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Broken symmetries and hydrodynamics of superfluid $^3P_2$-neutron-star matter

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We investigate spontaneously broken symmetries and their consequences for the static and dynamic behavior of $^3P_2$-neutron-star matter. Apart from a broken gauge symmetry, which is characteristic of any superfluid, the total rotational symmetry of spin and orbit space is spontaneously broken, a property unique to $^3P_2$-neutron-star matter. The additional hydrodynamic variables are discussed, especially their commutator relations and their anholonomity properties. A complete set of nonlinear hydrodynamic equations is derived, and a procedure for guaranteeing total angular momentum conservation locally is presented. As unusual hydrodynamic excitations we find three pairs of damped spin-orbit waves which reflect the intricate coupling of orbit and spin space in $^3P_2$-neutron-star matter. Its hydrodynamics shares some properties with biaxial nematic liquid crystals on the one hand and with ^{3}He-B on the other. Finally, we discuss the consequences of a reversible current connecting the vortices of the magnetic field with the density of linear momentum which can influence the magnetohydrodynamics of a neutron star.

I. INTRODUCTION

As has become clear during the last few years[1-3] there exists very probably a $^3P_2$ superfluid phase of neutrons in the interior of neutron stars. In the last decade many investigations concerning the superfluid state have been carried out. In these publications emphasis has been put on questions such as cooling mechanisms,[4] derivation of gap equations and generalized Gorkov equations,[5-9] generalized Ginzburg-Landau free energies,[6,2] and renormalization-group calculations of the phase transitions.[10,11] Very recently Golo[12] considered spatial configurations of the order parameter of $^3P_2$-neutron-star matter in the presence of a superfluid current.

The purpose of this paper is to clarify the question of which continuous symmetries are spontaneously broken in $^3P_2$-neutron-star matter and to derive the hydrodynamic equations for this quantum fluid. Our considerations are relevant to the dynamics of neutron stars because the hydrodynamic equations give a macroscopic description (including magnetic fields) of $^3P_2$-neutron-star matter. We find a coupling term between the density of linear momentum and the vorticity of the magnetization density. In stationary situations an inhomogeneous magnetic field gives rise to a superfluid counterflow. The hydrodynamic modes we find may influence the transport properties and the dissipative mechanisms of the neutron star.

As is well known the hydrodynamic regime is confined to small frequencies ω and to small wave vectors k, or to phrase it more precisely the conditions $kl_c \ll 1$ and $\omega\tau_c \ll 1$ are assumed to hold where τ_c is the largest microscopic (collision) time and l_c is the largest distance between two collisions on the microscopic level. Since the $^3P_2$ pairing is believed to occur at a neutron density of roughly 10^{14}–$10^{15}\,\mathrm{g\,cm^{-3}}$,[1] the Fermi wave vector is $\approx 2\times 10^{+13}\,\mathrm{cm}^{-1}$. Because $T_c/T_F \approx 10^{-3}$,[2] the coherence length is estimated as 5×10^{-11} cm (the London penetration depth is smaller) and the hydrodynamic theory is applicable down to 10^{-10} cm. To derive the nonlinear hydrodynamic equations we use exclusively general symmetry arguments and local thermodynamics. To get further insight into some of the parameters involved we include at some points considerations obtained from an application of the projector formalism of Mori to hydrodynamic systems. The derivation of phenomenological hydrodynamic equations is a rather classical approach and it has been applied to many hydrodynamic systems, e.g., to magnets,[13] uniaxial nematic liquid crystals,[14-16] crystals,[15-17] smectic liquid crystals,[15,16] cholesteric liquid crystals,[15,16,18-20] biaxial nematics,[20,21] superfluid ^{4}He,[22] to the various types of discotic liquid crystals,[23,24,21] and to the superfluid phases of ^{3}He.[25-41] The latter class of hydrodynamic systems shares with $^3P_2$-neutron-star matter the existence of an $S=1$ pairing in spin space (besides the property "superfluidity"). As a different approach to derive linear hydrodynamic equations one can use correlation functions, the projector formalism, and the Kubo relations. This approach may be used as a bridge to fully microscopic techniques such as the Boltzmann equation. It has been pursued for simple fluids,[42] ^{4}He superfluid,[43] nematic liquid crystals,[44,45] and the superfluid phases of ^{3}He.[46,38,40]

This paper is organized as follows.

In Sec. II we discuss the order parameter and identify the generators of the spontaneously broken continuous symmetries. As a feature unique to superfluid $^3P_2$-neutron-star matter (among all superfluid systems studied so far) we find that *total* angular momentum but neither orbital angular momentum nor magnetization serve as the generator of the broken symmetries. The latter fact is due to the strong spin-orbit coupling in the Hamiltonian of $^3P_2$-neutron-star matter, which destroys rotational invariance in spin space alone as well as rotational invariance in orbit space alone. Only combined rotations of spin and orbit space are symmetry elements of the Hamiltonian. However, this symmetry is spontaneously broken by the existence of the $^3P_2$ order parameter giving rise to additional hydrodynamic variables and excitations. The transformation properties under spatial inversion, time reversal, and Galilei transformations of the hydrodynamic variables characterizing the broken symmetries are also discussed in Sec. II.

In Sec. III we derive the general nonlinear hydrodynamic equations for an infinitely extended $^3P_2$ superfluid without impurities. Starting from the Gibbs relation and the conservation and quasiconservation laws we consider first the free energy including third-order terms. The necessity to include the latter class of contributions has been found for the superfluid phases of ^{3}He[47–49, 39] and has been applied to the various types of liquid crystals by the authors.[16, 20, 21] As has been stressed in Ref. 39 fourth-order terms must be kept in the free energy to preserve positivity although those do not contribute to the final hydrodynamic equations. From the third-order free energy the thermodynamic conjugate quantities can be obtained by differentiation. In the next step we present the irreversible and the reversible currents and quasicurrents which are expanded in the thermodynamic conjugates. Furthermore, we propose a novel procedure to guarantee conservation of total angular momentum density locally although the total angular momentum density is not a local quantity. This is achieved by relating the source term in the dynamical equation for the magnetization density (which is not conserved) to the antisymmetric part of the stress tensor. In order to fulfill the requirements connected with the nonnegativity of the entropy production, the hydrodynamic currents are expanded into the thermodynamic force conjugate to the total angular momentum. i.e., into the sum of the internal magnetic field and the vorticity. The latter quantity is local and may therefore occur in the hydrodynamic equations. Similarities and differences between the hydrodynamic equations of $^3P_2$ superfluid star matter and those of biaxial nematic liquid crystals and superfluid ^{3}He-B are pointed out.

We find that a large number of phenomenological parameters are involved in the hydrodynamic equations derived reflecting the biaxiality and the fact that spin and orbit indices can be interchanged at will in superfluid $^3P_2$-neutron-star matter. The approximate microscopic evaluation along the lines discussed, e.g., in Refs. 50 and 51, of some of the parameters involved is in progress now and will be reported elsewhere.

Up to linear order in the wave vector, we find as normal modes first and second sound and three spin-orbit waves. The sound waves are mixed together (as in any superfluid), and separately the three spin-orbit waves are coupled. The latter excitations reflect the spontaneously broken total rotational invariance. However, these spin-orbit waves are overdamped due to the nonconservation of the magnetization density. The other modes have quadratic dispersion relations $\omega(\vec{k}) \sim k^2$. Of course, all excitations reflect the biaxiality of the system. At the end of Sec. III we discuss the reversible coupling of the density of linear momentum to the vorticity of the magnetization density. The latter can be viewed, therefore, as an additional superfluid velocity, which is not irrotational (contrary to the usual superfluid velocity). Thus there is a close connection of orbital rotational dynamics and spin dynamics.

In Sec. IV we discuss the case of a strong magnetic field and we find that the hydrodynamic equations become uniaxial while the number of hydrodynamic variables and modes remain unchanged. In addition we discuss the other special case (no magentic field) in which the structure of the order parameter induces a uniaxiality in the hydrodynamic equations, but in which the number of "symmetry" variables is reduced by one.

II. ORDER PARAMETER AND BROKEN SYMMETRIES

The spontaneous symmetry breakdown due to triplet pairing can be represented by a 3×3 matrix of anomalous expectation values,[46]

$$\langle \hat{T}_{ij}(|\vec{r}|, \vec{x})\rangle = \mathrm{Tr}\hat{\rho}_\eta \hat{T}_{ij}(|\vec{r}|, \vec{x}) \neq 0\,, \tag{2.1}$$

where $\hat{\rho}_\eta$ is the density operator in equilibrium,

$$\hat{\rho}_\eta = Z^{-1}\exp\Big[-\beta\Big(\hat{\mathcal{H}}_0 - \eta\int d^3x[\lambda^*_{ij}(\vec{x})\hat{A}_{ij}(\vec{x}) + \lambda_{ij}(\vec{x})\hat{A}^\dagger_{ij}(\vec{x})]\Big)\Big]\,, \tag{2.2}$$

with

$$\hat{\mathcal{H}}_0 = \hat{H} - \mu m\hat{N} - \vec{v}^n\cdot\hat{\vec{P}} - \gamma\vec{h}\cdot\hat{\vec{S}}\,, \tag{2.3}$$

and where $\hat{N}$ is the operator for the total particle number, $\hat{\vec{P}}$ is the operator for total linear momentum, and $\hat{\vec{S}}$ is the operator of the total spin. The Lagrange parameters $\mu, \vec{v}^n$ and $\vec{h}, \eta\lambda_{ij}$ are determined from the constraints on the expectation values of the associated variables. For $\hat{H}$ we have

$$\hat{H} = \int d^3x \left(-\frac{\hbar^2}{2m}\hat{\psi}^\dagger_\alpha(\vec{x})\nabla^2\hat{\psi}_\alpha(\vec{x})\right) + \int d^3x\, d^3x' \hat{\psi}^\dagger_\alpha(\vec{x})\hat{\psi}^\dagger_\beta(\vec{x}')V_{\alpha\beta\gamma\delta}(\vec{x},\vec{x}')\hat{\psi}_\gamma(\vec{x}')\hat{\psi}_\delta(\vec{x}) . \quad (2.4)$$

The order-parameter operator is defined by

$$\hat{T}_{ij}(|\vec{r}|,\vec{x}) = \frac{3}{4\pi}\int d\Omega_r \hat{\psi}_\alpha\left(\vec{x}-\frac{\vec{r}}{2}\right)(\sigma_i\sigma_2)_{\alpha\beta}\frac{r_j}{|\vec{r}|}\hat{\psi}_\beta\left(\vec{x}+\frac{\vec{r}}{2}\right), \quad (2.5)$$

where an integration has been performed over the solid angle of the relative coordinate of the Cooper pairs.

The structure of the condensate can be characterized by the structure matrix $A_{\mu i}$ (Ref. 3) where

$$\langle \hat{T}_{\mu i}(|\vec{r}|,\vec{x})\rangle = F(|\vec{r}|,\vec{x})A_{\mu i}(\vec{x}) , \quad (2.6)$$

$$A_{\mu i}A^*_{\mu i} = 1 . \quad (2.7)$$

Like in superfluid ^{3}He the complex amplitude $F(|\vec{r}|,\vec{x})$ of the condensate describes the degree of ordering and is therefore a microscopic variable,[46] whereas the structure of the condensate is a macroscopic property. The matrix A_{ij} is the expectation value of the operator

$$\hat{A}_{ij}(\vec{x}) = \int_0^\infty r^2F^*(|\vec{r}|,x)\hat{T}_{ij}(|\vec{r}|,\vec{x})\,dr , \quad (2.8)$$

where

$$\int_0^\infty r^2|F|^2dr = 1 . \quad (2.9)$$

In $^3P_2$-neutron-star matter $A_{\mu i}$ is in general[3] a complex, traceless, symmetric matrix which transforms under rotations of the orbital coordinates like a vector with respect to the index i and under rotations in spin space like a vector with respect to the index μ. For the normalized unitary equilibrium order parameter $A^0_{\mu i}$ we have[3]

$$A^0_{\mu i} = \mathfrak{N}e^{i\varphi}n_{\mu i} , \quad n_{\mu i} = \hat{u}_\mu\hat{u}_i + r\hat{v}_\mu\hat{v}_i - (1+r)\hat{w}_\mu\hat{w}_i , \quad (2.10)$$

where $\mathfrak{N} = [1+r^2+(1+r)^2]^{-1/2}$ and where the parameter r represents the excess degeneracy of the unitary manifold. It can be chosen in such a way[3] that the Ginzburg-Landau free energy associated with the $^3P_2$ superfluid is minimized. One obtains in this way $r=-\frac{1}{2}$ without external field if sixth-order terms are taken into account[3] and $r=-1$ if an external magnetic field is included.[3] In the most general case one has $-1<r<-\frac{1}{2}$ and we will study this case in detail in the following whereas the results for the end points of the interval $[-1, -\frac{1}{2}]$ will be included in our presentation for completeness. In Eq. (2.10) φ denotes the phase and $\hat{u}$, $\hat{v}$, $\hat{w}$ form an orthogonal triad of real unit vectors.

The order-parameter matrix $A^0_{\mu i}$ is symmetric in real- and spin-space indices; in addition it is possible to replace any of the unit vectors of the triad ($\hat{u}$, $\hat{v}$, $\hat{w}$) by its negative values (e.g., $\hat{u}_\mu$ by $-\hat{u}_\mu$) without changing the value of $A^0_{\mu i}$, i.e., $\hat{u}$, $\hat{v}$, $\hat{w}$ are headless as in nematic liquid crystals. It is not possible to distinguish between spin-space and real-space variables, since only combined rotations of spin and orbit space are symmetry elements of the Hamiltonian.

As candidates for hydrodynamic variables characterizing spontaneously broken continuous symmetries we have to consider the phase variable φ (as in any superfluid) and the deviations from the equilibrium values of the triad of unit vectors $\hat{u}$, $\hat{v}$, $\hat{w}$ in real and spin space, respectively, or equivalently the rotation angles about these axes.

Since the matrix $n_{\mu i}$ in (2.10) is traceless and symmetric and real it contains at most five independent variations. However, assuming that $\hat{u}$, $\hat{v}$, $\hat{w}$ remain orthogonal unit vectors[3] even in nonequilibrium situations leads to the conclusion that $n_{\mu i}$ can contain only three independent variations and it remains to be clarified which continuous symmetries are broken by these variables, i.e., to find the corresponding generators.

We take as variables

$$\delta\hat{\varphi} = \frac{1}{2i}(A^{*0}_{\mu i}\hat{A}_{\mu i} - A^0_{\mu i}\hat{A}^\dagger_{\mu i}) \quad (2.11)$$

and

$$\delta\hat{\psi}_1 = \tfrac{1}{4}(2r+1)^{-2}\vec{u}\cdot\vec{\hat{\Theta}} , \quad \delta\hat{\psi}_2 = \tfrac{1}{4}(r+2)^{-2}\vec{v}\cdot\vec{\hat{\Theta}} , \quad \delta\hat{\psi}_3 = \tfrac{1}{4}(r-1)^{-2}\vec{w}\cdot\vec{\hat{\Theta}} , \quad (2.12)$$

where

$$\hat{\Theta}_i = \epsilon_{ijk}(A^0_{\mu j}\hat{A}^\dagger_{\mu k} + \text{H.c.}) .$$

We find for the behavior under gauge transformations

$$\langle[\delta\hat{\varphi},\hat{N}]\rangle = -2i , \quad (2.13)$$

$$\langle[\delta\hat{\psi}_i,\hat{N}]\rangle = 0 , \quad (2.14)$$

i.e., as expected the phase variable $\delta\hat{\varphi}$ characterizes broken gauge symmetry. For the commutators with the magnetization density and the operator of angular momentum we have ($r\neq-\frac{1}{2}$)

$$\langle[\delta\hat{\varphi},\hat{L}_i]\rangle = 0 , \quad (2.15)$$

$$\langle[\delta\hat{\psi}_i,\hat{L}_j]\rangle = 2i\left(u_iu_j\frac{r(1+r)}{(1+2r)^2} + v_iv_j\frac{1+r}{(2+r)^2} - w_iw_j\frac{r}{(1-r)^2}\right) , \quad (2.16)$$

and

$$\langle[\delta\hat{\varphi},\hat{S}_\alpha]\rangle=0\,, \tag{2.17}$$

$$\langle[\delta\hat{\psi}_i,\hat{S}_j]\rangle=i\left(u_iu_j\frac{1+2r+2r^2}{(1+2r)^2}+v_iv_j\frac{2+2r+r^2}{(2+r)^2}+w_iw_j\frac{1+r^2}{(1-r)^2}\right), \tag{2.18}$$

i.e., we arrive at the conclusion that $\delta\hat{\varphi}$ is a scalar with respect to rotations in real space and spin space (2.15) and (2.17). As the most important information it turns out from Eqs. (2.16) and (2.18) that neither angular momentum nor magnetization serve as generators of a spontaneously broken symmetry. If we calculate, however, the commutators of the total angular momentum $J_i=L_i+S_i$ with $\delta\hat{\psi}_i$ and $\delta\hat{\varphi}$ we obtain the results

$$\langle[\delta\hat{\varphi},\hat{J}_i]\rangle=0\,, \tag{2.19}$$

$$\langle[\delta\hat{\psi}_i,\hat{J}_j]\rangle=i\delta_{ij}\,. \tag{2.20}$$

From (2.20) we are forced to the conclusion that $\delta\hat{\psi}_i$ and $\hat{J}_j$ are conjugate variables, i.e., total angular momentum serves as the generator of a spontaneously broken symmetry. This can easily be understood, since the Hamiltonian shows rotational invariance with respect to simultaneous rotations of spin and real space, while the actual $^3P_2$ superfluid does not, due to the existence of the triad $\hat{u}$, $\hat{v}$, $\hat{w}$. Therefore, total rotational invariance is spontaneously broken. This result is unique to $^3P_2$-neutron-star matter when compared to all other superfluid systems studied so far. It has to be contrasted, e.g., to the case of superfluid ^{3}He-B. In this phase relative spin-orbit symmetry is spontaneously broken, i.e.,

$$\langle[\delta\hat{\theta}_i,\hat{L}_j]\rangle=-\langle[\delta\hat{\theta}_i,\hat{S}_j]\rangle \tag{2.21}$$

or

$$\langle[\delta\hat{\theta}_i,\hat{J}_j]\rangle=0\,, \tag{2.22}$$

whereas in other hydrodynamic systems with broken rotational symmetries (e.g., ^{3}He-A, nematics, antiferromagnets) these symmetries are spontaneously broken in real space or spin space separately. To summarize, we have identified the hydrodynamic variables characterizing spontaneously broken continuous symmetries of $^3P_2$-neutron-star matter: One variable, the phase deviation $\delta\varphi$ or the superfluid velocity v_i^s, characterizes the broken gauge invariance and three variables ($\delta\psi_i$) are connected to the spontaneously broken total rotational invariance. Some further symmetry properties will be important in the following. Under time reversal we have

$$\hat{A}_{ij}\to\hat{A}_{ij}^\dagger\,,\quad A_{ij}\to A_{ij}^*\,. \tag{2.23}$$

Equations (2.11) and (2.12) imply then

$$\delta\hat{\psi}_i\to\delta\hat{\psi}_i\,,\quad \delta\hat{\varphi}\to-\delta\hat{\varphi}\,,$$

whereas the behavior of $\hat{u}_i$, $\hat{v}_i$, $\hat{w}_i$ remains unspecified. Under spatial inversion all four operators $\delta\hat{\varphi}$, $\delta\hat{\psi}_i$ have even parity. Under Galilei transformations generated by

$$\hat{G}=\frac{i}{\hbar}\delta\vec{v}^n\cdot\int d^3x\,\hat{\psi}_\alpha^\dagger m\vec{x}\hat{\psi}_\alpha\,, \tag{2.24}$$

the structure matrix $\hat{A}_{ij}$ transforms according to[46]

$$[\hat{G},\hat{A}_{ij}(\vec{x})]=\hat{A}_{ij}\exp\left(-\frac{2i}{\hbar}m\delta\vec{v}\cdot\vec{x}\right). \tag{2.25}$$

Therefore,

$$\hat{\vec{v}}^s(\vec{x})=\frac{\hbar}{2m}\nabla\delta\hat{\varphi}(\vec{x})$$

transforms as a velocity under Galilei transformations whereas $\delta\hat{\psi}_i$ are left invariant. We close this section with a discussion of the special cases $r=-\frac{1}{2}$ and $r=-1$ in Eq. (2.10). The considerations with respect to the phase remain unchanged. For the behavior of $\delta\hat{\psi}_i$ we have for $r=-\frac{1}{2}$,

$$\langle[\delta\hat{\psi}_i,\hat{J}_j]\rangle=9i(v_iv_j+w_iw_j)\,. \tag{2.26}$$

For $r=-1$ we obtain

$$\langle[\delta\hat{\psi}_i,\hat{L}_j]\rangle=\frac{i}{2}w_iw_j$$

and (2.27)

$$\langle[\delta\hat{\psi}_i,\hat{S}_j]\rangle=i(u_iu_j+v_iv_j+\tfrac{1}{2}w_iw_j)\,.$$

Thus we find that for $r=-\frac{1}{2}$ and $r=-1$ it is also the operator of total angular momentum which serves as a generator of the broken symmetries. It seems worth noticing that for $r=-\frac{1}{2}$ there is a close connection of the order parameter $n_{\mu i}$ with that of uniaxial nematic liquid crystals and as will be shown in Sec. IV the hydrodynamic equations for this value of r show an uniaxiality of the transport parameters and static susceptibilities whereas for other values of r ($-1<r<-\frac{1}{2}$) a biaxiality emerges. In addition we find that there exist for $r=-\frac{1}{2}$ (as in nematics) only two variables characterizing spontaneously broken continuous symmetries. For $r=-1$, the case of strong magnetic field,[3] we find three hydrodynamic variables characterizing broken symmetries and as will be shown in Sec. IV the hydrodynamic equations are uniaxial for that case.

III. HYDRODYNAMIC EQUATIONS

As conserved quantities we have the density ρ, the energy density $\mathcal{E}$ (or the entropy density σ), and the density of linear momentum $\vec{g}$.

In addition we have as conserved quantity the total angular momentum J_i which is the sum of angular momentum L_i and $(1/\gamma)M_i$, the magnetization. However, it is important to notice that J_i is a nonlocal quantity. Therefore, we cannot add to the Gibbs relation (which reflects local thermodynamic equilibrium) a term $\sim dJ_i$. Instead we consider local changes of the magnetization dM_i and changes of the orbital angular momentum dL_i, which can locally be expressed by changes of the linear momentum dg_i.[32] Of course, we have then to ensure that only the sum $(1/\gamma)dM_i+dL_i$ occurs and never the difference. As hydrodynamic variables characterizing spontaneously broken continuous symmetries we have to keep in our list the quantities $\delta\psi_i$ as introduced in Sec. II which describe the broken total rotational invariance. These hydrodynamic variables can be interpreted as deviations from the equilibrium values of the triad $\hat{u}$, $\hat{v}$, $\hat{w}$. Only inhomogeneous rotations lead to forces, which tend to restore the original state. For small gradients, these forces are small and the appropriate dynamics is slow. This is the hydrodynamic nature of the variables due to the spontaneously broken total rotational symmetry. From the changes $\delta\vec{u}$, $\delta\vec{v}$, $\delta\vec{w}$ only three are independent, since $\vec{u}^2=\vec{v}^2=\vec{w}^2=1$ and $\vec{u}\cdot\vec{v}=\vec{u}\cdot\vec{w}=\vec{v}\cdot\vec{w}=0$. We choose the components $\vec{v}\cdot\delta\vec{w}$, $(\vec{w}\times\vec{v})\cdot\delta\vec{w}$, $(\vec{v}\times\vec{w})\cdot\delta\vec{v}$, which can be viewed as the components of a vector $\delta\vec{\psi}$. Note that these hydrodynamic variables are defined only locally, but *not* globally. Consequences of this fact will be given below. The hydrodynamic variables defined above are related to the operators $\delta\hat{\psi}_i$ defined in Sec. II by

$$\begin{aligned}\vec{v}\cdot\delta\vec{w}&=\langle\delta\hat{\psi}_1\rangle_{\text{noneq}}\equiv\delta\psi_1\,,\\(\vec{w}\times\vec{v})\cdot\delta\vec{w}&=\langle\delta\psi_2\rangle_{\text{noneq}}\equiv\delta\psi_2\,,\\(\vec{v}\times\vec{w})\cdot\delta\vec{v}&=\langle\delta\psi_3\rangle_{\text{noneq}}\equiv\delta\psi_3\,.\end{aligned}\tag{3.1}$$

The starting point of any hydrodynamic theory is the Gibbs relation, which expresses the assumption of local thermodynamic equilibrium:

$$\begin{aligned}d\mathcal{E}=&\,T\,d\sigma+\mu\,d\rho+\vec{v}^n\cdot d\vec{g}+\phi_{ij}d(\nabla_j\psi_i)\\&+h_i\,d\psi_i+\vec{H}\cdot d\vec{M}+\tau_i\,d\nabla_i\rho+\Theta_i\,d\nabla_i\sigma\\&+\Lambda_{ij}d\nabla_iM_j+\lambda_i^s\,d\nabla_i\varphi\,,\end{aligned}\tag{3.2}$$

and for the pressure we obtain $(p\equiv -dE/dV)$,

$$\begin{aligned}p=&-\mathcal{E}+\mu\rho+T\sigma+\vec{v}^n\cdot\vec{g}+\vec{M}\cdot\vec{H}+\tau_i\nabla_i\rho\\&+\Theta_i\nabla_i\sigma+\Lambda_{ij}\nabla_iM_j\,.\end{aligned}$$

In writing down Eqs. (3.1) and (3.2) we have made use of the fact that indices in spin and real space can be interchanged at will (cf. Sec. II for a detailed discussion of this point). Furthermore, we have kept gradients of the conserved quantities ρ, σ, and M_i in (3.1) and (3.2). The necessity to do this has occurred previously while studying the superfluid phases of ^{3}He and the various types of liquid crystals.[16,20,21,39,47–49]

In Eqs. (3.2) expressions such as $d\psi_i$ or $\nabla_j\psi_i$ have to be interpreted, e.g., as $(\vec{v}\times\vec{w})\cdot d\vec{\psi}$ or $(\vec{v}\times\vec{w})\cdot\nabla_j\vec{\psi}$ according to our choice of hydrodynamic variables.

The second derivatives $d\nabla_j\psi_i$ occurring in Eq. (3.2) must be handled with care, since the hydrodynamic variables are globally not defined. That means that second derivatives of the hydrodynamic variables are not interchangeable, $d\nabla_j\psi_i\neq\nabla_j d\psi_i$. Instead one finds

$$\partial_1(\partial_2\vec{\psi})-\partial_2(\partial_1\vec{\psi})=(\partial_1\vec{\psi})\times(\partial_2\vec{\psi})\,,\tag{3.3}$$

where ∂_1, ∂_2 are any first-order differential operators and where $\delta\vec{\psi}$ is the vector of the hydrodynamic variables $(\vec{v}\cdot\delta\vec{w},\ [\vec{w}\times\vec{v}]\cdot\delta\vec{w},\ [\vec{v}\times\vec{w}]\cdot\delta\vec{v})$.

Analogous anholonomity relations have been given previously, e.g., by the authors for biaxial nematics.[21] In this case angular momentum serves as the generator of the broken symmetries. The first occurrence of noncommuntativity relations for hydrodynamic variables has been noticed by Mermin and Ho for ^{3}He-A (Ref. 30) without external field and subsequently for the other superfluid phases of ^{3}He. For the phase $\delta\varphi$ (or the superfluid velocity v_i^s) there exists no anholonomity relation, i.e.,

$$\nabla\times\vec{v}^s\equiv 0\tag{3.4}$$

in $^3P_2$-neutron-star matter.

The conservation and quasiconservation laws take the form

$$\frac{\partial}{\partial t}\rho+\nabla\cdot\vec{g}=0\,,\tag{3.5}$$

$$\frac{\partial}{\partial t}\sigma+\nabla\cdot\vec{q}=\frac{R}{T}\,,\tag{3.6}$$

$$\frac{\partial}{\partial t}g_i+\nabla_j\sigma_{ij}=0\,,\tag{3.7}$$

$$\frac{\partial}{\partial t}\mathcal{E}+\nabla\cdot\vec{j}^{\mathcal{E}}=0\,,\tag{3.8}$$

$$\frac{\partial}{\partial t}M_i+\nabla_j j^M_{ij}=Q^M_i\,,\tag{3.9}$$

$$\frac{\partial}{\partial t}\psi_i+X_i=0\,,\tag{3.10}$$

$$\frac{\partial}{\partial t}\varphi+I_\varphi=0\,.\tag{3.11}$$

Of course, $\delta\psi_i$ and $\delta\varphi$ are not conserved and Eqs. (3.10) and (3.11) are simple balance equations. The source term in Eq. (3.6) is the entropy production, which is zero or positive on the reversible and irreversible levels, respectively. As already stated the magnetization is not conserved.

Q_i^M is related to the antisymmetric part of the stress tensor which acts as the source term of the conservation law for the orbital angular momentum density. For the time derivative $\dot{J}_i$ one finds

$$\frac{\partial}{\partial t}J_i=\frac{\partial}{\partial t}(\vec{r}\times\vec{g})_i+\frac{2}{\gamma}\frac{\partial}{\partial t}M_i\,, \tag{3.12}$$

and with the help of (3.7) and (3.9)

$$\frac{\partial}{\partial t}J_i=-\nabla_p\Xi_{ip}+\epsilon_{ijk}\sigma_{kj}+\frac{2}{\gamma}Q_i^M\,, \tag{3.13}$$

where

$$\Xi_{ip}=\epsilon_{ijk}r_j\sigma_{kp}+\frac{2}{\gamma}j_{ip}^M\,.$$

The total angular momentum conservation requires, therefore,

$$\epsilon_{ijk}\sigma_{jk}=\frac{2}{\gamma}Q_i^M\,. \tag{3.14}$$

Thus, Eq. (3.14) together with (3.9) is a local formulation for the conservation law for total angular momentum. To close the system of hydrodynamic equations we have (1) to express the thermodynamic conjugate quantities as defined in Eq. (3.2) in terms of the hydrodynamic variables and (2) to relate the currents and quasicurrents as introduced in Eqs. (3.5)–(3.11) to the thermodynamic conjugate quantities.

A. Statics

For convenience we will express $\mathcal{E}$ in terms of $\vec{v}^n$ instead of $\vec{g}$. Then we find for $F\equiv\mathcal{E}-\vec{g}\cdot\vec{v}^n$ up to third-order terms (quantities such as $\delta\rho$, $\nabla_i\varphi$, $\nabla_i\psi_j$ are counted as first-order terms whereas $\nabla_i\sigma$, $\nabla_i\rho$ are said to be of second order)

$$F=F^h+F^g+F^{(3)}\,, \tag{3.15}$$

where F^h contains the homogeneous part (magnet, simple fluid), F^g is the gradient part due to the broken symmetries, and $F^{(3)}$ contains cubic terms (higher-gradient order terms), which gives rise to nonlinear susceptibilities:

$$F^h=\tfrac{1}{2}\chi_{ij}\delta m_i\delta m_j+\tfrac{1}{2}T_0C_v^{-1}(\delta\sigma)^2+\zeta_4(\delta\rho)^2+\zeta_3(\delta\rho)(\delta\sigma)\,, \tag{3.16}$$

$$\begin{aligned}F^g=&\tfrac{1}{2}\rho_{ij}^s(v_i^n-\nabla_i\varphi)(v_j^n-\nabla_j\varphi)-\tfrac{1}{2}\rho\vec{v}^{n2}+A_1(\delta_{ik}^3\nabla_i w_k)^2+A_2(v_iv_k\nabla_i w_k)^2+A_3\delta_{ik}^3v_iv_l(\nabla_k w_i)(\nabla_l w_j)\\&+A_4\delta_{il}^3v_jv_k(\nabla_i w_j)(\nabla_l w_k)+A_5\delta_{il}^3v_jv_k(\nabla_j w_i)(\nabla_k w_l)+A_6\delta_{il}^3v_jv_k(\nabla_j w_i)(\nabla_k w_l)+A_7\delta_{ik}^3v_jv_l(\nabla_j w_i)(\nabla_l w_k)\\&+A_8w_jw_lv_iv_k(\nabla_j w_i)(\nabla_l w_k)+(A_9\delta_{ik}^3\delta_{jl}^3+A_{10}\delta_{ik}^3w_jw_l+A_{11}\delta_{ik}^3v_jv_l)(\nabla_j v_i)(\nabla_l v_k)\\&+\delta_{il}^3w_jw_k[A_{12}(\nabla_j w_k)(\nabla_i v_l)+A_{13}(\nabla_i w_k)(\nabla_j v_l)+A_{14}(\nabla_k w_i)(\nabla_j v_l)+A_{15}(\nabla_j w_i)(\nabla_k v_l)]\\&+D_{kl}(\nabla_i M_k)\epsilon_{lij}(\nabla_j\varphi-v_j^n)\,,\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}F^{(3)}\equiv&B_{ijk}^{(\alpha)}(\nabla_i w_j)(\nabla_k\rho)+C_{ijk}^{(\alpha)}(\nabla_i v_j)(\nabla_k\rho)+B_{ijk}^{(\beta)}(\nabla_i w_j)(\nabla_k\sigma)+C_{ijk}^{(\beta)}(\nabla_i v_j)(\nabla_k\sigma)\\&+(A_{ij}\delta\rho+B_{ij}\delta\sigma)(\nabla_i\varphi-v_i^n)(\nabla_j\varphi-v_j^n)+\zeta_{ijkl}M_k(\nabla_l\varphi-v_l^n)\nabla_j w_i+\xi_{ijkl}M_k(\nabla_l\varphi-v_l^n)\nabla_j v_i\,,\end{aligned} \tag{3.18}$$

and where $\delta_{kl}^3=\delta_{kl}-v_kv_l-w_kw_l$ and D_{kl}, A_{kl}, B_{kl}, and ρ_{ij}^s are of the biaxial form

$$t_{ij}=t_1\delta_{ij}^3+t_2v_iv_j+t_3w_kw_l\,, \tag{3.19}$$

whereas for $B_{ijk}^{(\alpha,\beta)}$ and $C_{ijk}^{(\alpha,\beta)}$ we find

$$\begin{aligned}C_{ijk}^{(\alpha,\beta)}&=v_i(C_1^{(\alpha,\beta)}\delta_{jk}^3)+C_2^{(\alpha,\beta)}v_k\delta_{ij}^3\,,\\B_{ijk}^{(\alpha,\beta)}&=w_i(B_1^{(\alpha,\beta)}\delta_{jk}^3+B_2^{(\alpha,\beta)}v_kv_j)\\&\quad+w_k(B_3^{(\alpha,\beta)}\delta_{ij}^3+B_4^{(\alpha,\beta)}v_iv_j)\,.\end{aligned} \tag{3.20}$$

The tensors ζ_{ijkl} and ξ_{ijkl} have been included for completeness and bring into play a plethora of further terms.

The equations of state are obtained from Eqs. (3.20)–(3.15) via differentiation:

$$\begin{aligned}\tilde{\mu}&\equiv\mu-\nabla_i\tau_i=\frac{\partial F}{\partial\rho}-\nabla_i\frac{\partial F}{\partial\nabla_i\rho}\,,\\\tilde{T}&\equiv T-\nabla_i\Theta_i=\frac{\partial F}{\partial\sigma}-\nabla_i\frac{\partial F}{\partial\nabla_i\sigma}\,,\\\tilde{H}_i&\equiv H_i-\nabla_j\Lambda_{ji}=\frac{\partial F}{\partial M_i}-\nabla_j\frac{\partial F}{\partial\nabla_j M_i}\,,\\\vec{g}&=-\frac{\partial F}{\partial\vec{v}^n}\,,\quad\vec{\lambda}^s=\frac{\partial F}{\partial\vec{\nabla}\varphi}\,,\\h_i&=\frac{\partial F}{\partial\psi_i}\,,\quad\phi_{ij}=\frac{\partial F}{\partial\nabla_j\psi_i}\,,\end{aligned} \tag{3.21}$$

with

$$\vec{\psi}=(\vec{v}\cdot\delta\vec{w},\ (\vec{w}\times\vec{v})\cdot\delta\vec{w},\ (\vec{v}\times\vec{w})\cdot\delta\vec{v})\,.$$

The energy density $\mathcal{E}$ defined above shows the correct Galilean transformation behavior $\mathcal{E}\to\mathcal{E}+\frac{1}{2}\rho U^2+\vec{U}\cdot\vec{g}$. By explicit differentiation we find in addition $\vec{g}=\rho\vec{v}^n+\vec{\lambda}^s$. As is obvious from the first three equations of (3.21) it is most convenient to expand the currents and quasicurrents into the renormalized quantities $\tilde{\mu}$, $\tilde{T}$, and $\tilde{H}_i$.

The terms involving the coefficients $A_1,\ldots,A_{15}$ in Eq. (3.17) are identical in structure with the corresponding terms in biaxial nematic liquid crystals, the term proportional to D_{kl} in F^g describes a coupling of the vorticity of the magnetic field and the velocity fields $\nabla_j\varphi$ and v_j^n. [Some consequences of this term will be indicated after Eq. (3.31)] and the other contributions to F^g are that of a usual superfluid (apart from the biaxial form of the tensor of the superfluid densities). Concerning the structure of the third-order contributions contained in $F^{(3)}$ it seems worthwhile to notice that there exist terms which describe couplings of the superfluid velocity v_i^s and the variables characterizing the broken total rotational invariance: ζ_{ijkl}, ξ_{ijkl} (apart from terms which also exist for liquid crystals and the superfluid phases of ^{3}He). After having specified completely the static behavior we turn now to the description of the dynamics, i.e., the investigation of the irreversible and reversible currents and quasicurrents.

B. Dynamics

Inserting the equations of motion (3.5)–(3.11) into the Gibbs relation (3.2) one obtains for the entropy production the relation

$$-R=-\nabla_i S_i+g_i\nabla_i\tilde{\mu}+\sigma_{ik}\nabla_i v_k^n-\tilde{H}_i(\nabla_k j_{ik}^m-Q_i^m)+q_i\nabla_i\tilde{T}+I_\varphi\nabla_i\lambda_i^s+\Pi_i X_i\,, \tag{3.22}$$

where S_i is a generalized energy current (whose specific form we do not need to know), the renormalized thermodynamics forces $\tilde{T}$, $\tilde{\mu}$, $\tilde{H}_i$ are defined by (3.21) and $\Pi_i\equiv h_i-\nabla_j\phi_{ij}+\phi_{lj}\epsilon_{lim}\nabla_j\psi_m$; the last term of this relation is due to the anholonomity condition (3.3).

Making use of the conservation of total angular momentum (3.14) we can express Q_i^M by the antisymmetric part of σ_{ij}. Splitting σ_{ij} in a symmetric and an antisymmetric part we then obtain

$$\sigma_{ik}\nabla_i v_k^n-\tilde{H}_i(\nabla_k j_{ik}^M-Q_i^M)=\sigma_{ik}^{\text{sym}}(\nabla_i v_k^M)^{\text{sym}}+\sigma_{ik}^{\text{anti}}\epsilon_{jik}\Omega_j-\tilde{H}_i\nabla_k j_{ik}^M$$

with

$$\Omega_j\equiv-\omega_j+\tfrac{1}{2}\gamma\tilde{H}_j\quad(\vec{\omega}\equiv\tfrac{1}{2}\nabla\times\vec{v}^n).$$

Thereby we have obtained $\vec{\Omega}$, the thermodynamic conjugate to the total angular momentum $\vec{J}$, as a thermodynamic force; $\vec{\Omega}$ is a local quantity in contrast to $\vec{J}$. If we now invoke the behavior of the currents under a Galilean transformation with constant velocity $\vec{U}$

$$\begin{aligned}&g_i\to\rho U_i+g_i\,,\\&q_i\to\sigma U_i+q_i\,,\\&\sigma_{ij}\to\rho U_iU_j+g_iU_j+g_jU_i+\sigma_{ij}\,,\\&j_{ik}^M\to M_iU_k+j_{ik}^M\,,\\&I_\varphi\to U_i\nabla_i\varphi+I_\varphi\,,\\&X_i\to U_k\nabla_k\psi_i+X_i\,,\end{aligned}\tag{3.23}$$

and the pressure term $p\delta_{ij}$ in the stress tensor, we obtain after a lengthy and puzzling rearrangement of terms the relation

$$-R=-\nabla_i S_i'+q_i'\nabla_i\tilde{T}+g_i'\nabla_i\tilde{\mu}+\sigma_{ij}'^{\text{sym}}(\nabla_j v_i^n)^{\text{sym}}+\epsilon_{ijk}\sigma_{jk}'\Omega_i+j_{ik}'^M\nabla_k\tilde{H}_i+I_\varphi'\nabla_i\lambda_i^s-X_i'\Pi_i\,, \tag{3.24}$$

which relates the yet unknown parts of the currents (denoted by a prime) to the (renormalized) thermodynamic forces. The currents, which enter the dynamic equations (3.5)–(3.11) are thereby related to the primed quantities by

$$\begin{aligned}&g_i=\rho v_i^n+g_i'\,,\\&q_i=\sigma v_i^n+q_i'\,,\\&2\sigma_{ij}^{\text{sym}}=\delta_{ij}(p-\nabla_k\rho\tau_k-\nabla_k\sigma\theta_k)\\&\qquad+g_iv_j^n+\lambda_j^s\nabla_i\varphi+\tau_j\nabla_i\varphi+\tau_j\nabla_i\rho\\&\qquad+\theta_j\nabla_i\sigma+\phi_{kj}\nabla_i\psi_k-M_k\nabla_i\Lambda_{jk}+\sigma_{ij}'^{\text{sym}}+(i\leftrightarrow j)\,,\\&2\sigma_{ij}^{\text{anti}}=\nabla_jM_k\Lambda_{ik}+H_iM_j+\Lambda_{pi}\nabla_pM_j+\nabla_k\phi_{ik}\psi_j\\&\qquad+\sigma_{ij}'^{\text{anti}}-(i\leftrightarrow j)\,,\\&j_{ik}^M=v_k^nM_i+\tfrac{1}{2}\gamma(\epsilon_{ljk}M_j\Lambda_{il}-\epsilon_{lkj}M_j\Lambda_{lj}-\epsilon_{lji}\phi_{lk}\psi_j)+j_{ik}'^M\,,\\&I_\varphi=v_i^n\nabla_i\varphi+I_\varphi'\,,\\&X_i=v_k^n\nabla_k\psi_i+\epsilon_{ijk}\omega_k\psi_j+X_i'.\end{aligned}\tag{3.25}$$

The primed parts of the currents must be Galilean invariant and may therefore not depend on $\vec{v}^n$ (or $\vec{\nabla}\varphi$). The term H_iM_j in $\sigma_{ij}^{\text{anti}}$ gives the well-known Larmor term in the dynamic equation for M_i. Taking into account the static relation $\vec{g}=\rho\vec{v}^n+\vec{\lambda}^s$ and expanding the remaining parts of the currents into the thermodynamic forces we close the dynamics.

On the reversible level ($R=0$) we obtain with the help of (3.24)

$$\begin{aligned}&g_i'=\lambda_i^s,\quad q_i'=0,\quad I_\varphi'=\tilde{\mu},\quad j_{ik}'^M=0,\\&\sigma_{ij}'^{\text{sym}}=\alpha_{ijk}\Pi_k,\quad\epsilon_{ijk}\sigma_{jk}'=\gamma_{ij}\Pi_j\,,\\&X_i'=\gamma_{ji}\Omega_j+\alpha_{kji}(\nabla_jv_k^n)^{\text{sym}}\,.\end{aligned}\tag{3.26}$$

If Eqs. (3.26) are written down in their linearized form, $\gamma_{ij}=\gamma_1\delta^3_{ij}+\gamma_2 v_i v_j+\gamma_3 w_i w_j$ can be evaluated rigorously via the commutators considered in Sec. II, Eq. (2.20), and we find explicitly

$$\gamma_1=\gamma_2=\gamma_3=1 \quad \text{or} \quad \gamma_{ij}=\delta_{ij}\,, \tag{3.27}$$

i.e., the generator property of the total angular momentum is an inherent property of Eqs. (3.26).

The tensor α_{ijk} is symmetric in the two last indices and contains three phenomenological parameters α_1, α_2, α_3:

$$\alpha_{jki}=(\alpha_1 v_i v_k+\alpha_2\delta^3_{ik})w_j+\alpha_3\delta^3_{ik}v_j+(j\leftrightarrow k)\,. \tag{3.28}$$

When the linearized hydrodynamic equations are considered in the framework of the projector formalism (introduced into hydrodynamics by D. Forster[15]) it is found that the memory matrix, which contains the noninstantaneous, collisional effects, yields contributions in the hydrodynamic regime to the phenomenological parameters α_1, α_2, and α_3. As has been noticed by the authors[21] all hydrodynamic systems studied so far which can be characterized by broken rotational symmetries in real space (^{3}He-A, ^{3}He-A$_1$, nematics, biaxial nematics, smectics C, ^{3}He-A in high magnetic fields) allow for the existence of such contributions whereas all other hydrodynamic systems (smectics A, simple fluids, ferromagnets, and antiferromagnets, etc.) do not show these terms. From the existence of such terms in the present system, $^3P_2$-neutron-star matter, we may conclude that the same holds for systems with broken total rotational invariance (which involves, of course, a contribution from broken rotational invariance in real space).

For the irreversible currents, we find with the help of Eq. (3.24) using symmetry arguments and the constraint $R>0$

$$\begin{aligned}
&g_i'=0\,,\\
&q_i'=-\kappa_{ij}\nabla_j\tilde{T}-\xi_{kji}\nabla_j\Pi_k\,,\\
&\sigma_{ij}'^{\,\text{sym}}=\nu_{ijkl}(\nabla_k v_l^n)^{\text{sym}}-\xi_{ij}\nabla_k\lambda_k^s\,,\\
&\sigma_{ij}'\epsilon_{kij}=\mu_{kjli}\nabla_j\nabla_l\Omega_i-\mu_{ki}\Omega_i\,,\\
&I_\varphi'=-\zeta\nabla_i\lambda_i^s-\xi_{ij}(\nabla_j v_i^n)^{\text{sym}}\,,\\
&X_i'=\bar{\zeta}_{ij}\Pi_j-\xi_{ijk}\nabla_k\nabla_j\tilde{T}\,,\\
&j_{ik}'^{M}=-\eta_{iklj}\nabla_l\tilde{H}_j\,.
\end{aligned} \tag{3.29}$$

The second-rank tensors κ_{ij}, ξ_{ij}, $\bar{\zeta}_{ij}$, μ_{ik}, the third-rank tensor ξ_{kji}, and the fourth-rank tensors η_{ijkl}, ν_{ijkl}, μ_{ijkl} are of the form

$$\begin{aligned}
A_{ij}&=A_1\delta^3_{ij}+A_2 v_i v_j+A_3 w_i w_j\,,\\
B_{ijk}&=(B_1 v_i v_k+B_2\delta^3_{ik})w_j+B_3\delta^3_{ik}v_j+(j\leftrightarrow k)\,,\\
C_{ijkl}&=C_1 v_i v_j v_k v_l+C_2 w_i w_j w_k w_l+C_3\delta^3_{ij}\delta^3_{kl}\\
&\quad+C_4 v_i v_j w_k w_l+C_5\delta^3_{kl}v_i v_j+C_6 w_i w_j\delta^3_{kl}\\
&\quad+C_7 v_i w_j w_l v_k+C_8 v_i v_k\delta^3_{jl}+C_9 w_i w_k\delta^3_{jl}\,,
\end{aligned} \tag{3.30}$$

respectively.

Next we discuss briefly the spectrum of the normal modes in $^3P_2$-neutron-star matter, which is obtained from the linearized hydrodynamic equations. Since we have 12 variables, we have to expect a total of 12 normal modes, damped or propagating excitations. Three of them are nonhydrodynamic $[\omega(k\to 0)\neq 0]$, since the three components of $\vec{m}$ are not conserved. The system under investigation is complicated by the fact that it is biaxial, and this feature leads to an intricate coupling of all variables involved. Nevertheless, one can extract some qualitative results.

Restricting for the moment to linear order in $\vec{k}$, the variables ρ, σ, φ, and $\nabla\cdot\vec{g}$ decouple from the rest and build up first and second sounds. Their velocities are given by the following equations:

$$C_1^2C_2^2=\frac{\sigma_0^2\rho_0^2T_0\hat{\rho}^2}{\rho_0\kappa_T C_P+\alpha^2T_0}\,,$$

$$C_1^2+C_2^2=\frac{1}{\rho_0\kappa_T C_P+\alpha^2T_0}\,(C_P+\rho_0^3\kappa_T\sigma_0^2T_0\hat{\rho}^2)$$

with

$$\hat{\rho}^2=\left(\frac{\rho_1^s}{\rho_1^n}\right)\left(\frac{k_1^2}{k^2}\right)+\left(\frac{\rho_2^s}{\rho_2^n}\right)\left(\frac{k_2^2}{k^2}\right)+\left(\frac{\rho_3^s}{\rho_3^n}\right)\left(\frac{k_3^2}{k^2}\right)\,,$$

$$\kappa_T\equiv\frac{1}{\rho_0}\left(\frac{\partial\rho}{\partial p}\right)_T,\quad \alpha=-\frac{1}{\rho_0}\left(\frac{\partial\rho}{\partial T}\right)_P,\quad C_P=T_0\left(\frac{\partial\rho\sigma}{\partial T}\right)_P.$$

The other variables ψ_i, m_i, and $\nabla\times\vec{g}$ produce three (coupled) pairs of spin-orbit waves. The term "spin-orbit waves" is chosen, because these modes arise from the coupling of the variables characterizing the broken total rotational invariance to the thermodynamic conjugate of the generator of this symmetry. This situation has to be contrasted to superfluid ^{3}He-B, where spin-orbit waves describe oscillations of spin relative to orbit space.[28,35] However, these waves are overdamped in $^3P_2$-neutron-star matter due to the nonconservation of $\vec{m}$. Their frequency spectra are of the form $\omega^2=C_{so}^2k^2+i\omega D$, where C_{so}^2 depend on the direction of $\vec{k}$, and where the relaxation constants D are proportional to the coefficients μ_1, μ_2, μ_3. If an external magnetic field is present, the Larmor precession changes two of the spin-orbit waves into one with a gap $\omega^2=\omega_L'^2+C_{so}'^2k^2$

$+i\omega D'$ [where ω_L' is the Larmor frequency shifted due to finite relaxation times $\omega_L'^2 \equiv (\gamma H)^2 - (\gamma/2)^2(\mu_1\chi_1 - \mu_2\chi_2)^2$; for the case of very strong magnetic fields $\omega_L' = \omega_L$, because $\mu_1\chi_1 = \mu_2\chi_2$] and one with $\omega^2 = 0$ (up to linear order in $\vec{k}$).

In order k^2 all variables are coupled. All one can say is, that the sound waves (and spin-orbit waves) acquire a damping proportional to k^2 and that those modes with $\omega = 0$ in linear order now have dispersion relations $\omega \sim Ek^2$. Thereby E is a complex function of reversible and irreversible transport parameters and static susceptibilities as well as of the direction of $\vec{k}$. Whether these modes are propagating ($\mathrm{Re}E \gg \mathrm{Im}E$) or diffusive ($\mathrm{Im}E \gg \mathrm{Re}E$) depends on the actual values of the hydrodynamic parameters involved. All modes reflect strongly in both, the velocities and the damping coefficients, the biaxial character of $^3P_2$-neutron-star matter. Furthermore, we would like to stress that the spectrum of the normal modes of superfluid $^3P_2$-neutron-star matter is different from the corresponding excitations in a biaxial nematic liquid crystal, and this may be traced back to the role played by the magnetization density which is absent in the latter system.

Before closing this section on the hydrodynamic equations we turn to the discussion of an interesting stationary solution of the hydrodynamic equations.

If we confine ourselves for the moment to linear reversible terms and if we set all time derivatives equal to zero we obtain from Eqs. (3.5)–(3.12), (3.15)–(3.18), and (3.21), (3.24)–(3.28)

$$\begin{aligned}
&\nabla_i[\rho_{ij}^n v_j^n + \rho_{ij}^s v_j^s + D_{ij}\epsilon_{jkl}\nabla_k M_l] = 0\,,\\
&\Omega_i + \alpha_{kji}(\nabla_j v_k^n)^{\text{sym}} = 0\,,\\
&\nabla_i v_i^n = 0\,,\\
&\nabla_j(p\delta_{ij} + \alpha_{ijk}\Pi_k + \tfrac{1}{2}\epsilon_{ijk}\Pi_k) = 0\,.
\end{aligned} \tag{3.31}$$

We investigate a rotation about the z axis:

$$\omega_z \neq 0\,,\quad \omega_x = \omega_y = 0\,,\quad \nabla_i v_j^n + \nabla_j v_i^n = 0 \quad (i \neq j)\,.$$

We have then from Eq. (3.31)

$$\begin{aligned}
&\Omega_z \equiv 0 \text{ or } \tilde{H}_z = +\frac{2}{\gamma}\omega_z\,,\\
&\tilde{H}_x = 0\,,\ \tilde{H}_y = 0\,,\\
&(\rho_{xx}^n - \rho_{zz}^n)\nabla_x v_x^n + (\rho_{yy}^n - \rho_{zz}^n)\nabla_y v_y^n + \rho_{ij}\nabla_i\nabla_j\varphi\\
&\qquad + (D_{xx} - D_{yy})\nabla_x\nabla_y M_z = 0\,.
\end{aligned} \tag{3.32}$$

From Eq. (3.32) we find that an angular velocity gives rise to a magnetic field in the same direction and vice versa. If the rotation ω_z is inhomogeneous, it induces a superfluid counterflow $\nabla_i\varphi$, which is also determined by the vorticity of the magnetization (or magnetic field). The latter effect is absent for the uniaxial special cases $r = -\frac{1}{2}$ and $r = -1$, where $D_{xx} = D_{yy}$.

On the other hand, we find that in $^3P_2$-neutron-star matter magnetic fields can induce an angular velocity and—if they are inhomogeneous—a superfluid current as well.

IV. ON THE UNIAXIALITY OF THE HYDRODYNAMIC EQUATIONS FOR $r = -\frac{1}{2}$ AND $r = -1$

For $r = -\frac{1}{2}$ the structure matrix $A_{\mu i}^0 = e^{i\varphi} n_{\mu i}^0$ takes the form

$$\begin{aligned}
n_{\mu i}^0 &= (\tfrac{2}{3})^{1/2}(u_\mu u_i - \tfrac{1}{2}v_\mu v_i - \tfrac{1}{2}w_\mu w_i)\\
&= (\tfrac{3}{2})^{1/2}(u_\mu u_i - \tfrac{1}{3}\delta_{\mu i})\,,
\end{aligned} \tag{4.1}$$

and, as has been already pointed out in Sec. II, only two of the variables characterizing the broken total rotational invariance survive for this value of r, because $\Theta^1 = \vec{w}\cdot\delta\vec{v}$ looses its hydrodynamic behavior. To obtain the hydrodynamic equations for $r = -\frac{1}{2}$ we have just to delete from the hydrodynamic equations given in Sec. III the variable $\vec{w}\delta\vec{v}$ and the corresponding thermodynamic conjugate force. By inspection of the remaining hydrodynamic variables and equations one arrives immediately at the conclusion that the hydrodynamic equations become uniaxial because of the impossibility of constructing a second preferred direction.

In this respect the case of $^3P_2$-neutron-star matter with $r = -\frac{1}{2}$ resembles that of a uniaxial nematic liquid crystal. Owing to the uniaxiality the number of phenomenological parameters involved reduces considerably. The tensor of the superfluid densities ρ_{ij}^s contains, e.g., only two components instead of three, etc.

As has been discussed by Muzikar, Sauls, and Serene[3] the case $r = -\frac{1}{2}$ is realized (in Ginzburg-Landau type calculations) if sixth-order terms (but no magnetic fields) are taken into account, thus lifting the degeneracy which exists if only fourth-order terms are considered.

Thus we have shown above that this effect is accompanied on the hydrodynamic level by a reduction of the number of variables characterizing the spontaneously broken total rotational invariance.

This result has to be contrasted to the case $r = -1$, which is realized in the presence of a magnetic field (and sixth-order terms neglected).[3] For this value of r, $n_{\mu i}^0$ reads

$$n_{\mu i}^0 = \frac{1}{\sqrt{2}}(u_\mu u_i - v_\mu v_i)\,. \tag{4.2}$$

In Sec. II it was already shown that the number of hydrodynamic variables does not change if r is approaching the value −1. (Strictly speaking the fluctuations of $\hat{w}$ are no longer true hydrodynamic variables, since $\hat{w}$ is parallel to $\hat{H}$ and the modes due to $\delta\hat{w}$ acquire gaps proportional to H^2.) Nevertheless, we find that the hydrodynamic equations presented in Sec. III for $-1 < r < -\frac{1}{2}$ have to be modified. By inspection of Eq. (4.2) it is easily checked that it is not possible to construct a second preferred direction (apart from $\vec{w} \parallel \vec{H}$) and this result has already been obtained by Richardson.[6] Thus we obtain the hydrodynamic equations for $r = -1$ from the general equations given in Sec. III by switching the biaxiality to a uniaxiality via restrictions of the type

$$\rho^s_{uu} = \rho^s_{vv}\,, \quad \rho^n_{uu} = \rho^n_{vv}\,,$$

etc. Therefore, in both cases, $r = -\frac{1}{2}$ and $r = -1$, the hydrodynamic equations reflect the uniaxiality of the gap. In the former case, there are two hydrodynamic variables connected with spontaneously broken total rotational symmetry, while in the latter case, there are three.

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