n .
\end{aligned} \tag{6.2}$$

Furthermore the use of the asymptotic conditions (2.6) leads for these ten functions to the following conditions:

$$\begin{aligned}
&\lim_{\lambda\to\pm\infty} R(\lambda)H = \epsilon^a m_a\gamma_a \equiv H^{(0)} , \\
&\lim_{\lambda\to\pm\infty} R(\lambda)P_k = \epsilon^a m_a\gamma_a v_{ak} \equiv P_k^{(0)} , \\
&\lim_{\lambda\to\pm\infty} R(\lambda)J_k = \epsilon^a \delta_{ki}{}^j x_a^i m_a\gamma_a v_a^j \equiv J_k^{(0)} , \\
&\lim_{\lambda\to\pm\infty} R(\lambda)K_k = \epsilon^a m_a\gamma_a x_{ak} \equiv K_k^{(0)} .
\end{aligned} \tag{6.3}$$

These same techniques used before can now be applied, and within the framework of perturbation theory it can be shown that a unique solution can be found using a recurrent algorithm based upon the integral equations

$$\begin{aligned}
&H = H^{(0)} + \int_0^{-\infty} d\lambda\, R(\lambda) a_b^i \frac{\partial H}{\partial v_b^i} , \\
&P_k = P_k^{(0)} + \int_0^{-\infty} d\lambda\, R(\lambda) a_b^i \frac{\partial P_k}{\partial v_b^i} , \\
&J_k = J_k^{(0)} + \int_0^{-\infty} d\lambda\, R(\lambda)\left(a_b^i \frac{\partial J_k}{\partial v_b^i}\right), \\
&K_k = K_k^{(0)} + \int_0^{-\infty} d\lambda\, R(\lambda)\left(a_b^i \frac{\partial K_k}{\partial v_b^i} - P_k\right),
\end{aligned} \tag{6.4}$$

satisfying both (6.2) and (6.3).

VII. SPIN AND CENTER OF MASS OF AN ISOLATED SYSTEM

An isolated system of interacting particles can be characterized by the above-mentioned ten functions. H is usually identified with the energy, P_k with the total linear momentum, J_k with the total angular momentum, and K_j does not possess a clear physical interpretation.

These quantities are simply related with the four-dimensional quantities $J^{\lambda\mu}$ and P^{μ} associated with the same system through the relations

$$H=P^0,\quad P^i=P^i,\quad J^i=\tfrac{1}{2}\delta^{ijk}J_{jk},\quad K^i=J^{i0}. \tag{7.1}$$

The intrinsic angular momentum of an isolated system is usually defined using the Pauli-Lubanski four-vector[15]

$$W^{\alpha}=\tfrac{1}{2}\delta^{\alpha\beta\gamma\delta}P_{\beta}J_{\gamma\delta}\quad(\delta^{0123}=+1) \tag{7.2}$$

as

$$S^i=\frac{1}{M}\left(W^i-\frac{W^0P^i}{M+H}\right)\quad[M\equiv(H^2-P^2)^{1/2}], \tag{7.3}$$

because this vector has the following Poisson brackets:

$$\{S_i,S_j\}=\delta_{ijk}S_k,\quad \{S_i,P_j\}=0,$$
$$\{S_i,H\}=0,\quad \{J_i,S_j\}=\delta_{ijk}S_k,$$

and reduces to J_i in the $P_j=0$ reference frame.

Using this vector the total angular momentum can be written in the form

$$\vec{J}=\vec{Q}\times\vec{P}+\vec{S}, \tag{7.4}$$

where

$$\vec{Q}\equiv\frac{\vec{K}}{H}-\frac{\vec{P}\times\vec{S}}{H(M+H)}. \tag{7.5}$$

The vector $\vec{Q}$ is usually called center of spin and it has the following Poisson brackets with the other quantities:

$$\{Q_i,H\}=\frac{P_i}{H},\quad \{Q_i,P_j\}=\delta_{ij},\quad \{Q_i,Q_j\}=0,$$
$$\{Q_i,S_j\}=0,\quad \{Q_i,J_j\}=\delta_{ijk}Q_k,$$
$$\{K_i,Q_j\}=-\frac{Q_iP_j}{H}-\frac{\epsilon_{ijk}S_k}{M+H}\frac{(P\times S)_iP_j}{H(M+H)}.$$

This last Poisson bracket precludes the possibility of interpreting $\vec{Q}$ as the center of mass. However, there exists a vector $\vec{X}$ having the right Poisson brackets,[16]

$$\{P_i,X_j\}=-\delta_{ij},\quad \{H,X_j\}=-\frac{P_j}{H},$$
$$\{J_i,X_j\}=-\delta_{ijk}X_k,\quad \{K_i,X_j\}=-\frac{X_iP_j}{H}. \tag{7.6}$$

This vector is defined as

$$\vec{X}=\vec{Q}-\frac{\vec{P}\times\vec{S}}{M(M+H)}, \tag{7.7}$$

and following Pryce we shall call it the center of mass.

VIII. INTERACTION AMONG SYSTEMS OF PARTICLES

Let us consider a system of mutually interacting particles. In many cases this system can be considered as naturally divided into subsystems because of the fact that some of its constitutents remain tied together along the interaction process. This is what gives sense to the concepts of nuclei, atoms, molecules, and macroscopic objects in general.

Assume we are in one such situation. In order to avoid unnecessary complications we shall assume that the global system is divided into only two subsystems. The variables of the particles belonging to system 1 will be represented by $\vec{x}_a$, $\vec{v}_a$ and e_a will represent their charges. We shall take the notation $\vec{x}_A$, $\vec{v}_A$, e_A for the same variables of system 2.

For each subsystem we shall now construct the ten quantities H_i, $\vec{P}_i\vec{J}_i$, $\vec{K}_i$ $(i=1,2)$ as if it were an isolated system. Therefore H_1, $\vec{P}_1$, $\vec{J}_1$, $\vec{K}_1$ are functions of the variables e_a, $\vec{x}_a$, $\vec{v}_a$ only and H_2, $\vec{P}_2$, $\vec{J}_2$, $\vec{K}_2$ are functions of the variables e_A, $\vec{x}_A$, and $\vec{v}_A$ only. When these functions are known, we can define for each system the functions $\vec{X}_i$, $\vec{S}_i$ that we shall identify with the positions of the center of mass of each subsystem, and the intrinsic angular momentum, respectively,

$$\vec{X}_i=\frac{1}{M_i^2}\left[H_i\vec{K}_i-\vec{P}_i\times\vec{J}_i-\frac{(\vec{P}_i\cdot\vec{K}_i)\vec{P}_i}{H_i}\right], \tag{8.1}$$

$$\vec{S}_i=\frac{H_i}{M_i}\vec{J}_i-\frac{\vec{K}_i\times\vec{P}_i}{M_i}-\frac{(\vec{P}_i\cdot\vec{J}_i)\vec{P}_i}{M_i(M_i+H_i)}. \tag{8.2}$$

This identification is justified by their transformation properties and by their limits both in the case in which the two systems are far apart (free subsystem limit) and in the case of small velocities (classical limit).

The quantity

$$M_i=(H_i^2-P_i^2)^{1/2} \tag{8.3}$$

will be identified with the rest mass of the system i and $\vec{P}_i$ with its linear momentum.

The next step is to study the time evolution of these quantities. Let Λ_1 be any of the functions defined for the system 1. The time derivative of this function can be written as

$$\frac{d}{dt}\Lambda_1(\vec{x}_a,\vec{v}_a)=v_a^j\frac{\partial\Lambda_1}{\partial x_a^j}+a_a^j\frac{\partial\Lambda_1}{\partial v_a^j}. \tag{8.4}$$

The acceleration of the particle a can be decomposed in two parts

$$a_a^j = a_a^j(1) + a_a^j(1,2)\,, \tag{8.5}$$

where $a_a^j(1)$ is the acceleration that particle a would have if the second subsystem were absent, and $a_a^j(1,2)$ is defined by (8.5). Therefore (8.4) can be written as

$$\frac{d\Lambda_1}{dt} = v_a^j \frac{\partial \Lambda_1}{\partial x_a^j} + a_a^j(1)\frac{\partial \Lambda_1}{\partial v_a^j} + a_a^j(1,2)\frac{\partial \Lambda_1}{\partial v_a^j}\,. \tag{8.6}$$

We recognize here the vector field $\vec{H}_1$ of subsystem 1;

$$\vec{H}_1 = v_a^j \frac{\partial}{\partial x_a^j} + a_a^j(1)\frac{\partial}{\partial v_a^j}\,.$$

We can therefore write (8.4) in the form

$$\frac{d\Lambda_1}{dt} = \vec{H}_1(\Lambda_1) + \dot{\Lambda}_1\,, \tag{8.7}$$

where we have used the shorthand notation

$$\dot{\Lambda} \equiv a_a^j(1,2)\frac{\partial \Lambda_1}{\partial v_a^j}\,.$$

Since the accelerations are known within the framework of PRM, they are expressed as power series of the coupling constants. Calculations are therefore performed iteratively order by order. We shall now write the equations giving the derivatives of M, S, and X to the first order in the coupling constants. As it can easily be seen, the terms are of first order and therefore they can only be multiplied by zeroth-order terms. Omitting the subindex i which is immaterial here we find

$$\frac{dM}{dt} = \gamma(\dot{H} - \vec{V}\cdot\dot{\vec{P}})\,, \tag{8.8}$$

$$\frac{dS^j}{dt} = \gamma\left\{\dot{H}(\vec{X}\times\vec{V})^j - (\dot{\vec{K}}\times\vec{V})^j + \dot{J}^k\left(\delta_k^j - \frac{\gamma}{\gamma+1}V^jV_k\right) + \dot{P}_k\left[\delta^{jkn}X_n + \frac{1}{\gamma+1}\frac{S^kV^j - (\vec{S}\cdot\vec{V})\delta^{jk}}{M} - \frac{\gamma}{\gamma+1}(X\times V)^kV^j\right]\right\}, \tag{8.9}$$

$$\begin{aligned}\frac{d^2X^i}{dt} = \frac{\gamma}{M}\Bigg(&\ddot{K}_j(\delta^{ij} - V^iV^j) - \ddot{H}\left\{\frac{X^i}{\gamma^2} + \left[\left(\frac{\vec{S}}{M} + \vec{V}\times\vec{X}\right)\times\vec{V}\right]^i\right\}\\ &- \ddot{P}_j\left[\delta^{ijk}\frac{S_k}{M} - X^iX^j + (\vec{V}\cdot\vec{X})\delta^{ij} + \frac{(\vec{V}\times\vec{S})^jV^i}{M} - \frac{\gamma(\vec{V}\cdot\vec{S})\delta^{ijk}V_k}{(\gamma+1)M}\right]\\ &+ (\ddot{\vec{J}}\times V)^i - 2\frac{\dot{H}}{\gamma^2}V^i + \dot{P}_j[V^iV^j + (1-2V^2)\delta^{ij}]\Bigg)\,,\end{aligned} \tag{8.10}$$

where

$$\ddot{\Lambda} = \left(v_a^j\frac{\partial}{\partial x_a^j} + v_A^j\frac{\partial}{\partial x_A^j}\right)\dot{\Lambda}\,,$$

$$\vec{V} = \frac{d\vec{X}}{dt} \quad \text{and} \quad \gamma = (1-V^2)^{-1/2}\,.$$

The functions $\dot{\Lambda}$ and $\ddot{\Lambda}$ must now be written in terms of M_i, $\vec{X}_i$, $\vec{V}_i$, and $\vec{S}_i$ $(i=1,2)$ in order to have a closed system of differential equations. This, of course, cannot be achieved in general. However, if we introduce multipolar developments, and suitable assumptions, we may be able to find a closed system. We shall discuss this possibility in the next section.

IX. MULTIPOLAR DEVELOPMENTS

If we assume that the typical distance between the systems is much greater than the typical dimensions of a subsystem, multipolar developments make sense.

As the origin for the multipolar developments we take the center of mass of each subsystem, and we define the multipolar moments in the center-of-mass reference system.

Let $\vec{r}_a$ be the coordinate of particle a referred to this frame, then the multipolar moments appearing can be classified in quantities of the form

$$\vec{\delta} \equiv \epsilon^a e_a \vec{r}_a\,, \quad \vec{D} \equiv \epsilon^a e_a \vec{r}_a \otimes \vec{r}_a\,, \tag{9.1}$$

etc., called static moments, and quantities of the form

$$\begin{aligned}&\dot{\vec{\delta}} \equiv \epsilon^a e_a \vec{w}_a \quad \text{with } \vec{w}_a = \frac{d}{dt}\vec{r}_a\,,\\ &\vec{M} \equiv \epsilon^a e_a \vec{r}_a \otimes \vec{w}_a\,, \quad \text{etc.,}\end{aligned} \tag{9.2}$$

called kinetic moments.

As our purpose is to apply this formalism to the study of the electromagnetic, gravitational, and short-range scalar interaction among bodies of finite extent, we shall restrict ourselves to the case in which the subsystems can be considered as undeformable. Specifically we shall assume that each subsystem behaves as a rigid body and in order to avoid unnecessary complications we shall also assume spherical symmetry or only small departures from it. Therefore,

$$\vec{\delta} = \epsilon^a e_a \vec{r}_a = 0\,, \quad \frac{d}{dt}\vec{\delta} = \epsilon^a e_a \vec{w}_a = 0\,, \tag{9.3}$$

$$D^{ij} = D\delta^{ij},$$

$$\frac{d}{dt}\vec{D} = \epsilon^a e_a(\vec{r}_a \otimes \vec{w}_a + \vec{w}_a \otimes \vec{r}_a) = 0. \tag{9.4}$$

Hence

$$M^{ij} = \epsilon^a e_a r_a^i w_a^j = \tfrac{1}{2}\epsilon^a e_a(r_a^i w_a^j - r_a^j w_a^i) = \tfrac{1}{2}\delta^{ijk}\mu_k, \tag{9.5}$$

where

$$\vec{\mu} = \epsilon^a e_a \vec{r}_a \times \vec{w}_a.$$

In the case of small departures from spherical symmetry we can take

$$D^{ij} = D\delta^{ij} + \epsilon\gamma^{ij},$$

where ϵ is a small parameter. Then in a perturbation expansion (9.5) can again be used.

If a multipole development is introduced in (8.8), (8.9), and (8.10) under the preceding hypotheses, the functions Λ and $\ddot{\Lambda}$ can be written as functions of $\vec{X}_i$, $\vec{V}_i$, $\vec{S}_i$, $\vec{M}_i$, and $\vec{\mu}_i$. To obtain a closed system we must provide either an equation of motion for $\vec{\mu}_i$ or, as we shall do, a relation between $\vec{S}_i$ and $\vec{\mu}_i$.

If more general assumptions are used, equations giving the evolution of the multipolar moments considered must be provided.

X. INTEGRATION OF THE EQUATIONS OF MOTION

At first order in perturbation theory, under the hypotheses outlined in the preceding section, we can give a general procedure to find the general integral of the systems (8.8)–(8.10) in implicit form.

Assume that a relation between $\vec{S}_i$ and $\vec{\mu}_i$ has been given. The relation we shall use is $\vec{S}_i = \alpha\vec{\mu}_i$. Then the system becomes a closed system where the variables are $\vec{M}_i$, $\vec{X}_i$, $\vec{V}_i$, $\vec{S}_i$. The integration is performed by means of the Hamilton-Jacobi coordinates defined in Sec. II. The ten invariant functions associated with the total system can be written as

$$\vec{H} = \epsilon^a E_a + \epsilon^A E_A,$$

$$\vec{P} = \epsilon^a \vec{p}_a + \epsilon^A \vec{p}_A,$$

$$\vec{J} = \epsilon^a \vec{p}_a \times \vec{p}_a + \epsilon^A \vec{q}_A \times \vec{p}_A,$$

$$\vec{K} = \epsilon^a E_a \vec{q}_a + \epsilon^A E_A \vec{q}_A.$$

We introduce the notations

$$\mathcal{E}_1 = \epsilon^a E_a, \quad \vec{\mathcal{P}}_1 = \epsilon^a \vec{p}_a, \quad \vec{\mathcal{J}}_1 = \epsilon^a \vec{q}_a \times \vec{p}_a, \quad \vec{\mathcal{K}}_1 = \epsilon^a E_a \vec{q}_a,$$

$$\mathcal{E}_2 = \epsilon^A E_A, \quad \vec{\mathcal{P}}_2 = \epsilon^A \vec{p}_A, \quad \vec{\mathcal{J}}_2 = \epsilon^A \vec{q}_A \times \vec{p}_A, \quad \vec{\mathcal{K}}_2 = \epsilon^A E_A \vec{q}_A.$$

These quantities satisfy the equations

$$\frac{d}{dt}\mathcal{E}_i = 0, \quad \frac{d}{dt}\vec{\mathcal{P}}_i = 0, \quad \frac{d}{dt}\vec{\mathcal{J}}_i = 0, \quad \frac{d}{dt}\vec{\mathcal{K}}_i = \vec{\mathcal{P}}_i. \tag{10.1}$$

We now define the functions $\mathcal{L}_i\mathcal{S}_i$ through the equations

$$\vec{\mathcal{J}}_i = \vec{\mathcal{L}}_i \times \vec{\mathcal{P}}_i + \vec{\mathcal{S}}_i, \quad \vec{\mathcal{K}}_i = \mathcal{E}_i\vec{\mathcal{L}}_i + \frac{\vec{\mathcal{P}}_i \times \vec{\mathcal{S}}_i}{\mathfrak{M}_i + \mathcal{E}_i}$$

and it can easily be proved that

$$\frac{d\vec{\mathcal{S}}_i}{dt} = 0, \quad \frac{d\vec{\mathcal{L}}_i}{dt} = \vec{\mathcal{P}}_i. \tag{10.2}$$

To first order in perturbation theory the quantities we have just defined have the general expressions

$$\mathcal{E}_i = H_i + \mathcal{E}_i(1,2), \quad \vec{\mathcal{P}}_i = \vec{P}_i + \vec{\mathcal{P}}_i(1,2),$$

$$\vec{\mathcal{L}}_i = Q_i + \vec{\mathcal{L}}_i(1,2), \quad \vec{\mathcal{S}}_i = \vec{S}_i + \vec{\mathcal{S}}_i(1,2).$$

Performing now a multipolar development in $\mathcal{E}_i(1,2)$, $\vec{\mathcal{P}}_i(1,2)$, $\vec{\mathcal{L}}_i(1,2)$, $\vec{\mathcal{S}}_i(1,2)$ and substituting the resulting expressions in the general integral of (10.1) and (10.2), we get

$$\mathcal{E}_i = \mathcal{E}_i^{(0)}, \quad \vec{\mathcal{P}}_i = \vec{\mathcal{P}}_i^{(0)}, \qquad \vec{\mathcal{S}}_i = \vec{\mathcal{S}}_i^{(0)}, \quad \vec{\mathcal{L}}_i = \vec{\mathcal{L}}_i^{(0)} + t\vec{\mathcal{P}}_i^{(0)}. \tag{10.3}$$

We only have to write H_i, $\vec{P}_i$, $\vec{Q}_i$, $\vec{S}_i$ in terms of $\vec{X}_i$, $\vec{V}_i$, $\vec{S}_i$, M to get from (10.3) the general integral of (8.8)–(8.10).

XI. CONCLUSIONS

The Hamiltonian formulation developed in the first part of this work is valuable in itself, in the sense that it opens the door towards the quantization of predictive relativistic mechanics in its three-dimensional formalism. This possibility is now under investigation. It is our feeling that this formalism will be easier to quantize than the four-dimensional one because the physical interpretation of the quantities is simpler.

The usefulness of the equations we have derived for the motion of bodies of finite size in relativistic mechanics will be better discussed when applied to specific interactions, and we shall postpone the discussion until we have specific examples at hand. This will be the subject of a forthcoming paper.

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[1]B. Bakamjian and L. H. Thomas, Phys. Rev. 92, 1300 (1953).
[2]F. E. Close and H. Osborn, Phys. Rev. D 2, 2127 (1970).
[3]R. A. Krajcik and L. L. Foldy, Phys. Rev. Lett. 24, 545 (1970).
[4]R. A. Krajcik and L. L. Foldy, Phys. Rev. D 10, 1777 (1974).
[5]L. L. Foldy and R. A. Krajcik, Phys. Rev. D 12, 1700 (1975).
[6]L. L. Foldy, Phys. Rev. D 15, 3044 (1977).
[7]M. K. Liou, Phys. Rev. Lett. 21, 1255 (1976).
[8]G. L. Tindle, Phys. Rev. D 14, 2155 (1976).
[9]F. Coester and P. Havas, Phys. Rev. D 14, 2556 (1976).
[10]D. G. Currie, Phys. Rev. 142, 817 (1966).
[11]L. Bel, Ann. Inst. Henri Poincaré 14, 817 (1971).
[12]R. N. Hill, J. Math. Phys. 8, 201 (1967).
[13]D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. 35, 350 (1963).
[14]L. Bel and J. Martin, Ann. Inst. Henri Poincaré 22, 173 (1975).
[15]E. C. G. Sudarshan and N. Mukunda, *Classical Dynamics: A Modern Perspective* (Wiley, New York, 1974).
[16]M. H. L. Pryce, Proc. R. Soc. London A195, 62 (1948).