Geodetic precession in a Kerr field

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An exact calculation is presented for the geodetic precessional frequency of a gyroscope moving in an equatorial circular orbit in a Kerr field.

I. INTRODUCTION

The precession of a freely falling gyroscope following a circular orbit in Schwarzschild space time was calculated to the lowest order of approximation by Schiff.[1] An exact value of such a precessional frequency has recently been calculated by Sakina and Chiba.[2] Since the earth is a rotating body, it would be more appropriate to calculate the exact precessional frequency in a Kerr space-time. In this paper, we present an exact calculation for an equatorial circular orbit in a Kerr field.

II. DETAILED CALCULATIONS

The equation of the parallel propagator $\tilde{g}^i{}_{j_0}$ generating parallel vector fields along a curve C: $x^i = x^i(\tau)$ in a Riemannian manifold of n dimensions is[2-4]

$$\frac{\delta \tilde{g}^i{}_{j_0}}{\delta u} = \frac{d\tilde{g}^i{}_{j_0}}{du} + \Gamma^i_{lk}\frac{dx^l}{du}\tilde{g}^k{}_{j_0} = 0 \ . \tag{1}$$

Equation (1) has been solved by Sakina and Chiba[2] for a circular orbit in the Schwarzschild space-time and an exact value of the precessional frequency obtained using the concept of the parallel propagator. In this paper, Eq. (1) has been solved for an equatorial circular orbit in a Kerr field and a generalized value of the precessional frequency obtained which reduces to the value for the Schwarzschild space-time in the absence of rotation.

In Kerr space-time the metric form is

$$ds^2 = \left(1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\right) dt^2 - \frac{r^2 + a^2\cos^2\theta}{r^2 - 2mr + a^2}\, dr^2 - (r^2 + a^2\cos^2\theta)d\theta^2 - \left(r^2 + a^2 + \frac{2mra^2\sin^2\theta}{r^2 + a^2\cos^2\theta}\right)\sin^2\theta d\phi^2 + \frac{4amr\sin^2\theta}{r^2 + a^2\cos^2\theta}\, d\phi\, dt \ , \tag{2}$$

where a is the angular momentum per unit mass.

The circular orbit lying in the equatorial plane with an affine parameter τ is given by

$$r = r_0\ , \quad \theta = \frac{\pi}{2}\ , \quad \phi = \left(\frac{\epsilon Q + LP}{PR + Q^2}\right)\tau\ , \quad t = \left(\frac{\epsilon R - LQ}{PR + Q^2}\right)\tau\ , \tag{3}$$

where

$$b = m/r_0\ ,$$

$$P = 1 - 2b\ ,$$

$$Q = 2ab\ ,$$

and

$$R = r_0{}^2 + a^2 + 2a^2 b\ .$$

ϵ and L are constants related to the energy and angular momentum, respectively, of an orbiting body and are[5]

$$\epsilon = \frac{1 - 2b \pm ab^{1/2}/r_0}{(1 - 3b \pm 2ab^{1/2}/r_0)^{1/2}}\ ,$$

$$L = \frac{\pm r_0 b^{1/2}(1 \mp 2ab^{1/2}/r_0 + a^2/r_0{}^2)}{(1 - 3b \pm 2ab^{1/2}/r_0)^{1/2}}\ ,$$

where the upper sign refers to corotation and the lower sign to counterrotation of the orbiting body.

The equation of the parallel propagator (1) is split as follows making use of (2) and (3):

$$\frac{d\tilde{g}^1{}_{j_0}}{d\tau} + \left(\Gamma^1_{33}\frac{\epsilon Q + LP}{PR + Q^2} + \Gamma^1_{43}\frac{\epsilon R - LQ}{PR + Q^2}\right)\tilde{g}^3{}_{j_0} + \left(\Gamma^1_{34}\frac{\epsilon Q + LP}{PR + Q^2} + \Gamma^1_{44}\frac{\epsilon R - LQ}{PR + Q^2}\right)\tilde{g}^4{}_{j_0}\ , \tag{4}$$

$$\frac{d\tilde{g}^2{}_{j_0}}{d\tau}=0\,, \tag{5}$$

$$\frac{d\tilde{g}^3{}_{j_0}}{d\tau}+\left(\Gamma^3_{31}\,\frac{\epsilon Q+LP}{PR+Q^2}+\Gamma^3_{41}\,\frac{\epsilon R-LQ}{PR+Q^2}\right)\tilde{g}^1{}_{j_0}=0\,, \tag{6}$$

$$\frac{d\tilde{g}^4{}_{j_0}}{d\tau}+\left(\Gamma^4_{31}\,\frac{\epsilon Q+LP}{PR+Q^2}+\Gamma^4_{41}\,\frac{\epsilon R-LQ}{PR+Q^2}\right)\tilde{g}^1{}_{j_0}=0\,, \tag{7}$$

where $x^1=r$, $x^2=\theta$, $x^3=\phi$, and $x^4=t$ and $j=1,2,3,4$. In order to find $\tilde{g}^1{}_{j_0}$ we differentiate (4), substitute (6) and (7) in the result, and obtain finally

$$\frac{d^2\tilde{g}^1{}_{j_0}}{d\tau^2}+\alpha^2\tilde{g}^1{}_{j_0}=0\,, \tag{8}$$

where

$$\alpha^2=-\frac{(\epsilon Q+LP)^2}{(PR+Q^2)^2}(\Gamma^1_{33}\Gamma^3_{31}+\Gamma^1_{34}\Gamma^4_{31})-\frac{(\epsilon R-LQ)^2}{(PR+Q^2)^2}(\Gamma^1_{43}\Gamma^3_{41}+\Gamma^1_{44}\Gamma^4_{41})$$
$$-\frac{(\epsilon Q+LP)(\epsilon R-LQ)}{(PR+Q^2)2}(\Gamma^3_{31}\Gamma^3_{41}+\Gamma^1_{43}\Gamma^3_{31}+\Gamma^1_{34}\Gamma^4_{41}+\Gamma^1_{44}\Gamma^4_{31})\,.$$

Equations (8) and (5) may be solved easily to obtain $\tilde{g}^1{}_{j_0}$ and $\tilde{g}^2{}_{j_0}$. The values of the other components of the parallel propagator are calculated by substituting the value of $\tilde{g}^1{}_{j_0}$ in (6) and (7). Thus the parallel propagator on the timelike circular orbit is given by

$$\tilde{g}^i{}_{j_0}=\begin{bmatrix}\cos\alpha\tau & 0 & -\frac{A}{\alpha}\sin\alpha\tau & -\frac{B}{\alpha}\sin\alpha\tau\\ 0 & 1 & 0 & 0\\ -\frac{C}{\alpha}\sin\alpha\tau & 0 & 1+\frac{2E}{\alpha^2}\sin^2\frac{\alpha\tau}{2} & \frac{2F}{\alpha^2}\sin^2\frac{\alpha\tau}{2}\\ -\frac{D}{\alpha}\sin\alpha\tau & 0 & \frac{2G}{\alpha^2}\sin^2\frac{\alpha\tau}{2} & 1+\frac{2H}{\alpha^2}\sin^2\frac{\alpha\tau}{2}\end{bmatrix}, \tag{9}$$

which reduces to the value in the Schwarzschild space-time in the absence of rotation.

Here

$$A=\Gamma^1_{33}\,\frac{\epsilon Q+LP}{PR+Q^2}+\Gamma^1_{43}\,\frac{\epsilon R-LQ}{PR+Q^2}\,,$$

$$B=\Gamma^1_{34}\,\frac{\epsilon Q+LP}{PR+Q^2}+\Gamma^1_{44}\,\frac{\epsilon R-LQ}{PR+Q^2}\,,$$

$$C=\Gamma^3_{31}\,\frac{\epsilon Q+LP}{PR+Q^2}+\Gamma^3_{41}\,\frac{\epsilon R-LQ}{PR+Q^2}\,,$$

$$D=\Gamma^4_{31}\,\frac{\epsilon Q+LP}{PR+Q^2}+\Gamma^4_{41}\,\frac{\epsilon R-LQ}{PR+Q^2}\,, \tag{10}$$

$$E=\Gamma^3_{31}\Gamma^1_{33}\frac{(\epsilon Q+LP)^2}{(PR+Q^2)^2}+\Gamma^3_{31}\Gamma^1_{43}\frac{(\epsilon Q+LP)(\epsilon R-LQ)}{(PR+Q^2)^2}+\Gamma^3_{41}\Gamma^1_{33}\frac{(\epsilon Q+LP)(\epsilon R-LQ)}{(PR+Q^2)^2}+\Gamma^3_{41}\Gamma^1_{43}\frac{(\epsilon R-LQ)^2}{(PR+Q^2)^2}\,,$$

$$F=\Gamma^3_{31}\Gamma^1_{34}\frac{(\epsilon Q+LP)^2}{(PR+Q^2)^2}+\Gamma^3_{31}\Gamma^1_{44}\frac{(\epsilon Q+LP)(\epsilon R-LQ)}{(PR+Q^2)^2}+\Gamma^3_{41}\Gamma^1_{34}\frac{(\epsilon Q+LP)(\epsilon R-LQ)}{(PR+Q^2)^2}+\Gamma^3_{41}\Gamma^1_{44}\frac{(\epsilon R-LQ)^2}{(PR+Q^2)^2}\,,$$

$$G=\Gamma^4_{31}\Gamma^1_{33}\frac{(\epsilon Q+LP)^2}{(PR+Q^2)^2}+\Gamma^4_{31}\Gamma^1_{43}\frac{(\epsilon Q+LP)(\epsilon R-LQ)}{(PR+Q^2)^2}+\Gamma^4_{41}\Gamma^1_{33}\frac{(\epsilon Q+LP)(\epsilon R-LQ)}{(PR+Q^2)^2}+\Gamma^4_{41}\Gamma^1_{43}\frac{(\epsilon R-LQ)^2}{(PR+Q^2)^2}\,,$$

$$H=\Gamma^4_{31}\Gamma^1_{34}\frac{(\epsilon Q+LP)^2}{(PR+Q^2)^2}+\Gamma^4_{31}\Gamma^1_{44}\frac{(\epsilon Q+LP)(\epsilon R-LQ)}{(PR+Q^2)^2}+\Gamma^4_{41}\Gamma^1_{34}\frac{(\epsilon Q+LP)(\epsilon R-LQ)}{(PR+Q^2)^2}+\Gamma^4_{41}\Gamma^1_{44}\frac{(\epsilon R-LQ)^2}{(PR+Q^2)^2}\,.$$

We now consider, with the use of this parallel propagator, the geodetic precession of a gyroscope in free fall which moves along a timelike circular orbit. We shall identify the affine parameter with the proper time.

Now, the comoving frame $\{\vec{e}_{\hat{i}}\}$ and its dual basis attached to the gyroscope are given by

$$\vec{e}_{\hat{i}}=S_i{}^k\vec{e}_k,\quad \vec{w}^{\hat{i}}=t^i{}_k\vec{w}^k.$$

The transformation matrices $S_i{}^k$ and $t^i{}_k$ are as follows:

$$S_i{}^k=\begin{bmatrix}\left(1-2b+\frac{a^2}{r_0^2}\right)^{1/2} & 0 & 0 & 0\\ 0 & \frac{1}{r_0} & 0 & 0\\ 0 & 0 & \dfrac{C\left(1-2b+\frac{a^2}{r_0^2}\right)^{1/2}}{\alpha} & \dfrac{D\left(1-2b+\frac{a^2}{r_0^2}\right)^{1/2}}{\alpha}\\ 0 & 0 & \dfrac{B}{\left(1-2b+\frac{a^2}{r_0^2}\right)r_0\alpha} & \dfrac{-A}{\left(1-2b+\frac{a^2}{r_0^2}\right)r_0\alpha}\end{bmatrix}, \tag{11}$$

$$t^{i}{}_{k}=\begin{bmatrix} \dfrac{1}{\left(1-2b+\dfrac{a^2}{r_0{}^2}\right)^{1/2}} & 0 & 0 & 0 \\ 0 & r_0 & 0 & 0 \\ 0 & 0 & \dfrac{-A}{\left(1-2b+\dfrac{a^2}{r_0{}^2}\right)^{1/2}\alpha} & \dfrac{-B}{\left(1-2b+\dfrac{a^2}{r_0{}^2}\right)^{1/2}\alpha} \\ 0 & 0 & \dfrac{-D\left(1-2b+\dfrac{a^2}{r_0{}^2}\right)r_0}{\alpha} & \dfrac{C\left(1-2b+\dfrac{a^2}{r_0{}^2}\right)r_0}{\alpha} \end{bmatrix} . \tag{12}$$

The parallel propagator in the comoving frame is then

$$\tilde{g}^{\hat{i}}{}_{\hat{j}}=t^{i}{}_{k}S^{l_0}{}_{j_0}\tilde{g}^{k}{}_{l_0}=\begin{bmatrix} \cos\alpha\tau & 0 & \sin\alpha\tau & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha\tau & 0 & \cos\alpha\tau & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{13}$$

The following relation holds for the spin vector S^i of the gyroscope which is defined to remain orthogonal to the velocity vector $(dx^i/d\tau)$ [i.e., $S_i(dx^i/d\tau)=0$]:

$$S^4=\frac{r_0{}^2+a^2+2a^2b}{(1-2b)}\,\frac{(\epsilon Q+LP)}{(\epsilon R-LQ)}\,S^3 . \tag{14}$$

The S^4 component of the spin vector in the comoving frame is given by

$$S^{\hat{4}}=\frac{-D\left(1-2b+\dfrac{a^2}{r_0{}^2}\right)r_0{}^2}{\sqrt{b}}S^3+\frac{C\left(1-2b+\dfrac{a^2}{r_0{}^2}\right)r_0{}^2}{\sqrt{b}} \times\frac{(r_0{}^2+a^2+2a^2b)}{(1-2b)}\,\frac{(\epsilon Q+LP)}{(\epsilon R-LQ)}S^3$$

using (11) and (14). With $S^{\hat{\alpha}}{}_0(\alpha=1,2,3)$ as the spin vector at the point $P_0(\tau=0)$ on the circular orbit and $S^{\hat{\alpha}}$ as the parallel transport of $S^{\hat{\alpha}}{}_0$ at the point

$$P[\tau=\tau_P=2\pi(PR+Q^2)/(\epsilon Q+LP)],$$

where the gyroscope returns to the initial spatial point along the orbit, the angle between $S^{\hat{\alpha}}{}_0$ and $S^{\hat{\alpha}}$ is given in this case by

$$\cos\delta_K=\frac{S^{\hat{\alpha}}{}_0S_{\hat{\alpha}}}{\|S^{\hat{\alpha}}{}_0\|\|S^{\hat{\alpha}}\|}=\cos\alpha\tau_P . \tag{15}$$

Since $\alpha\tau_P=2\pi\alpha(PR+Q^2)/(\epsilon Q+LP)$ we get

$$\delta_K=\pm\alpha\tau_P+2n\pi\ ,\quad n=0,\pm1,\pm2 . \tag{16}$$

Also as $\alpha(PR+Q^2)/(\epsilon Q+LP)\to1$, $\delta_K\to0$ and we have $\delta_K>0$, so that

$$\delta_K=2\pi\left[1-\alpha\frac{(PR+Q^2)}{(\epsilon Q+LP)}\right] . \tag{17}$$

Hence the geodetic processional frequency of the gyroscope on an equatorial circular orbit in a Kerr space-time is

$$\|\vec{\Omega}_k\|=\frac{\delta_k}{\tau_p}=\frac{(\epsilon Q+LP)}{(PR+Q^2)}-\alpha , \tag{18}$$

and its direction is in the $\vec{e}_{\hat{2}}$ direction. This value of the precessional frequency reduces, to first order in a, to the value

$$\|\vec{\Omega}_k\|=\|\vec{\Omega}_s\|\mp\frac{ab}{r_0{}^2(1-3b)^{3/2}} . \tag{19}$$

For $a=0$, it is equal to the value $\|\vec{\Omega}_s\|$ obtained by Sakina and Chiba.[2] A further approximation leads to Schiff's formula[1]

TABLE I. We give the values of $\|\vec{\Omega}_s\|$ and Kerr correction [to first order in a according to Eq. (19)] for the Earth, the Sun, and a typical neutron star. The circular orbit has been assumed to be very close to the equatorial surface of each astrophysical object.

Object	Mass of the object m	Period of rotation of the object	Radius of circular orbit	$\|\vec{\Omega}_s\|$ (in gravitational unit)	Kerr correction according to Eq. (19) (in gravitational unit)
Earth	5.98×10^{27} g	1 d	6.38×10^{8} cm	8.27×10^{-23}	$\mp6.74\times10^{-25}$
Sun	1.99×10^{33} g	25 d	6.96×10^{10} cm	6.27×10^{-20}	$\mp8.22\times10^{-23}$
Neutron star	1.39×10^{33} g	10^{-3} sec	9.6×10^{5} cm	7.35×10^{-8}	$\mp1.67\times10^{-8}$

$$||\vec{\Omega}|| = \frac{3}{2}\left(\frac{m}{r_0^2}\right)\left(\frac{m}{r_0}\right)^{1/2}. \tag{20}$$

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