Electromagnetic waves in a Bianchi type-I universe

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A general formalism is developed to deal with electromagnetic waves in a metric of the form $-ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \Sigma_{i=1}^3 A_i{}^2(t)(dx^i)^2$. The propagation problem is reduced to the integration of a second-order differential equation determining the time evolution of the $F_{\mu\nu}$ tensor. The observable electric and magnetic fields are obtained referring $F_{\mu\nu}$ to a suitable orthonormal tetrad and are shown to consist of a superposition of spectral components, each labeled by a set of parameters k_i and transverse to a time-varying direction specified by $k_i/A_i(t)$. The ratio between energy and spin angular momentum for a single spectral component is found to be $(k_ik^i)^{1/2}$, which in the WKB limit is the angular frequency of the waves. The high-frequency solutions for the fields are used to discuss the possible effects of anisotropy in the expansion of the universe after decoupling on polarization and intensity distributions of the microwave background. Exact solutions are given for waves propagating along the coordinate axes of the general Kasner spacetimes and along any direction in the flat Kasner spacetime. Propagation out of the singularity is seen to alter considerably amplitude and phase relationships between the fields. Waveforms traveling along the coordinate axes without a backward tail are constructed.

INTRODUCTION

Electromagnetic wave propagation in gravitational fields has been studied for several reasons. Many astrophysical situations (pulsars, quasars, collapsing stars) involve strong electromagnetic and gravitational fields in interaction. The interpretation of the observed characteristics of the microwave background requires an understanding of the effect of a cosmological metric on electromagnetic waves. The characteristics of the electromagnetic radiation scattered by a black hole could furnish an indirect way of detecting the presence of these collapsed objects. The predictions of geometrical optics in curved spacetime led to some famous tests of Einstein's theory of gravitation. An understanding of the electromagnetic-gravitational coupling beyond the geometrical optics limit could lead to further tests, while providing additional insight into the theory.

The electromagnetic-gravitational interaction is described by the Einstein-Maxwell equations, which take into account the effect of the electromagnetic stress-energy tensor on the geometry of spacetime. If this effect is disregarded, one is led to a useful approximation, in which the electromagnetic field is considered to propagate in a given background metric without affecting it. The problem is then reduced to solving Maxwell's equations in a given curved spacetime.

This approach is followed in the present work. The background metric is assumed to be of the form $-ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \sum_{i=1}^3 A_i{}^2(t)(dx^i)^2$, where the A_i are given non-negative functions of time. These models describe anisotropically expanding universes which are homogeneous, spatially flat, and admit no rotational matter. They are usually referred to as the diagonal Bianchi type-I models. Several exact solutions of Einstein's equations of this form have been found.[1,2] They have the interesting property that the expansion rates generally become isotropic at large times, and are therefore compatible with the present-day high degree of large-scale isotropy of the Hubble expansion.

Previous studies of electromagnetic wave propagation in these spaces have been carried out in the framework of geometrical optics.[3-6] They were motivated by the interest in the possible effects of anisotropy in the expansion rates of the universe on the polarization characteristics of the microwave background.

This paper consists of three sections and two appendices. In Sec. I a general formalism is developed to deal with the electromagnetic field in a diagonal Bianchi type-I metric. The observable electric and magnetic fields are shown to consist in general of a superposition of spectral components, each labeled by a set of parameters k_i and transverse to the time-varying direction $k_i/A_i(t)$. Expressions for the total energy, total momentum, and total spin angular momentum of the electromagnetic field are obtained. A very simple relationship is found to hold for the ratio between magnitude of spin angular momentum and energy of every spectral component, namely, it equals the reciprocal of a quantity $k\mu$, which in the high-frequency limit is the angular frequency of the waves.

In Sec. II the high-frequency solutions for the fields are given. In agreement with Caderni *et al.*,[5] it is shown that the polarization vector of a given wave undergoes parallel transport on top of a two-dimensional sphere of constant curvature as the direction of propagation changes due to anisotropy in the expansion. The high-frequency fields are then used to relate observations of intensity and polarization distributions made at two different times by means of a receiver comoving with the background metric. The implications of these results for the microwave background are discussed. A simple estimate of the errors thus introduced is obtained in Appendix B, on the basis of the higher-order corrections to the fields given in Appendix A.

In Sec. III the Kasner spacetimes,[7] for which $A_i(t)=t^{p_i}$ with $\sum_{i=1}^3 p_i=\sum_{i=1}^3 p_i^2=1$ are considered. They are exact solutions of Einstein's equations with no sources. Electromagnetic waves in Kasner spacetimes have been the subject of a previous work by Goorjian,[8] who obtained the asymptotic behavior of the stress-energy tensor near the singularity for waves propagating along the coordinate axes. In the present work exact solutions are obtained for waves propagating in any direction in the flat Kasner model, for which $p_1=p_2=0$ and $p_3=1$, and for waves propagating along the coordinate axes of the general Kasner models. It is shown that propagation out of the time singularity alters in a significant way amplitude and phase relationships between the field components, thus rendering an early polarization state of the field inaccessible to an observer at late times. It is shown that suitable relations between the initial distributions of the field components lead to waveforms traveling in a definite direction along the coordinate axes of the Kasner models without a backward tail.

I. GENERAL FORMALISM

In the absence of sources, electromagnetic waves propagating in a gravitational field are described by the covariant Maxwell equations

$$F^{\alpha\beta}{}_{;\beta}=0 \tag{1.1}$$

and

$$(*F)^{\alpha\beta}{}_{;\beta}=0\,. \tag{1.2}$$

We use the following conventions: $g_{\alpha\beta}$ has signature +2, $g=\det(g_{\alpha\beta})$, the Minkowskian metric $\eta_{\alpha\beta}=\mathrm{diag}(-1,1,1,1)$, $\epsilon^{0123}=1$, and $\epsilon_{123}=1$. Semicolons denote covariant derivatives, Greek indices run from zero to three, Latin indices run from one to three, and repeated indices are summed over.

In Eq. (1.1), $F^{\alpha\beta}$ denotes the electromagnetic field tensor, and Eq. (1.2) is written in terms of the dual-field tensor

$$(*F)^{\alpha\beta}=\frac{1}{2\sqrt{-g}}\epsilon^{\alpha\beta\gamma\eta}F_{\gamma\eta}\,, \tag{1.3}$$

in order to make the duality of Maxwell's equations explicit.

We restrict our attention to homogeneous cosmological models described by diagonal metrics of the form $[A_i(t)\geq 0]$

$$-ds^2=-dt^2+\sum_{j=1}^{3}A_j^2(t)(dx^j)^2\,, \tag{1.4}$$

which are a particular case of the Bianchi type-I spaces. These nonstationary and spatially flat metrics can be used to describe a universe undergoing anisotropic expansion.

The translational invariance of the metric given in Eq. (1.4) makes it convenient to introduce the spatial Fourier transforms of the components of the electromagnetic field tensor, defined by

$$F^{\alpha\beta}(t,x^i)=\int\frac{d^3k}{(2\pi)^3}e^{ik_jx^j}f^{\alpha\beta}(t,k_i)$$

and (1.5)

$$(*F)(t,x^i)=\int\frac{d^3k}{(2\pi)^3}e^{ik_jx^j}(*f)^{\alpha\beta}(t,k_i)\,,$$

where $d^3k=dk_1dk_2dk_3$. Equations (1.1) and (1.2) then reduce to

$$k_jf^{0j}=0\,, \tag{1.6a}$$

$$k_j(*f)^{j0}=0\,, \tag{1.6b}$$

$$\frac{d}{dt}(\sqrt{-g}f^{0j})=-i\epsilon_{jml}k_lA_m^2(t)(*f)^{m0}\,, \tag{1.6c}$$

$$\frac{d}{dt}[\sqrt{-g}(*f)^{j0}]=i\epsilon_{jml}k_lA_m^2(t)f^{0m}\,. \tag{1.6d}$$

The choice of f^{0j} and $(*f)^{j0}$ as field variables serves the purpose of stressing the analogy between Eqs. (1.6) and their flat-space counterparts. In flat space f^{0j} and $(*f)^{j0}$ are just the spatial Fourier transforms of the Cartesian components of the $\vec{E}$ and $\vec{B}$ fields, respectively.

The evident symmetry of Eqs. (1.6) allows considerable simplification if the quantities

$$S_j^{\pm}=\sqrt{-g}[f^{0j}\pm i(*f)^{j0}] \tag{1.7}$$

are introduced. In flat space $S_j^{\pm}$ are just field amplitudes for circularly polarized waves. Equations (1.6a)–(1.6d) then decouple into two almost identical sets involving either S_j^+ or S_j^-.

Further simplification is achieved by introducing the variables

$$S_\delta^{\pm}=\cos\delta\cos\xi S_1^{\pm}+\cos\delta\sin\xi S_2^{\pm}-\sin\delta S_3^{\pm}$$

and (1.8)

$$S_\xi^{\pm}=-\sin\xi S_1^{\pm}+\cos\xi S_2^{\pm}\,,$$

where the angles δ and ξ are defined by

$$(k_1, k_2, k_3) = k(\sin\delta\cos\xi,\ \sin\delta\sin\xi,\ \cos\delta). \quad (1.9)$$

Equations (1.6a) and (1.6b) are then automatically satisfied and Eqs. (1.6c) and (1.6d) yield

$$\pm\frac{dS_\delta^\pm}{dt} = -kaS_\delta^\pm - kbS_\xi^\pm$$

and (1.10)

$$\pm\frac{dS_\xi^\pm}{dt} = kcS_\delta^\pm + kaS_\xi^\pm,$$

where

$$a = \frac{\cos\delta\cos\xi\sin\xi}{\sqrt{-g}}[A_2{}^2(t) - A_1{}^2(t)],$$

$$b = \frac{1}{\sqrt{-g}}[A_2{}^2(t)\cos^2\xi + A_1{}^2(t)\sin^2\xi],$$

and (1.11)

$$c = \frac{1}{\sqrt{-g}}[A_1{}^2(t)\cos^2\delta\cos^2\xi + A_2{}^2(t)\cos^2\delta\sin^2\xi + A_3{}^2(t)\sin^2\delta].$$

In order to proceed further in the analysis, a digression about basis vectors is necessary. As emphasized by Mo,[9] a metric $g_{\mu\nu}$ defines natural covariant unit vectors $\vec{e}_\mu$, with contravariant components $e_\mu^\alpha = \delta_\mu^\alpha$, and natural contravariant unit vectors $\vec{e}^\mu$, with covariant components $e_\alpha^\mu = \delta_\alpha^\mu$, such that, for example, $\vec{e}_\mu\cdot\vec{e}_\nu = g_{\mu\nu}$. Every tensor results from the contraction of contravariant indices with covariant basis vectors and of covariant indices with contravariant basis vectors. All this is in close analogy with tensors in flat space. The difference lies in the condition $\vec{e}_\mu\cdot\vec{e}_\nu = g_{\mu\nu}$ as compared with $\vec{e}_\mu\cdot\vec{e}_\nu = \eta_{\mu\nu}$, valid in flat space referred to Cartesian coordinates.

The absence of off-diagonal terms in the metric given in Eq. (1.4) makes the natural basis vectors orthogonal, but the lengths of the spatial basis vectors $\vec{e}_i$ vary in time according to $\vec{e}_i\cdot\vec{e}_i = A_i{}^2(t)$. Therefore, owing to the time variation of the basis vectors, tensor components do not carry full information about the fields. This difficulty is easily removed in our case by introducing the tetrad basis vectors $\vec{e}_{(0)} = \vec{e}_0$ and $\vec{e}_{(i)} = \vec{e}_i/A_i(t)$, such that $\vec{e}_{(\mu)}\cdot\vec{e}_{(\nu)} = \eta_{\mu\nu}$, and by expressing tensors in terms of their components with respect to the $\vec{e}_{(\mu)}$ basis. In what follows tetrad indices will be denoted by enclosing them within parentheses. The raising and lowering of tetrad indices is done with the matrices $\eta^{\alpha\beta}$ and $\eta_{\alpha\beta}$, respectively, and the tetrad components of a tensor are defined as its projections into the vectors that form the tetrad. For example $\vec{T} = T^{\alpha\beta}\vec{e}_\alpha\vec{e}_\beta = T^{(\alpha)(\beta)}\vec{e}_{(\alpha)}\vec{e}_{(\beta)}$, where $T^{(\alpha)(\beta)} = e_\mu^{(\alpha)}e_\nu^{(\beta)}T^{\mu\nu}$. In general this procedure must be applied locally, as it involves the components of the $g_{\mu\nu}$ tensor, which can depend on the four spacetime coordinates. In our case, however, the independence of the metric coefficients given in Eq. (1.4) on x^i makes it possible to obtain tetrad components with the use of the same orthonormal tetrad throughout space at any instant of time.

One is therefore led to identify the Cartesian components $E^{(i)}$ of the electric field with $F^{(0)(i)}$ and the Cartesian components $B^{(i)}$ of the magnetic field with $(*F)^{(i)(0)}$. With the help of Eqs. (1.5) one then gets

$$\vec{E} = \int\frac{d^3k}{(2\pi)^3}e^{ik_jx^j}\sum_{l=1}^{3}A_l(t)f^{0l}\vec{e}_{(l)}$$

and (1.12)

$$\vec{B} = \int\frac{d^3k}{(2\pi)^3}e^{ik_jx^j}\sum_{l=1}^{3}A_l(t)(*f)^{l0}\vec{e}_{(l)}.$$

Because of Eqs. (1.6a) and (1.6b), the spectral components of $\vec{E}$ and $\vec{B}$ corresponding to k_1, k_2, and k_3 are orthogonal to the time-varying direction specified by $k_{(i)} = k_i/A_i(t)$. It is therefore convenient to introduce the time-varying unit vectors

$$\vec{e}_r = \sin\theta\cos\phi\,\vec{e}_{(1)} + \sin\theta\sin\phi\,\vec{e}_{(2)} + \cos\theta\,\vec{e}_{(3)},$$

$$\vec{e}_\theta = \cos\theta\cos\phi\,\vec{e}_{(1)} + \cos\theta\sin\phi\,\vec{e}_{(2)} - \sin\theta\,\vec{e}_{(3)},$$

and (1.13)

$$\vec{e}_\phi = -\sin\phi\,\vec{e}_{(1)} + \cos\phi\,\vec{e}_{(2)},$$

where the angles θ and ϕ are defined by

$$(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) = \mu^{-1}\left(\frac{\sin\delta\cos\xi}{A_1}, \frac{\sin\delta\sin\xi}{A_2}, \frac{\cos\delta}{A_3}\right) \quad (1.14)$$

and

$$\mu = \left(\frac{\sin^2\delta\cos^2\xi}{A_1{}^2} + \frac{\sin^2\delta\sin^2\xi}{A_2{}^2} + \frac{\cos^2\delta}{A_3{}^2}\right)^{1/2}. \quad (1.15)$$

In this way $\vec{e}_r$ is always along the time-varying direction specified by $k_{(i)}$. One can now write the fields $\vec{E}$ and $\vec{B}$ of Eq. (1.12) using the $(\vec{e}_\theta, \vec{e}_\phi)$ basis. The manipulations proceed through the use of Eqs. (1.7), (1.8), (1.10), (1.11), (1.14), and (1.15). The result is remarkably simple, namely,

$$\vec{E} = \frac{1}{2(-g)^{1/4}}\int\frac{d^3k}{(2\pi)^3}\frac{e^{ik_jx^j}}{b^{1/2}}\left[\mu(S_\delta^+ + S_\delta^-)\vec{e}_\theta - \frac{1}{k}\frac{d}{dt}(S_\delta^+ - S_\delta^-)\vec{e}_\phi\right] \quad (1.16)$$

and

$$\vec{B}=\frac{1}{2(-g)^{1/4}}\int\frac{d^3k}{(2\pi)^3}\frac{e^{ik_jx^j}}{b^{1/2}}\left[\mu(S_6^+-S_6^-)\vec{e}_\theta -\frac{1}{k}\frac{d}{dt}(S_6^++S_6^-)\vec{e}_\phi\right].$$

Reality of the fields then demands that

$$S_6^+(-k_i)=S_6^{-*}(k_i)\,. \tag{1.17}$$

Moreover, as $S_\xi^\pm$ does not appear in Eqs. (1.16), it is convenient to derive from Eqs. (1.10) second-order equations for $S_6^\pm$:

$$\frac{d^2S_6^\pm}{dt^2}-\frac{1}{b}\frac{db}{dt}\frac{dS_6^\pm}{dt}+\left[k^2\mu^2\pm kb\frac{d}{dt}\left(\frac{a}{b}\right)\right]S_6^\pm=0\,. \tag{1.18}$$

The general propagation problem has thus been reduced to the integration of Eq. (1.18) for S_6^+ and S_6^-, subject to the condition (1.17). The fields appear as superpositions of spectral components, each transverse to a time-varying direction (θ,ϕ). The two quantities S_6^+ and S_6^- satisfy different equations, unless one or more of the k_i's is equal to zero or $A_1(t)=A_2(t)$. If any two scale factors are equal, relabeling of the coordinate axes allows one to use two variables $S_6'^\pm$ that satisfy the same differential equation.

The time evolution of the direction (θ,ϕ) is determined by Eqs. (1.14) and (1.15). A wave propagating along a coordinate axis will keep propagating along that coordinate axis. A wave propagating in a coordinate plane will keep propagating in that coordinate plane, but will tilt toward the axis with smaller Hubble constant $H_i=dA_i/A_idt$. A wave propagating outside the coordinate planes will tilt toward the one coordinate axis whose Hubble constant is the smallest.

The tetrad components of the stress-energy tensor for the electromagnetic field

$$T^{(\alpha)(\beta)}=\frac{1}{4\pi}\left[F^{(\alpha)}{}_{(\gamma)}F^{(\gamma)(\beta)}-\tfrac{1}{4}F_{(\gamma)(\sigma)}F^{(\gamma)(\sigma)}\eta^{\alpha\beta}\right] \tag{1.19}$$

can be simply expressed in terms of the quantities $\mathcal{G}^{(j)}=E^{(j)}+iB^{(j)}$ and their complex conjugates. One finds

$$T^{(0)(0)}=\frac{1}{8\pi}\mathcal{G}^{(j)}\mathcal{G}^{(j)*}\,, \tag{1.20}$$

$$T^{(0)(j)}=\frac{1}{8\pi i}\epsilon_{jkl}\mathcal{G}^{(k)*}\mathcal{G}^{(l)}\,, \tag{1.21}$$

and

$$T^{(j)(k)}=\frac{1}{8\pi}\left[-(\mathcal{G}^{(j)}\mathcal{G}^{(k)*}+\mathcal{G}^{(j)*}\mathcal{G}^{(k)})+\mathcal{G}^{(l)}\mathcal{G}^{(l)*}\delta^{jk}\right]. \tag{1.22}$$

At any instant of time, the total energy of the electromagnetic field is obtained integrating $\sqrt{-g}T^{00}$ throughout space:

$$E=\int\sqrt{-g}T^{00}d^3x\,, \tag{1.23}$$

where $d^3x=dx^1dx^2dx^3$. It correctly behaves as a scalar under spatial rotations. Use of Eqs. (1.16) and (1.20) yields

$$E=\frac{1}{16\pi}\int\frac{d^3k}{(2\pi)^3}\left[\frac{\mu^2}{b}(S_6^+S_6^{+*}+S_6^-S_6^{-*})+\frac{1}{k^2b}\left(\frac{dS_6^+}{dt}\frac{dS_6^{+*}}{dt}+\frac{dS_6^-}{dt}\frac{dS_6^{-*}}{dt}\right)\right]. \tag{1.24}$$

The total energy of the electromagnetic field is not constant in time owing to the exchange of energy between the electromagnetic and the gravitational fields.

At any instant of time, the total momentum of the electromagnetic field is obtained integrating $\sqrt{-g}T^{0i}$ throughout space:

$$P^i=\int T^{0i}\sqrt{-g}d^3x\,. \tag{1.25}$$

Even though P^i is not in general a tensor, it behaves as a vector under spatial rotations. Use of the orthonormal tetrad leads to

$$\vec{P}=\int T^{(0)(i)}\vec{e}_{(i)}\sqrt{-g}d^3x\,. \tag{1.26}$$

Equations (1.16) and (1.21) then yield

$$\vec{P}=\frac{1}{4\pi}\int\frac{d^3k}{(2\pi)^3}\alpha(k_i)\sum_{l=1}^{3}\frac{k_l}{A_l(t)}\vec{e}_{(l)}\,, \tag{1.27}$$

where

$$\alpha(k_i)=\frac{1}{k^2b}\mathrm{Im}\left(S_6^+\frac{dS_6^{+*}}{dt}\right) \tag{1.28}$$

is constant in time on account of Eq. (1.18). The result expressed by Eq. (1.27) is remarkably simple, namely, $P^{(i)}\propto 1/A_i$. Therefore, the total momentum of the electromagnetic field in general tends to align itself with the coordinate axis with smallest Hubble constant.

At any instant of time, one can identify the total angular momentum of the electromagnetic field with

$$L_j=\tfrac{1}{2}\sqrt{-g}\epsilon_{jlm}J^{lm}\,, \tag{1.29}$$

where

$$J^{lm}=\int(x^lT^{m0}-x^mT^{l0})\sqrt{-g}d^3x\,. \tag{1.30}$$

J^{lm} defines a tensor under spatial rotations and L_j is the associated pseudovector angular momentum. Use of the orthonomal tetrad leads to

$$\vec{L}=L^{(j)}\vec{e}_{(j)}=\frac{1}{4\pi}\int x^{(k)}(E^{(j)}B^{(k)}-E^{(k)}B^{(j)})\vec{e}_{(j)}\sqrt{-g}d^3x\,. \tag{1.31}$$

The integral in Eq. (1.31) can be evaluated by means of Eqs. (1.16). In analogy with the usual procedure for classical electromagnetic waves in flat space, one obtains the total spin angular momentum $\vec{\sigma}$ of the electromagnetic field by retaining for each wave of the spectrum the component of the angular momentum parallel to the direction of propagation $\vec{e}_r$. After some algebra one gets

$$\vec{\sigma}=\frac{1}{16\pi}\int\frac{d^3k}{(2\pi)^3}\vec{e}_r\left[\frac{\mu^2}{b}(S_\delta^+S_\delta^{+*}-S_\delta^-S_\delta^{-*})+\frac{1}{k^2b}\left(\frac{dS_\delta^+}{dt}\frac{dS_\delta^{+*}}{dt}-\frac{dS_\delta^-}{dt}\frac{dS_\delta^{-*}}{dt}\right)\right]. \tag{1.32}$$

According to Eq. (1.32), S_δ^+ and S_δ^- correspond to waves giving contributions of opposite sign to the spin angular momentum. In particular, if one restricts his attention to a single spectral component, comparison with Eq. (1.24) shows that

$$\frac{\vec{\sigma}}{E}=\pm\frac{1}{k\mu}\vec{e}_r, \tag{1.33}$$

where the "plus" sign applies to waves generated by S_δ^+ and the "minus" sign applies to waves generated by S_δ^-. This result is in complete analogy with the corresponding one for circularly polarized waves in flat space: $k\mu$ is the time component of the wave vector k_α as obtained for example by solving the eikonal equation for the metric given in Eq. (1.4). In flat space $k\mu$ is the frequency of the circularly polarized waves generated by S_δ^+ and S_δ^-. In what follows waves generated by S_δ^+ will be referred to as + waves and waves generated by S_δ^- will be referred to as − waves.

In order to proceed further with the physical interpretation of the waves, it is of interest to consider in some detail a single spectral component of the fields. Denoting by $\vec{E}^+$ and $\vec{B}^+$ the fields generated by S_δ^+ and by $\vec{E}^-$ and $\vec{B}^-$ the fields generated by S_δ^-, it follows from Eqs. (1.16) that one can write

$$E_\theta^\pm \pm iB_\theta^\pm = e^{ik_jx^j}\frac{\mu}{b^{1/2}(-g)^{1/4}}S_\delta^\pm$$

and (1.34)

$$E_\phi^\pm \pm iB_\phi^\pm = \mp e^{ik_jx^j}\frac{1}{kb^{1/2}(-g)^{1/4}}\frac{dS_\delta^\pm}{dt},$$

where it is understood that the $\vec{E}$ and $\vec{B}$ fields are obtained taking the real and imaginary parts of the right-hand sides.

In analogy with flat space, it is convenient to write the general solution of Eq. (1.18) as

$$S_\delta^\pm = C_1^\pm y^\pm + C_2^\pm y^{\pm *}. \tag{1.35}$$

The two complex-conjugate solutions in Eq. (1.35) are the analogs of the exponentials $e^{-i\omega t}$ and $e^{i\omega t}$ of flat space. The analogy is indeed supported by further results. Equations (1.34), with the help of the normalization condition

$$\mathrm{Im}\left(y^\pm\frac{dy^{\pm *}}{dt}\right)=kb \tag{1.36}$$

for $y^\pm$, yield

$$T^{(0)(i)}=\frac{1}{4\pi k}\frac{1}{\sqrt{-g}}(|C_1^\pm|^2-|C_2^\pm|^2)k^{(i)}. \tag{1.37}$$

The two complex-conjugate solutions in Eq. (1.35) are therefore amplitudes for waves carrying power in the direction of $k^{(i)}$ and in the direction of $-k^{(i)}$, respectively, without interfering with each other. This result is just what would be obtained for two circularly polarized waves propagating in opposite directions in flat space. The power flux through an element of area transverse to $k^{(i)}$ and marked by comoving objects (objects with constant x^i) can also be obtained from Eq. (1.37). The result,

$$P^\pm \propto \mu^2(|C_1^\pm|^2-|C_2^\pm|^2), \tag{1.38}$$

can be readily given an interpretation in terms of the "frequency" $k\mu$ of the radiation and of the effective scale factor in the $\vec{e}_r$ direction, proportional to $1/\mu$. Furthermore, if one denotes by $\psi^\pm$ the phase of $y^\pm$, one can show that

$$\frac{d\psi^\pm}{dt}=-\frac{kb}{|y^\pm|^2}, \tag{1.39}$$

where use has been made of Eq. (1.36). As b is positive, the phases $\psi^\pm$ are monotonically decreasing functions of time, just as $-\omega t$ is.

Equations (1.33), (1.37), and (1.39) provide the physical interpretation for S_δ^+ and S_δ^-. $S_\delta^+(k_i)$ generates a wave with positive helicity carrying power along $k_{(i)}$ and a wave with negative helicity carrying power along $-k_{(i)}$. $S_\delta^-(k_i)$ generates a wave with negative helicity carrying power along $k_{(i)}$ and a wave with positive helicity carrying power along $-k_{(i)}$.

The phase delay $\Delta^\pm$ between the θ and ϕ components of either the electric or the magnetic fields of waves generated by y^+ or y^- ($C_2^\pm=0$) can be evaluated from Eqs. (1.34) and (1.36). One finds

$$\tan\Delta^\pm=-\frac{2kb}{(d/dt)|y^\pm|^2}. \tag{1.40}$$

According to Eq. (1.40), the phase delay is in general not constant in time and differs from $\pm\pi/2$, which is the phase delay for circularly polarized waves in flat space. Moreover, the phase delay depends on the spin angular momentum, being different in general for waves generated by y^+ or y^-. The deviation of the phase delay from $\pm\pi/2$ is due to the anisotropy in the expansion rates, and is absent in an isotropically expanding universe $[A_i(t)=R(t)]$. In this case Eqs. (1.18) reduce to

$$\frac{d^2S_6^\pm}{dt^2}+\frac{1}{R}\frac{dR}{dt}\frac{dS_6^\pm}{dt}+\frac{k^2}{R^2}S_6^\pm=0\,, \tag{1.41}$$

and are exactly solved by

$$S_6^\pm=C_1^\pm\exp\left(-ik\int^t\frac{d\tau}{R(\tau)}\right)+C_2^\pm\exp\left(ik\int^t\frac{d\tau}{R(\tau)}\right). \tag{1.42}$$

This simple result is just what one expects on the basis of the conformal invariance of Maxwell's equations. It also holds for an axisymmetric universe $(A_1=A_2)$ for waves propagating along the symmetry axis, provided $R(\tau)$ is replaced by $A_3(\tau)$ in Eq. (1.42).

II. HIGH-FREQUENCY WAVES. APPLICATION TO THE MICROWAVE BACKGROUND

As remarked in Sec. I, S_6^+ and S_6^- satisfy in general different equations. The WKB solutions of Eqs. (1.18) provide direct insight into this point. Moreover, they are of practical interest, as they give an accurate description of the free propagation of the microwave background after decoupling, owing to the high frequency of the radiation and to the long time scale of the cosmic evolution.

If the new variable $r=\int b\,dt$ is introduced and Eqs. (1.18) are rewritten in the more convenient form

$$\frac{d^2S_6^\pm}{dr^2}+\left[\left(\frac{k\mu}{b}\right)^2\pm k\frac{d}{dr}\left(\frac{a}{b}\right)\right]S_6^\pm=0\,, \tag{2.1}$$

the general solution in the WKB approximation is found to be

$$S_6^\pm=C_1^\pm\left(\frac{b}{\mu}\right)^{1/2}\exp\left(-ik\int^t\mu\,dt\mp i\lambda\right)+C_2^\pm\left(\frac{b}{\mu}\right)^{1/2}\exp\left(ik\int^t\mu\,dt\pm i\lambda\right), \tag{2.2}$$

where

$$\lambda=\int^t d\tau\frac{b}{2\mu}\frac{d}{d\tau}\left(\frac{a}{b}\right)=\cos\delta\cos\xi\sin\xi\int^t d\tau\left(\frac{1}{A_2}\frac{dA_2}{d\tau}-\frac{1}{A_1}\frac{dA_1}{d\tau}\right)\Big/\left(A_3{}^2b\mu\right). \tag{2.3}$$

It is possible to find a simple expression for the time derivative of λ. Making use of Eqs. (2.3) and (1.14) one can prove that

$$\frac{d\lambda}{dt}=-\cos\theta\frac{d\phi}{dt}. \tag{2.4}$$

λ is therefore a purely geometrical parameter, independent of k, whose evolution is determined by the time rate of change of the azimuthal angle ϕ of the direction of propagation of a given wave.

The expressions for the fields, obtained from Eq. (1.34), are

$$E_\theta^\pm\pm iB_\theta^\pm=\frac{\mu^{1/2}}{(-g)^{1/4}}[C_1^\pm\exp(i\eta\mp i\lambda)+C_2^\pm\exp(i\tau\pm i\lambda)]$$

and

$$E_\phi^\pm\pm iB_\phi^\pm=\pm\frac{i\mu^{1/2}}{(-g)^{1/4}}[C_1^\pm\exp(i\eta\mp i\lambda)-C_2^\pm\exp(i\tau\pm i\lambda)]\,, \tag{2.5}$$

where

$$\eta=k_jx^j-k\int^t\mu\,dt$$

and

$$\tau=k_jx^j+k\int^t\mu\,dt\,. \tag{2.6}$$

It is seen from Eq. (2.5) and (2.6) that in the WKB limit S_6^+ generates a left circularly polarized wave propagating along $k_{(i)}$ and a right circularly polarized wave propagating along $-k_{(i)}$. The field vectors of both waves, as seen from the $\vec{e}_r$ direction, undergo a counterclockwise rotation with respect to the $(\vec{e}_\theta,\vec{e}_\phi)$ basis with angular velocity $k\mu+d\lambda/dt$. In the same way, S_6^- generates a right circularly polarized wave propagating along $k_{(i)}$ and a left circularly polarized wave propagating along $-k_{(i)}$. The field vectors of both waves, as seen from the $\vec{e}_r$ direction, undergo a clockwise rotation with respect to the $(\vec{e}_\theta,\vec{e}_\phi)$ basis with angular velocity $k\mu-d\lambda/dt$.

As time goes by, the direction of propagation changes, and the unit vectors $\vec{e}_\theta$ and $\vec{e}_\phi$, with respect to which the previous angular velocities were measured, change as well. They rotate clockwise, as seen from the $\vec{e}_r$ direction, with an angular velocity $d\lambda/dt$ with respect to the local fixed $(\vec{e}_\theta,\vec{e}_\phi)$ frame. It follows that, with respect to the local fixed $(\vec{e}_\theta,\vec{e}_\phi)$ frame, the field vectors of a + wave rotate counterclockwise and the field vectors of a − wave rotate clockwise with the same angular velocity $\omega=k\mu$, as seen from the $\vec{e}_r$ direction. This is a sensible result in the light of Eq. (1.33). It also follows from the previous discussion that the superposition of two waves of equal amplitudes, phases and directions of propagation, and characterized by opposite states of circular polarization yields a wave with electric and magnetic field vectors that do not rotate with respect to the local fixed $(\vec{e}_\theta, \vec{e}_\phi)$ frame. Consequently, as time goes by, the directions of the field vectors are parallel transported on top of the celestial sphere. More quantitatively, Eqs. (2.5) for $E_\theta^\pm$ and $E_\phi^\pm$, where for definiteness $C_2^\pm$ is taken to be zero, lead to introduce the two quantities $m_\theta^\pm = C_1^\pm \exp(\mp i\lambda)$ and $m_\phi^\pm = \pm i C_1^\pm \exp(\mp i\lambda)$, which can be identified with the θ and ϕ components of a constant length polarization vector. Equation (2.4) then implies that $m_\theta^\pm$ and $m_\phi^\pm$ satisfy the parallel transport equations in a tetrad frame on top of a sphere, as previously recognized by Caderni *et al.*[5] on the basis of a perturbative treatment of the parallel transport equations of geometrical optics.

The fields given in Eqs. (2.5) can now be used to relate observations of intensity and polarization distributions made at two different times by means of a receiver comoving with the background metric given in Eq. (1.4). A simple estimate of the errors introduced by using the WKB fields (2.5) is given in Appendix B. It is shown there that the evolution laws for intensity and polarization to be derived in this section apply with a high degree of accuracy to the free propagation of the microwave background after decoupling.

Consider first the problem of evolving an intensity distribution. To this end the receiver can be thought of as recording the time-averaged power carried by all the waves approaching it within a narrow cone $d\Omega$ about a direction (θ, ϕ) and within a narrow frequency interval $d\omega$ about a frequency ω. The averaging is done over a time interval long compared to the mean period of the waves, but short compared to the time scale of evolution of the background metric.

The most general wave propagating along $-k_{(i)}$ is a superposition of a + wave and a − wave [$C_1^\pm = 0$ in Eqs. (2.5)]. According to Eq. (1.21), it carries the time-averaged power

$$\langle (T^{(0)(i)} T^{(0)(i)})^{1/2} \rangle = \frac{\mu}{4\pi\sqrt{-g}} (|C_2^+|^2 + |C_2^-|^2). \tag{2.7}$$

The power received by the instrument per unit solid angle and per unit frequency is then obtained multiplying the expression given in Eq. (2.7) by the Jacobian of the transformation from $(\cos\delta, \xi, k)$ to $(\cos\theta, \phi, \omega)$. The result is

$$\frac{dP}{d\Omega\, d\omega} = \frac{1}{4\pi} \frac{|C_2^+|^2 + |C_2^-|^2}{[A_1^2 \sin^2\theta \cos^2\phi + A_2^2 \sin^2\theta \sin^2\phi + A_3^2 \cos^2\theta]^{3/2}}, \tag{2.8}$$

where use has been made of

$$\begin{aligned} \omega &= k\mu \\ &= k(A_1^2 \sin^2\theta \cos^2\phi + A_2^2 \sin^2\theta \sin^2\phi + A_3^2 \cos^2\theta)^{-1/2}, \end{aligned} \tag{2.9}$$

which follows from Eqs. (1.14) and (1.15).

Intensity distributions at two different times can be related by writing Eq. (2.8) for t and t', and eliminating the factors $(|C_2^+|^2 + |C_2^-|^2)$ from the two by requiring that (θ, ϕ, ω) and $(\theta', \phi', \omega')$ be observation angles and angular frequencies corresponding to equal values of (δ, ξ, k) at t and t', respectively. The relations between (θ, ϕ, ω) and $(\theta', \phi', \omega')$, obtained using Eq. (1.14) for both t and t' and Eq. (2.9), are

$$\begin{aligned} \cos\theta' &= v_3 \cos\theta (v_1^2 \sin^2\theta \cos^2\phi + v_2^2 \sin^2\theta \sin^2\phi + v_3^2 \cos^2\theta)^{-1/2}, \\ \cos\phi' &= v_1 \cos\phi (v_1^2 \cos^2\phi + v_2^2 \sin^2\phi)^{-1/2}, \\ \sin\phi' &= v_2 \sin\phi (v_1^2 \cos^2\phi + v_2^2 \sin^2\phi)^{-1/2}, \end{aligned} \tag{2.10}$$

and

$$\omega' = \omega (v_1^2 \sin^2\theta \cos^2\phi + v_2^2 \sin^2\theta \sin^2\phi + v_3^2 \cos^2\theta)^{1/2}, \tag{2.11}$$

where $v_i = A_i(t)/A_i(t')$. One finally finds

$$\frac{dP_t}{d\Omega\, d\omega} = \left[\!\left[\frac{dP_{t'}}{d\Omega\, d\omega} \right]\!\right] (v_1^2 \sin^2\theta \cos^2\phi + v_2^2 \sin^2\theta \sin^2\phi + v_3^2 \cos^2\theta)^{-3/2}, \tag{2.12}$$

where $dP_t/d\Omega\, d\omega$ and $dP_{t'}/d\Omega\, d\omega$ denote the intensity distributions at times t and t', respectively, and the barred braces indicate that the arguments of $dP_{t'}/d\Omega\, d\omega$ have to be transformed according to Eqs. (2.10) and (2.11).

In the particular case of initially isotropic radiation, Eq. (2.12) yields a result previously obtained by Thorne[10] by analyzing the effect of anisotropic expansion on blackbody radiation.

As an example of the use of Eq. (2.12), consider an axisymmetric universe ($A_1 = A_2$) and assume the intensity distribution after decoupling to be[3] $dP_{t'}/d\Omega\, d\omega = 1 + \epsilon \sin^2\theta'$. Equation (2.12), with $t - t' \sim 1/H$, where H denotes the mean Hubble constant, $\Delta H/H = (H_3 - H_1)/H \ll 1$ and $R(t)$ denotes the average scale factor, then gives

$$\frac{dP_t}{d\Omega\, d\omega} \approx \left[\frac{R(t')}{R(t)}\right]^3 \left[1 + \left(\epsilon + \frac{3\Delta H}{H}\right)\sin^2\theta\right]. \qquad (2.13)$$

Therefore, as is known,[11] anisotropic expansion introduces an additional quadrupole term in the intensity.

Consider now the problem of evolving a polarization distribution. For any two orthogonal directions $\vec{e}_u$ and $\vec{e}_v$ in a plane transverse to a direction (θ, ϕ), the degree of linear polarization $D_{u,v}$ can be defined as the difference between the fractions of the time-averaged power per unit solid angle and per unit frequency received by instruments accepting only waves linearly polarized along $\vec{e}_u$ and along $\vec{e}_v$, respectively,

$$D_{u,v} = \frac{|P_u - P_v|}{P_u + P_v}. \qquad (2.14)$$

As in general different choices for the set of axes $(\vec{e}_u, \vec{e}_v)$ correspond to different values for $D_{u,v}$, an actual observation is better described in terms of the maximum value attained by $D_{u,v}$ as the set of axes is rotated in all possible ways in the transverse plane. This quantity can be properly referred to as the degree of linear polarization D.

The most general wave propagating along $-k_{(i)}$ is a superposition of a + wave and a − wave [$C_1^\pm = 0$ in Eqs. (2.5)]. One then finds

$$D = \frac{2|C_2^+ C_2^-|}{|C_2^+|^2 + |C_2^-|^2}. \qquad (2.15)$$

Following steps similar to the ones that led to Eq. (2.12) then yields

$$D_t = [\![D_{t'}]\!] , \qquad (2.16)$$

where D_t and $D_{t'}$ denote the values of the degree of linear polarization at t and t', respectively, and the effect of the barred braces is defined below Eq. (2.12).

Therefore, to this order, radiation with no linear polarization ($D = 0$) does not aquire any linear polarization under anisotropic expansion. An initial polarization, however, is altered in its angular dependence according to Eqs. (2.10) and in its frequency distribution according to Eq. (2.11). It follows from Eq. (2.16) that the change in the degree of linear polarization amounts only to a redistribution of its initial values in angle and frequency. Therefore, the maximum linear polarization is conserved in time. Consequently, one can infer that, as long as anisotropic expansion is the dominant mechanism, the present experimental upper bound on the degree of linear polarization of the microwave background,[12] if applicable over the whole spectrum and for all directions, implies the same upper bound on the polarization after decoupling.

As an example of the use of Eq. (2.16), consider again the axisymmetric model universe mentioned above and assume that after last scattering $D_{t'} = \epsilon \sin^2\theta'$ (Ref. 3). One then finds

$$D_t \approx \epsilon \sin^2\theta \left(1 - \frac{2\Delta H}{H}\cos^2\theta\right). \qquad (2.17)$$

In contrast with the result obtained for the intensity distribution in Eq. (2.13), the effect of anisotropic expansion on the polarization is negligible, as it involves the product of ϵ and $\Delta H/H$. This result is quite general and reflects the absence in Eq. (2.16) of the angle-dependent factor of Eq. (2.12). One can therefore conclude that a small anisotropy in the expansion rates of the universe after decoupling could only have a negligible effect on the polarization distribution of the microwave background.

So far this section has dealt with the interpretation of the properties of the lowest-order WKB solutions of Eqs. (2.1). The effects derived from the fields given in Eqs. (2.5) are to be regarded as geometrical optics effects, as they could also be derived in the framework of geometrical optics from the parallel transport equations for the polarization vector and from the propagation equation for the scalar amplitude. More accurate solutions of Eqs. (2.1) are of interest, as they illuminate the nature of the electromagnetic-gravitational coupling beyond the geometrical optics limit. They also provide a means of estimating the size of the corrections to the lowest-order results presented so far. This is done in Appendix B. As is well known,[13] the WKB procedure can be carried further to yield successive terms of the asymptotic representations of the high-frequency solutions of Eqs. (2.1). The relevant equations and the expressions for $S_6^\pm$ are given in Appendix A.

As remarked in Sec. I, the phase delay between the θ and ϕ components of either the electric or the magnetic fields of waves generated by y^+ or y^- differs in general from $\pm\pi/2$. To lowest order $|y^\pm|^2 = b/\mu$, and for an axisymmetric model ($A_1 = A_2$), Eq. (1.40) yields

$$\tan\Delta^\pm = 2k\mu \Big/ \left[\sin^2\theta\left(\frac{1}{A_3}\frac{dA_3}{dt} - \frac{1}{A_1}\frac{dA_1}{dt}\right)\right]. \qquad (2.18)$$

The maximum deviation of $\Delta^\pm$ from $\pm\pi/2$ is thus determined by the ratio between the angular anisotropy of the expansion rates and the frequency of the radiation, and vanishes along the axis of symmetry, as expected from the results obtained at the end of Sec. I.

The corrections to the energy content of a spectral component of the electromagnetic field generated by $y^{\pm}$ (or $y^{\pm *}$) can be obtained by means of the expressions for $S_\delta^{\pm}$ given in Appendix A and of Eqs. (1.20) and (1.34). The result is

$$T_{(0)(0)}=(T_{(0)(i)}T_{(0)(i)})^{1/2}\left\{1+\frac{1}{8k^2\mu^2}\left[\left(2\frac{d\lambda}{dt}\right)^2+\left(\frac{1}{b}\frac{db}{dt}-\frac{1}{\mu}\frac{d\mu}{dt}\right)^2\right]\right\}, \quad (2.19)$$

where $T_{(0)(i)}$ is given by Eq. (1.37) with $C_2^{\pm}=0$ (or $C_1^{\pm}=0$). Therefore, unlike the situation for plane waves in flat space, total energy and total momentum do not coincide. Moreover, the lowest-order nonvanishing correction to $T_{(0)(0)}$, of order $(\Delta H/\omega)^2$, is always positive for an anisotropic model. For an isotropic model, energy and momentum coincide for every spectral component since, due to the conformal invariance of Maxwell's equations, the lowest-order WKB fields are in this case the exact solutions (1.42) of Eqs. (2.1).

III. WAVES IN KASNER SPACETIMES

Exact solutions for electromagnetic perturbations in a given spacetime of the form (1.4) provide an interesting and complete example of the effect of anisotropy on electromagnetic fields and illustrate the formalism developed in Sec. I.

The Kasner spacetimes[7,14] are a convenient choice. They are described by the class of metrics

$$-ds^2=-dt^2+t^{2p_1}(dx^1)^2+t^{2p_2}(dx^2)^2+t^{2p_3}(dx^3)^2\,, \quad (3.1)$$

where

$$p_1+p_2+p_3=p_1{}^2+p_2{}^2+p_3{}^2=1\,. \quad (3.2)$$

They are exact solutions of Einstein's equations in empty space and are of the form (1.4). They are also asymptotic solutions of Einstein's equations near the time singularity in the presence of a homogeneous matter distribution.

The ranges of the three quantities p_1, p_2, and p_3 are most conveniently described by means of the Khalatnikov-Lifshitz parametrization, according to which

$$p_1(u)=\frac{-u}{1+u+u^2}\,,\quad p_2(u)=\frac{1+u}{1+u+u^2}\,,\quad p_3(u)=\frac{u(1+u)}{1+u+u^2}\,, \quad (3.3)$$

where $u\geq 1$. This choice corresponds to $-\frac{1}{3}\leq p_1\leq 0\leq p_2\leq \frac{2}{3}\leq p_3\leq 1$.

For any choice of p_1, p_2, and p_3 within their specified ranges, the metric (3.1) represents an expanding universe for which the volume element increases like $\sqrt{-g}=t$. Two cases need be distinguished. If $p_1=p_2=0$ and $p_3=1$ the spacetime is actually flat, as a simple coordinate transformation brings the metric to the Minkowskian form.[14] In all other cases the spacetime is curved and the model represents a universe expanding along two axes and contracting along the third one.

Consider first the flat Kasner universe, which corresponds to $p_1=p_2=0$ and $p_3=1$. In this case Eqs. (1.18) reduce to

$$\frac{d^2S_\delta^{\pm}}{dt^2}+\frac{1}{t}\frac{dS_\delta^{\pm}}{dt}+\left(k^2\sin^2\delta+\frac{k^2\cos^2\delta}{t^2}\right)S_\delta^{\pm}=0\,. \quad (3.4)$$

For $\delta=0,\pi$ the general solution of Eq. (3.4) is

$$S_\delta^{\pm}=C_1^{\pm}\exp(-ik\ln t)+C_2^{\pm}\exp(ik\ln t)\,. \quad (3.5)$$

In this case, Eqs. (3.5), (1.34), (1.20), and (1.21) yield

$$T^{(0)(0)}=\frac{1}{4\pi t^2}(|C_1^{\pm}|^2+|C_2^{\pm}|^2)$$

and (3.6)

$$T^{(0)(3)}=\frac{1}{4\pi t^2}(|C_1^{\pm}|^2-|C_2^{\pm}|^2)\,.$$

These simple results are just what one expects on the basis of the remarks made at the end of Sec. I. For any direction outside the x^3 axis ($\delta\neq 0,\pi$), Eqs. (3.4) are exactly solved by

$$S_\delta^{\pm}=C_1^{\pm}e^{\nu\pi/2}H_{i\nu}^{(2)}(kt\sin\delta)+C_2^{\pm}e^{-\nu\pi/2}H_{i\nu}^{(1)}(kt\sin\delta)\,. \quad (3.7)$$

Here $H_{i\nu}^{(1)}$ and $H_{i\nu}^{(2)}$ denote Hankel functions of the first and second kind with purely imaginary order, $\nu=k\cos\delta$ and the exponential factors have been introduced in order that the two solutions be complex conjugates of each other. Equations (1.34) then yield

$$E_\theta^{\pm}\pm iB_\theta^{\pm}=\mu\, e^{ik_jx^j}(C_1^{\pm}e^{\nu\pi/2}H_{i\nu}^{(2)}+C_2^{\pm}e^{-\nu\pi/2}H_{i\nu}^{(1)})$$

and (3.8)

$$E_\phi^{\pm}\pm iB_\phi^{\pm}=\mp\sin\delta\, e^{ik_jx^j}(C_1^{\pm}e^{\nu\pi/2}H_{i\nu}^{(2)\prime}+C_2^{\pm}e^{-\nu\pi/2}H_{i\nu}^{(1)\prime})\,,$$

where primes denote derivatives with respect to the full argument of the Hankel functions, which has been omitted for brevity. It then follows from Eqs. (1.20) and (1.21) that

$$T^{(0)(0)}=\frac{e^{-\nu\pi}}{8\pi}(|C_1^{\pm}|^2+|C_2^{\pm}|^2)(\mu^2|H_{i\nu}^{(1)}|^2+\sin^2\delta\,|H_{i\nu}^{(1)\prime}|^2)+(\text{interference terms})$$

and (3.9)

$$T^{(0)(i)}=\frac{1}{2\pi^2k^2t}(|C_1^{\pm}|^2-|C_2^{\pm}|^2)k^{(i)}\,.$$

As $t\to\infty$, apart from interference terms,

$$T^{(0)(0)}\sim\frac{\sin\delta}{2\pi^2kt}(|C_1^\pm|^2+|C_2^\pm|^2)\left[1+\frac{1}{2t^2\sin^2\delta}\left(\cos^2\delta+\frac{1}{4k^2}\right)\right]. \tag{3.10}$$

The agreement between this result and Eq. (2.19), which gives the high-frequency behavior of $T^{(0)(0)}$, is a consequence of a general property of the Kasner spacetimes. It is due to the fact that the Hubble constants decrease in time as $1/t$. This is in fact enough to ensure that, for sufficiently large times, the WKB solutions of Sec. II give a very accurate description of the fields.

In order to examine more in detail the behavior of the fields, it is convenient to set for definiteness $C_1^\pm=0$. The most general E_θ and E_ϕ fields then result from the superposition of a + wave and a − wave. Letting $\alpha=(C_2^+ + C_2^-)\exp(-\nu\pi/2)$ and $\beta=(C_2^- - C_2^+)\exp(-\nu\pi/2)$, with $\alpha=|\alpha|e^{i\phi_\alpha}$ and $\beta=|\beta|e^{i\phi_\beta}$, one can write

$$E_\theta=\mu\,|\alpha|M_{i\nu}\cos(k_jx^j+\psi_{i\nu}+\phi_\alpha)$$

and (3.11)

$$E_\phi=\sin\delta\,|\beta|N_{i\nu}\cos(k_jx^j+\chi_{i\nu}+\phi_\beta)\,,$$

where $M_{i\nu}$ and $\psi_{i\nu}$ denote modulus and phase of $H_{i\nu}^{(1)}$, and $N_{i\nu}$ and $\chi_{i\nu}$ denote modulus and phase of $H_{i\nu}^{(1)\prime}$. The behavior of the fields is then suitably described in terms of the two functions R and Δ, where

$$R=\frac{|\alpha|}{|\beta|}\,\frac{\mu}{\sin\delta}\,\frac{M_{i\nu}}{N_{i\nu}} \tag{3.12}$$

is the ratio between the moduli of E_θ and E_ϕ, and

$$\Delta=\psi_{i\nu}-\chi_{i\nu}+\phi_\alpha-\phi_\beta \tag{3.13}$$

is the phase difference between E_θ and E_ϕ. As $t\to\infty$

$$R\sim\frac{|\alpha|}{|\beta|}\left(1-\frac{1}{4k^2t^2\sin^2\delta}\right)$$

and (3.14)

$$\Delta\sim\frac{\pi}{2}-\frac{1}{2kt\sin\delta}+\phi_\alpha-\phi_\beta\,.$$

The ratio R and the phase difference Δ approach constant values as $t\to\infty$, as expected for WKB waves. As $t\to0$ the behavior is rather complicated, and outside the (x^1,x^2) plane $(\delta\neq\pi/2)$

$$R\sim\frac{|\alpha|}{|\beta|}\left\{\frac{\cosh\pi\nu+\cos[2\nu\ln(kt\sin\delta)-2\theta_\Gamma]}{\cosh\pi\nu-\cos[2\nu\ln(kt\sin\delta)-2\theta_\Gamma]}\right\}^{1/2}, \tag{3.15}$$

where θ_Γ denotes the phase of the Γ function $\Gamma(i\nu)$. As $t\to0$ the phase difference Δ undergoes an infinite number of oscillations between the two values

$$\Delta_{1,2}=-\frac{\pi}{2}\pm\tan^{-1}\left(\frac{1}{\sinh\pi\nu}\right)+\phi_\alpha-\phi_\beta\,. \tag{3.16}$$

In the (x^1,x^2) plane ν equals zero. The asymptotic behavior of R and Δ as $t\to\infty$ is still described by Eqs. (3.14), while as $t\to0$

$$R\sim\frac{|\alpha|}{|\beta|}kt\,|\ln(kt)|$$

and (3.17)

$$\Delta\sim-\pi+\phi_\alpha-\phi_\beta\,.$$

The previous results illustrate the effect of anisotropy on electromagnetic waves. In particular, the time dependence of the two quantities R and Δ implies that propagation from early times affects in a significant way the polarization of electromagnetic waves. An early polarization state is thus inaccessible to an observer at late times.

Consider now the general Kasner spacetimes, for which p_1, p_2, and p_3 lie within the ranges specified below Eq. (3.3) and the combination $p_1=p_2=0$, $p_3=1$ is excluded. In this case Eqs. (1.18) are extremely complicated for $k_{(i)}$ lying outside the coordinate axes. If $k_{(i)}$ lies along a coordinate axis, however, they reduce to a rather tractable form. For definiteness, take $k_{(i)}$ lying along the x^1 axis. The cases of $k_{(i)}$ lying along the x^2 and x^3 axes can be treated similarly. Equations (1.18) then become

$$\frac{d^2S_\delta^\pm}{dt^2}+\frac{1-2p_2}{t}\,\frac{dS_\delta^\pm}{dt}+k^2t^{-2p_1}S_\delta^\pm=0\,, \tag{3.18}$$

and are exactly solved by

$$S_\delta^\pm=C_1^\pm t^{p_2}H^{(2)}_{p_2/(1-p_1)}\left(\frac{kt^{1-p_1}}{1-p_1}\right)+C_2^\pm t^{p_2}H^{(1)}_{p_2/(1-p_1)}\left(\frac{kt^{1-p_1}}{1-p_1}\right), \tag{3.19}$$

where $H^{(1)}$ and $H^{(2)}$ denote Hankel functions of the first and second kind, respectively. Equations (1.34) then yield

$$E_3^\pm\pm iB_3^\pm=-t^{-p_1}e^{ikx^1}(C_1^\pm H_\nu^{(2)}+C_2^\pm H_\nu^{(1)})$$

and (3.20)

$$E_2^\pm\pm iB_2^\pm=\mp t^{-p_1}e^{ikx^1}(C_1^\pm H_{\nu-1}^{(2)}+C_2^\pm H_{\nu-1}^{(1)})\,,$$

where $\nu=p_2/(1-p_1)$ ranges between 0 and $\frac{1}{2}$ on account of Eq. (3.3) and the arguments of the Hankel functions have been omitted for brevity. One then finds

$$T^{(0)(0)}=\frac{1}{8\pi t^{2p_1}}(|C_1^{\pm}|^2+|C_2^{\pm}|^2)(|H_\nu^{(1)}|^2+|H_{\nu-1}^{(1)}|^2)$$

$$+(\text{interference terms})$$

and (3.21)

$$T^{(0)(1)}=\frac{(1-p_1)}{2\pi^2 k}t^{-(1+p_1)}(|C_1^{\pm}|^2-|C_2^{\pm}|^2).$$

As $t\to\infty$, apart from interference terms,

$$T^{(0)(0)}\sim\frac{(1-p_1)t^{-(1+p_1)}}{2\pi^2 k}(|C_1^{\pm}|^2+|C_2^{\pm}|^2)\times\left[1+\frac{(p_2-p_3)^2 t^{-2(1-p_1)}}{8k^2}\right], \tag{3.22}$$

again in agreement with the high-frequency behavior of $T^{(0)(0)}$ described by Eq. (2.19).

In order to examine more in detail the behavior of the fields, it is convenient to set for definiteness $C_1^{\pm}=0$. The most general E_3 and E_2 fields then result from the superposition of a + wave and a − wave. Letting $\alpha=C_2^{+}+C_2^{-}$ and $\beta=C_2^{+}-C_2^{-}$ with $\alpha=|\alpha|e^{i\phi_\alpha}$ and $\beta=|\beta|e^{i\phi_\beta}$, one can write

$$E_3=t^{-p_1}|\alpha|M_\nu\cos(kx^1+\psi_\nu+\phi_\alpha)$$

and (3.23)

$$E_2=-t^{-p_1}|\beta|M_{\nu-1}\cos(kx^1+\psi_{\nu-1}+\phi_\beta),$$

where M_ν and ψ_ν denote modulus and phase of $H_\nu^{(1)}$ and $M_{\nu-1}$ and $\psi_{\nu-1}$ denote modulus and phase of $H_{\nu-1}^{(1)}$. The behavior of the fields is then suitably described in terms of the two functions R and Δ, where

$$R=\frac{|\alpha|}{|\beta|}\frac{M_\nu}{M_{\nu-1}} \tag{3.24}$$

is the ratio between the moduli of E_3 and E_2, and

$$\Delta=\psi_\nu-\psi_{\nu-1}+\phi_\alpha-\phi_\beta \tag{3.25}$$

is the phase difference between E_3 and E_2.

As $t\to\infty$

$$R\sim\frac{|\alpha|}{|\beta|}\left[1-\frac{(p_3{}^2-p_2{}^2)}{16k^2}t^{-2(1-p_1)}\right]$$

and (3.26)

$$\Delta\sim-\frac{\pi}{2}-\frac{(p_3-p_2)}{2k}t^{-(1-p_1)}+\phi_\alpha-\phi_\beta.$$

The ratio between the moduli and the phase difference approach constant values as $t\to\infty$, as expected for WKB waves. As $t\to 0$

$$R\sim\frac{|\alpha|}{|\beta|}\frac{\Gamma(\nu)}{\Gamma(1-\nu)}\left[\frac{k}{2(1-p_1)}\right]^{1-2\nu}t^{p_3-p_2}$$

and (3.27)

$$\Delta\sim\pi(\nu-1)+\phi_\alpha-\phi_\beta.$$

A few comments are in order here. As $t\to 0$, E_3 is suppressed with respect to E_2, and the phase difference between E_3 and E_2 tends to a constant value. Therefore, as the singularity is approached, at any point in space the electric field (and the magnetic field, also) aligns itself with the x^2 axis. Similar arguments lead to the conclusion that both for waves with $k_{(i)}$ lying along the x^2 axis and for waves with $k_{(i)}$ lying along the x^3 axis, the electric field (and the magnetic field, also) aligns itself with the x^1 axis as the singularity is approached. It follows that, near the singularity, the stress-energy tensor of the electromagnetic field is generally dominated by its diagonal elements. It describes a tension along the field lines and a pressure perpendicular to the field lines, acting in a way that tends to make the expansion rates isotropic. Moreover, since the modulus of $H_\nu^{(1)}$ is an even function of ν and is monotonically increasing with $|\nu|$,[15] for waves along the x^1 axis $M_\nu<M_{\nu-1}$. Therefore, R is always less than the value $|\alpha|/|\beta|$ which it approaches as $t\to\infty$. The phase difference Δ between the E_3 and E_2 components of the electric field increases monotonically in time, and the overall phase accumulated as a consequence of the propagation out of the singularity is $(\frac{1}{2}-\nu)\pi$, which, for the allowed range of ν, is constrained to lie between 0 and $\pi/2$. Again, propagation in a strongly anisotropic gravitational field is seen to alter in a non-negligible way phase and amplitude relationships between the field components.

The fields given in Eq. (3.20) can now be used to construct waveforms traveling along the x^1 axis. For this purpose k must be regarded as a Fourier transform parameter and must be allowed to run from $-\infty$ to $+\infty$. As the Hankel functions are analytic in the complex plane cut along the negative real axis, an ambiguity results in the definition of $H^{(1)}$ and $H^{(2)}$ for real negative values of their arguments. A way out of this difficulty is defining

$$f(t)=\begin{cases}H_\nu^{(2)}\left(\dfrac{kt^{1-p_1}}{1-p_1}\right) & (k>0),\\[2ex] H_\nu^{(1)}\left(\dfrac{|k|t^{1-p_1}}{1-p_1}\right) & (k<0).\end{cases} \tag{3.28}$$

Introducing, in analogy with Sec. I, the notation $\mathcal{G}^{(j)}=E^{(j)}+iB^{(j)}$, one can write

$$\mathcal{G}^{(3)}(x^1,t)=t^{-p_1}\int_{-\infty}^{+\infty}e^{ikx^1}dk[C_1 f(t)+C_2 f^*(t)]. \tag{3.29}$$

Equations (3.28) and (1.34) then imply that

$$\mathcal{G}^{(2)}(x^1,t)=t^{-p_1}\int_{-\infty}^{+\infty}e^{ikx^1}dk[C_1 g(t)+C_2 g^*(t)], \tag{3.30}$$

where

$$g(t)=\begin{cases} H^{(2)}_{\nu-1}\left(\dfrac{kt^{1-p_1}}{1-p_1}\right) & (k>0)\,, \\ -H^{(1)}_{\nu-1}\left(\dfrac{|k|t^{1-p_1}}{1-p_1}\right) & (k<0)\,. \end{cases} \tag{3.31}$$

The choice (3.28) for $f(t)$ is a convenient one, as the function bears a close similarity to the exponential $\exp(-ikt)$ of flat space electrodynamics. Indeed, comparison with the well-known flat space case, in which $\exp(-ikt)$ would be associated with waveforms traveling toward $x^1=+\infty$, suggests that a similar property may hold for $f(t)$. It is therefore interesting to construct the most general waveform corresponding to $C_2=0$. This requires a definite relationship between the initial amplitudes of $\mathcal{J}^{(2)}$ and $\mathcal{J}^{(3)}$, so that

$$C_1=\frac{1}{2\pi}\frac{\tau^{p_1}}{f(\tau)}\int_{-\infty}^{+\infty}\mathcal{J}^{(3)}(\xi,\tau)e^{-ik\xi}d\xi =\frac{1}{2\pi}\frac{\tau^{p_1}}{g(\tau)}\int_{-\infty}^{+\infty}\mathcal{J}^{(2)}(\xi,\tau)e^{-ik\xi}d\xi\,. \tag{3.32}$$

Equations (3.29), (3.30), and (3.32) then yield

$$\mathcal{J}^{(3)}(x^1,t)=\left(\frac{\tau}{t}\right)^{p_1}\int_{-\infty}^{+\infty}\mathcal{J}^{(2)}(\xi,\tau)K_3^+(x^1-\xi;t,\tau)d\xi$$

and (3.33)

$$\mathcal{J}^{(2)}(x^1,t)=\left(\frac{\tau}{t}\right)^{p_1}\int_{-\infty}^{+\infty}\mathcal{J}^{(2)}(\xi,\tau)K_2^+(x^1-\xi;t,\tau)d\xi\,,$$

where the two propagators are written for convenience in the form

$$K_3^+(x^1-\xi;t,\tau)=\frac{\partial}{\partial x}\int_{-\infty}^{+\infty}\frac{dk}{2\pi i}\frac{e^{ik(x^1-\xi)}}{k}\frac{f(t)}{g(\tau)}$$

and (3.34)

$$K_2^+(x^1-\xi;t,\tau)=\frac{\partial}{\partial x}\int_{-\infty}^{\infty}\frac{dk}{2\pi i}\frac{e^{ik(x^1-\xi)}}{k}\frac{g(t)}{g(\tau)}.$$

On account of the analytic properties of the Hankel functions,[16] the integrals in Eqs. (3.34) can be regarded as contour integrals in the complex k plane:

$$K_3^+(x^1-\xi;t,\tau)=\frac{\partial}{\partial x}\int_\gamma\frac{dk}{2\pi i}\frac{e^{ik(x^1-\xi)}}{k}\frac{H^{(2)}_\nu(kb)}{H^{(2)}_{\nu-1}(ka)}$$

and (3.35)

$$K_2^+(x^1-\xi;t,\tau)=\frac{\partial}{\partial x}\int_\gamma\frac{dk}{2\pi i}\frac{e^{ik(x^1-\xi)}}{k}\frac{H^{(2)}_{\nu-1}(kb)}{H^{(2)}_{\nu-1}(ka)}.$$

In Eqs. (3.35) the contour γ runs along the real axis of the complex k plane below the cut and is indented below the origin, $b=kt^{1-p_1}/(1-p_1)$ and $a=k\tau^{1-p_1}/(1-p_1)$. The fact that all the zeros of the Hankel functions of the second kind lie above the real axis of the complex k plane[17] is then sufficient to conclude that K_3^+ and K_2^+ vanish identically for $x^1-\xi<b-a$. The two propagators can also be written

$$K_3^+(x^1-\xi;t,\tau)=\frac{1}{i}\left(\frac{\tau}{t}\right)^{p_1}\left[\delta(x^1-\xi+a-b)\alpha_3\left(\frac{b}{a}\right)+\eta(x^1-\xi+a-b)\beta_3(x^1-\xi;a,b)\right]$$

and (3.36)

$$K_2^+(x^1-\xi;t,\tau)=\left(\frac{\tau}{t}\right)^{p_1}\left[\delta(x^1-\xi+a-b)\alpha_2\left(\frac{b}{a}\right)+\eta(x^1-\xi+a-b)\beta_2(x^1-\xi;a,b)\right].$$

In Eqs. (3.36) δ denotes Dirac's δ function, η denotes the step function [i.e., $\eta(u)=0$ for $u<0$ and $\eta(u)=1$ for $u>0$],

$$\alpha_3=\frac{1}{2}\left(\frac{a}{b}\right)^{1/2}+\frac{1}{\pi}\int_0^\infty\frac{dk}{k}\,\mathrm{Re}\left[e^{ik(b/a-1)}\frac{H^{(2)}_\nu(kb/a)}{H^{(2)}_{\nu-1}(k)}\right],$$

$$\beta_3=\frac{1}{\pi}\int_0^\infty dk\,\mathrm{Re}\left\{e^{ik(x^1-\xi)}\left[\frac{iH^{(2)}_\nu(kb)}{H^{(2)}_{\nu-1}(ka)}+\left(\frac{a}{b}\right)^{1/2}e^{ik(a-b)}\right]\right\}, \tag{3.37}$$

$$\alpha_2=\frac{1}{2}\left(\frac{a}{b}\right)^{1-\nu}+\frac{1}{2}\left(\frac{a}{b}\right)^{1/2}+\frac{1}{\pi}\int_0^\infty\frac{dk}{k}\,\mathrm{Im}\left[e^{ik(b/a-1)}\frac{H^{(2)}_{\nu-1}(kb/a)}{H^{(2)}_{\nu-1}(k)}\right],$$

and

$$\beta_2=\frac{1}{\pi}\int_0^\infty dk\,\mathrm{Re}\left\{e^{ik(x^1-\xi)}\left[\frac{H^{(2)}_{\nu-1}(kb)}{H^{(2)}_{\nu-1}(ka)}-e^{ik(a-b)}\left(\frac{a}{b}\right)^{1/2}\right]\right\}.$$

The field distributions corresponding to K_3^+ and K_2^+ therefore involve a pulse of variable amplitude centered at $x^1=\xi+b-a$ and a structure for $x^1>\xi+b-a$. It is worthwhile to remark that no backward tail is present.

In the same way it is possible to show that the case $C_1=0$ corresponds to waveforms traveling toward $x^1=-\infty$. In this case

$$\mathcal{J}^{(3)}(x^1,t)=\left(\frac{\tau}{t}\right)^{p_1}\int_{-\infty}^{+\infty}\mathcal{J}^{(2)}(\xi,\tau)K_3^-(x^1-\xi;t,\tau)d\xi$$

and (3.38)

$$\mathcal{J}^{(2)}(x^1,t)=\left(\frac{\tau}{t}\right)^{p_1}\int_{-\infty}^{\infty}\mathcal{J}^{(2)}(\xi,\tau)K_2^-(x^1-\xi;t,\tau)d\xi\,,$$

with

$$K_3^-(x^1-\xi;t,\tau)=\frac{\partial}{\partial x}\int_{\gamma'}\frac{dk}{2\pi i}\frac{e^{ik(x^1-\xi)}}{k}\frac{H^{(1)}_\nu(kb)}{H^{(1)}_{\nu-1}(ka)}$$

and (3.39)

$$K_2^-(x^1-\xi;t,\tau)=\frac{\partial}{\partial x}\int_{\gamma'}\frac{dk}{2\pi i}\frac{e^{ik(x^1-\xi)}}{k}\frac{H^{(1)}_{\nu-1}(kb)}{H^{(1)}_{\nu-1}(ka)}$$

and where the contour γ' runs along the real axis of the complex k plane above the cut and is indented above the origin. The fact that all the zeros of the Hankel functions of the first kind lie below the real axis of the complex k plane[17] is then sufficient to conclude that K_3^- and K_2^- vanish identically for $x^1 - \xi > a - b$. It then follows that the field distributions corresponding to K_3^- and K_2^- involve a pulse of variable amplitude centered at $x^1 = \xi + a - b$ and a structure for $x^1 < \xi + a - b$. Again, no backward tail is present.

The generalization of these results to waves propagating along the other coordinate axes is straightforward, as these cases involve Hankel functions as well, the main difference lying in the range of values for the index ν.

CONCLUSIONS

A general formalism has been developed to deal with electromagnetic waves propagating in a background metric of the form $-ds^2 = -dt^2 + \sum_{i=1}^3 A_i^2(t) \times (dx^i)^2$. The general solution of the propagation problem has been reduced to the integration of two linear second-order differential equations for the variables $S_\delta^\pm$ which determine the time evolution of the $F_{\mu\nu}$ tensor. These equations are in general different. They can be chosen to coincide for an axisymmetric model and, for an isotropic model, are exactly solvable in terms of simple trigonometric functions, as one would expect on the basis of the conformal invariance of Maxwell's equations. The electric and magnetic fields $\vec{E}$ and $\vec{B}$, as measured by an observer comoving with the background metric (x^i = constant), have been obtained from the $F_{\mu\nu}$ tensor by use of an orthonormal tetrad. The $\vec{E}$ and $\vec{B}$ fields are shown to be superpositions of spectral components, each labeled by given values of k_i, transverse to the time-varying direction specified by k_i/A_i, and expressible in the form of simple combinations of $S_\delta^\pm$ and their time derivatives.

Expressions for the total energy, total momentum, and total spin angular momentum of the electromagetic field have been obtained in terms of the contributions of the various spectral components, which involve bilinear combinations of $S_\delta^\pm$ and $dS_\delta^\pm/dt$.

For each spectral component the ratio of the corresponding contributions to the energy and to the spin angular momentum equals $k\mu$, which in the WKB limit is just the frequency of the waves. This result is in close analogy with the familiar flat-space one, to which it reduces if the metric coefficients are independent of time.

It has been shown that $S_\delta^+(k_i)$ generates a wave with positive helicity carrying power along $k_{(i)}$ and a wave with negative helicity carrying power along $-k_{(i)}$. $S_\delta^-(k_i)$ generates a wave with negative helicity carrying power along $k_{(i)}$ and a wave with positive helicity carrying power along $-k_{(i)}$.

In the high-frequency limit, S_δ^+ and S_δ^- generate circularly polarized waves. Their satisfying different equations has been shown to be connected to the time-varying nature of the unit vectors to which the fields are referred. On the basis of these considerations, it has been concluded that the polarization vector for a given wave satisfies the parallel transport equations on top of the celestial sphere. This result had been previously derived by Caderni *et al.*[5] from the parallel transport equations of geometrical optics.

The high-frequency fields have been used to relate observations of intensity and polarization distributions made at two different times by means of a receiver comoving with the background metric. These results have been shown to describe with a high degree of accuracy possible effects of anisotropic expansion after decoupling on polarization and intensity characteristics of the microwave background. In agreement with previous analyses,[11] it is seen that anisotropic metrics of the form discussed in this paper introduce a quadrupole term in the intensity distribution. Moreover, the maximum degree of linear polarization is conserved in time. One is therefore led to infer that the present experimental upper bound on the degree of linear polarization of the microwave background,[12] if applicable over the whole spectrum and for all angles, implies that after last scattering the radiation was unpolarized at least to one part in 3000, if the large-scale evolution of the universe is well described by a metric of the form considered in this paper.

In the high-frequency limit, energy density and momentum density of a spectral component of the electromagnetic field are shown to be equal only in the lowest-order approximation. The first correction to the energy density has been found explicitly and seen to be always positive.

The formalism has been used to study the behavior of electromagnetic fields in Kasner spacetimes. Exact solutions have been found for waves propagating in any direction in the flat Kasner model and along the coordinate axes of the general Kasner models. As $t \to \infty$, the solutions are well described by their WKB approximations, while near the singularity new and rather complicated effects arise. The fields move away from the most rapidly compressing direction, leading to a stress-energy tensor that tends to make the expansion rates isotropic. Moreover, the phase difference between two field components can vary considerably in time. Propagation out of the Kasner singularity therefore alters in a considerable way amplitude and phase relationships for the waves. Nonetheless, it has been shown that suitable relations between the initial distributions of the field components lead to waveforms traveling along the coordinate axes of the Kasner models without a backward tail, a result in close similarity with the usual flat-space one.

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APPENDIX A

The asymptotic representations of $S_\delta^\pm$ in the high-frequency limit can be obtained by introducing in Eq. (2.1) the formal expansion[13]

$$S_\delta^\pm = \exp\left(kS_0 + S_1 + \frac{S_2}{k} + \frac{S_3}{k^2} + \cdots\right) \tag{A1}$$

and by solving for $S_0, S_1, \ldots,$ recursively. The result for $S_\delta^\pm$, accurate up to terms in $1/k^2$ in the amplitude and up to terms in $1/k$ in the phase, is found to be

$$S = \left(\frac{b}{\mu}\right)^{1/2}\left\{1 \mp \frac{1}{2k\mu}\frac{d\lambda}{dt} + \frac{1}{k^2}\left[\frac{5}{8}\frac{1}{\mu^2}\left(\frac{d\lambda}{dt}\right)^2 + \phi_1\right]\right\} \times \exp\left(ik\int^t \mu\, dt \pm i\lambda + \frac{i}{k}\phi_2\right), \tag{A2}$$

where

$$\phi_1 = \frac{b^2}{\mu^3}\left\{\frac{1}{8}\frac{d}{dt}\left[\frac{1}{b}\frac{d}{dt}\left(\frac{\mu}{b}\right)\right] - \frac{3}{16\mu}\left[\frac{d}{dt}\left(\frac{b}{\mu}\right)\right]^2\right\} \tag{A3}$$

and

$$\phi_2 = -\int^t \left[2\mu\phi_1 + \frac{1}{2\mu}\left(\frac{d\lambda}{dt}\right)^2\right]dt. \tag{A4}$$

As is well known, the validity of the asymptotic representation (A2) rests on the conditions

$$|kS_0| \gg |S_1| \gg \frac{1}{k}|S_2| \gg \frac{1}{k^2}|S_3|, \tag{A5}$$

with the further requirement that the absolute value of the first neglected term be always much less than one. Analysis of (A2), (A3), and (A4) reveals that the conditions (A5) amount to

$$k\mu \sim \omega \gg \Delta H \gg \frac{1}{\omega}\left[(\Delta H)^2, \Delta H', H\Delta H\right] \gg \frac{1}{\omega^2}(\Delta H'', \ldots) \tag{A6}$$

where H denotes the average Hubble constant, ΔH denotes the typical angular anisotropy of the Hubble expansion, and primes denote time derivatives. For sufficiently large ω, the conditions (A6) are necessarily satisfied.

APPENDIX B

The size of the errors introduced by using the lowest-order WKB fields to describe the free propagation of the microwave background after decoupling can be estimated by evaluating the magnitude of the first neglected term in the asymptotic series (A1). For this purpose it is sufficient to consider an axisymmetric model ($A_1 = A_2$) and to compare with unity the quantity $|\phi_2|/k$ introduced in Eq. (A4), which can be put in the convenient form

$$\left|\frac{\phi_2}{k}\right| = \left|\left[\frac{1}{4k\mu}\left(\frac{\dot\mu}{\mu} - \frac{\dot b}{b}\right)\right]_{t_d}^{t_0} + \frac{1}{8k}\int_{t_d}^{t_0}\frac{dt}{\mu}\left(\frac{\dot\mu}{\mu} - \frac{\dot b}{b}\right)^2\right|, \tag{B1}$$

where t_d denotes the time of decoupling and t_0 denotes the present time. In order to obtain a simple numerical estimate it is sufficient to consider waves propagating in the symmetry plane ($\delta = \pi/2$) of a dust universe,[18] for which

$$A_1(t) = A_2(t) = \left(\frac{t}{t_0}\right)^{2/3}, \quad A_3(t) = \left(\frac{t}{t_0}\right)^{2/3}\left(1 - \frac{2}{3}\frac{\Delta H_0}{H_0}\frac{t_0}{t}\right), \tag{B2}$$

where $\Delta H_0/H_0$ denotes the present value of the fractional angular anisotropy of the Hubble expansion. Present experimental values[12] imply that both for the $H_{\rm I}$ and the $H_{\rm II}$ cases[3] $|(\Delta H_0/H_0)(t_0/t_d)| \ll 1$. The second term in Eq. (B1) is therefore negligible and

$$\left|\frac{\phi_2}{k}\right| \approx \frac{1}{6\omega_0 t}\left|\frac{\Delta H_0}{H_0}\right|\left(\frac{t_0}{t_d}\right)^{1/3}. \tag{B3}$$

Not surprisingly, this result agrees with the estimate of the first-order corrections to geometrical optics previously given by Anile and Breuer[6] and indicates that the WKB fields (2.5) describe with a high degree of accuracy the free propagation of the microwave background after decoupling.

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