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Collapse of radiating fluid spheres

Kayll Lake and Charles Hellaby
Department of Physics, Queen's University at Kingston, Ontario, Canada, K7L 3N6
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We show that the radiating counterpart to the Oppenheimer-Snyder problem can give rise to nakedly singular spacetimes. Possible classical collapse scenarios include those with regular initial conditions whose end states are not merely instantaneously singular.

One of the most important unresolved problems in classical general relativity is the cosmic censorship hypothesis.[1] This hypothesis (in its weak form) asserts that no physically reasonable matter field can give rise to singularities which are observable from the asymptotic regions of spacetime. The absence of a result in classical general relativity with the general implications of the cosmic censorship hypothesis would be a serious flaw in the theory. Despite various lines of evidence in favor of the hypothesis, there is as yet no formulation of it which is precise and complete enough to allow a clear proof or disproof. In view of this, it is certainly a worthwhile exercise to search for examples which, at least superficially, appear to violate the hypothesis. To date, these "counterexamples" have included the development of singularities of the "shell-crossing" type,[2] the cylindrical collapse of dust,[3] and the collapse of radiating spheres for which the entire mass of the sphere is radiated away leaving a nakedly singular spacetime.[4]

The purpose of this paper is to put forward some further examples of this latter type. Our results, which for the most part are analytic, are somewhat less contrived than those given previously as they depict the radiating counterpart to the familiar Oppenheimer-Snyder collapse scenario. More importantly, the naked singularities that we find are not merely "instantaneous" (as in the example given by Demianski and Lasota[4]).

We consider the collapse of a spherical distribution of matter with timelike boundary surface Σ and take the interior metric ($\mathscr{V}^-$) to be of the spatially flat Robertson-Walker form

$$ds_-{}^2=a^2(t)(dr^2+r^2d\Omega^2)-dt^2\ , \tag{1}$$

where $d\Omega^2\equiv d\theta^2+\sin^2\theta\, d\phi^2$. The exterior to Σ is described here by means of a radial flux of unpolarized radiation so that the exterior metric ($\mathscr{V}^+$) takes the outgoing Vaidya form[5]

$$ds_+{}^2=-2\,dv\,d\mathfrak{r}+\mathfrak{r}^2d\Omega^2-\left[1-\frac{2m(v)}{\mathfrak{r}}\right]dv^2\ . \tag{2}$$

Assuredly, the transition from isotropy to a radial flux of radiation accross Σ seems unphysical, or rather, physically unrealizable. We stress, however, that for the examples given below no standard energy or junction condition is in fact violated by this transition.

Necessary (but not sufficient) conditions for the junction of $\mathscr{V}^+$ onto $\mathscr{V}^-$ (for a perfect fluid interior) are given by[6] $\theta^+=\theta^-$, $\phi^+=\phi^-$,

$$\mathfrak{r}_\Sigma=a(t)r_\Sigma\ , \tag{3}$$

$$m(v)=\tfrac{4}{3}\pi a^3(t)r_\Sigma{}^3\rho\ , \tag{4}$$

$$a^2(t)\dot{r}_\Sigma{}^2+1=\dot{t}_\Sigma{}^2=\frac{1}{1-\epsilon^2}\ , \tag{5}$$

and

$$\left[1-\frac{2m(v)}{\mathfrak{r}_\Sigma}\right]\dot{v}_\Sigma{}^2+2\dot{v}_\Sigma\dot{\mathfrak{r}}_\Sigma=1\ , \tag{6}$$

where a dot denotes $d/d\tau$ (τ the proper time along Σ), ρ gives the total comoving interior energy density, and $\epsilon\equiv p/\rho$, where p gives the co-moving interior isotropic pressure. We set t decreasing to the future along Σ ($\dot{r}_\Sigma<0$) and choose the final condition[7]

$$r_\Sigma(t=0)=0\ . \tag{7}$$

First, we consider the simplest possible interior equation of state, $\epsilon=\text{const.}$[8] The Einstein equations then give $8\pi\rho = 3\gamma^2/t^2$ and $a(t) \propto t^{\gamma}$ with $\gamma \equiv 2/3(1+\epsilon)$. Equations (3), (4), and (5) with condition (7) give $\mathfrak{r}_{\Sigma} = 3\epsilon(1+\epsilon)t/(1+3\epsilon)$ and $m = 6\epsilon^3(1+\epsilon)t/(1+3\epsilon)^3$. From Eqs. (6) and (7) then[9]

$$v = -(1+3\epsilon)t \,, \tag{8}$$

so that the exterior history of Σ is given by[10]

$$\mathfrak{r}_{\Sigma} = \frac{-3\epsilon(1+\epsilon)}{(1+3\epsilon)^2} v \,, \tag{9}$$

and

$$m = \frac{-6\epsilon^3(1+\epsilon)}{(1+3\epsilon)^4} v \,. \tag{10}$$

With $u \equiv -\mathfrak{r}/v$, the radial ingoing null geodesic equation of the metric (2), with the dependence (10), takes the separated form

$$\frac{dv}{du} = \frac{uv}{(u-\alpha_+)(\alpha_- - u)} \,, \tag{11}$$

where $\alpha_{\pm} \equiv \frac{1}{4}\{1 \pm [1 - 96\epsilon^3(1+\epsilon)/(1+3\epsilon)^4]^{1/2}\}$. For $u \neq \alpha_{\pm}$ then, the ingoing null geodesics are given by[11]

$$A = \frac{|\mathfrak{r} + \alpha_- v|^{b_1}}{|\mathfrak{r} + \alpha_+ v|^{b_2}} \,, \tag{12}$$

where $b_1 = \alpha_-/(\alpha_+ - \alpha_-)$, $b_2 = \alpha_+/(\alpha_+ - \alpha_-)$, and A is a constant >0. Of the null surfaces $u = \alpha_{\pm}$, only the surface $u = \alpha_-$ is an event horizon. The surface $u = \alpha_+$ is merely a particle horizon (e.g., for Σ, see Fig. 1 below). The history of Σ is a homothetic Killing trajectory.[12] We note that the (radial) red-shift from Σ, as reckoned from spatial infinity for $v \leq 0$, is given by $(1-\epsilon^2)^{1/2}/(1+3\epsilon)$.

For the metric (2) [with the dependence (10)], the surface $v=0$ has a scalar polynomial singularity at $\mathfrak{r} = 0$.[13] Moreover, the (outgoing) null surface $v=0$ is not flat (unless $\epsilon = 0$). Any C^n continuation of the metric (2) across the null surface $v=0$ for $\epsilon \neq 0$ requires negative m for $n > 0$. All such continuations are nakedly singular. The collapse scenario described above then necessitates the development of a nakedly singular spacetime (which is not merely instantaneously singular[14]) unless a C° continuation is imposed on the null surface $v=0$. The history of Σ is summarized in Fig. 1. (For the figure, we have chosen the continuation so that the observed luminosity at spatial infinity remains constant.[5])

The example discussed above is not entirely regular since, according to Eq. (10), m diverges initially (as $v \to -\infty$).[15] We now show that this deficiency is due to the simple equation of state used throughout the history of Σ. For a mixture of noninteracting dust and radiation, the scale factor a can be written in terms of elementary functions.[16] For such a mixture the Einstein equations now give $8\pi\rho = 3\times 4^3(2\sqrt{\nu}+\mu\eta)^2/\eta^4(4\sqrt{\nu}+\mu\eta)^4$ and $a(t) \propto \sqrt{\nu}\eta + \mu\eta^2/4$ with $t = \sqrt{\nu}\eta^2/2 + \mu\eta^3/12$, where ν and μ are constants.[17] Now $\epsilon = 4\nu/3(2\sqrt{\nu}+\mu\eta)^2$, and Eqs. (3), (4), and (5) with condition (7) give

$$\mathfrak{r}_{\Sigma} = \frac{\sqrt{\nu}\eta^2(4\sqrt{\nu}+\mu\eta)}{6(2\sqrt{\nu}+\mu\eta)} \tag{13}$$

and

$$m = \frac{4\nu^{3/2}\eta^2}{27(4\sqrt{\nu}+\mu\eta)(2\sqrt{\nu}+\mu\eta)} \,. \tag{14}$$

Moreover, from Eqs. (6) and (7) we find

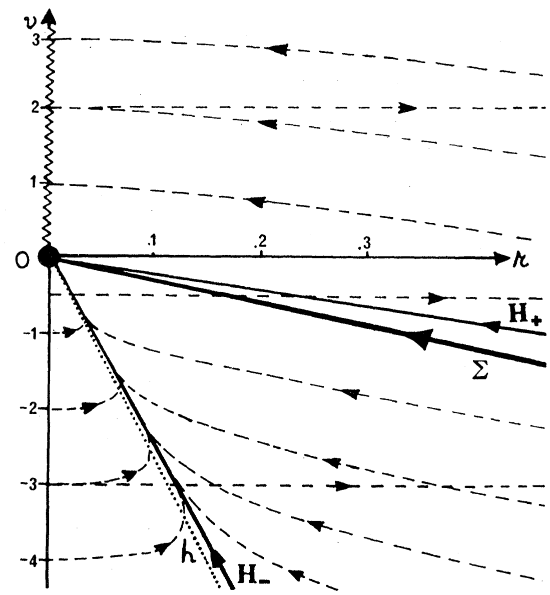

FIG. 1. The history of the boundary surface Σ in the Vaidya $(v, \mathfrak{r})$ coordinates for the case $\epsilon = \frac{1}{3}$. While only the geometry above Σ (that is, the exterior spacetime) is relevant to the collapse scenario described in the text, the complete Vaidya field is shown here for clarity. The exterior metric has been continued through the null surface $v=0$ by setting $dm/dv = -\frac{1}{54}$. The null surfaces $\mathfrak{r} = -\alpha_{\pm} v$ are labeled by $H_{\pm}$, and h gives the (spacelike) apparent horizon at $\mathfrak{r} = 2m(v)$. Whereas the radial outgoing null geodesics are given simply by $v = $const, the radial ingoing null geodesics (except for $\mathfrak{r} = -\alpha_{\pm} v$) satisfy Eq. (12) and are shown dashed. The surface $\mathfrak{r} = 0$ is timelike, singular, and naked for $v \geq 0$.

$$v=-\frac{(3\mu^2\eta^2+24\mu\sqrt{\nu}\eta+28\nu)\eta}{36\mu}+\frac{2\nu^{3/2}\eta}{\mu(\mu\eta+2\sqrt{\nu})}-\frac{4\nu^{3/2}}{\mu^2}\ln\left|\frac{\mu\eta}{2\sqrt{\nu}}+1\right|+\frac{128\nu^{3/2}}{27\mu^2}\ln\left|\frac{3\mu\eta}{8\sqrt{\nu}}+1\right| . \quad (15)$$

It is easy to show that in the limit $\mu\to 0$ we recover the case $\epsilon=\frac{1}{3}$. For $\mu\neq 0$ however, Eq. (14) shows that m is initially finite (for $\eta\to\infty$) and monotonically decreases to zero (at $v=\eta=0$). Moreover, $dm/dv\,|_{v=0}=-\frac{1}{54}$ (again as in the $\epsilon=\frac{1}{3}$ case, but now for all finite ν and μ) and so the null surface $v=0$ (which also contains a scalar polynomial singularity at $\mathbf{r}=0$) is not flat. The present case is, in the neighborhood of the null surface $v=0$, indistinguishable from the simple case $\epsilon=\frac{1}{3}$. The regularity of the initial conditions, however, distinguishes it from any $\epsilon=$const case.[18]

The "counterexamples" to cosmic censorship given above could be dismissed in a number of ways. The least satisfactory, we feel, would be the claim that the required transition at Σ seems physically unrealizable. Moreover, the collapse scenarios discussed here would, at the semiclassical level, be accompanied by a catastrophic flux of created particles which, near the end point of the collapse, would completely alter the classical picture given.[19] The cosmic censorship hypothesis is, however, formulated entirely at the classical level. Any dismissal based on a semiclassical particle flux is, therefore, unacceptable. What seems to us to be the culprit in the examples given is the interior equation of state. We do not believe that the equations of state used are oversimplified to the extent that they give unreliable results for the conventional picture of collapse.[20] Rather, they seem to indicate that the present "counterexamples" can be avoided (or rather, reduced to at best an instantaneous nature) only if ϵ (as $t\to 0$) necessarily approaches zero as, for example, in the Hagedorn equation of state.[21]

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[1]See, for example, the reviews by R. Penrose and by R. Geroch and G. T. Horowitz, in *General Relativity: An Einstein Centenary Survey,* edited by S. W. Hawking and W. Israel (Cambridge Univ. Press, Cambridge, 1979); and by F. J. Tipler, C. J. S. Clarke, and G. F. R. Ellis, in *General Relativity and Gravitation,* edited by A. Held (Plenum, New York, 1980), Vol. 2.

[2]P. Yodzis, H. J. Seifert, and H. Müller zum Hagen, Commun. Math. Phys. 34, 135 (1973); 37, 29 (1974).

[3]E. P. T. Liang, Phys. Rev. D 10, 447 (1974); K. S. Thorne, in *Magic Without Magic,* edited by J. R. Klauder (Freeman, San Francisco, 1972). See also E. P. T. Liang, Gen. Relativ. Gravit. 10, 1051 (1979).

[4]The possibility that a star could radiate away its entire mass was, apparently, first considered by H. Bondi, Proc. R. Soc. London A281, 39 (1964). A detailed example has been studied by M. Demianski and J. P. Lasota, Astrophys. Lett. 1, 205 (1968). The significance of this example has been discussed by B. Steinmüller, A. R. King, and J. P. Lasota, Phys. Lett. 51A, 191 (1975).

[5]See, for example, R. W. Lindquist, R. A. Schwartz, and C. W. Misner, Phys. Rev. 137, 1364 (1965).

[6]K. Lake, Astrophys. J. 240, 744 (1980). The boundary surface junction conditions are rigorously completed with the continuity of $n_\alpha \delta u^\alpha/\delta\tau$ across Σ, where $\vec{n}$ and $\vec{u}$ are the unit normal and tangent four-vectors to Σ, respectively.

[7]If $r_\Sigma(t=0)\neq 0$, an event horizon forms.

[8]The junction of the metrics (1) and (2) for this equation of state has been considered previously (in a cosmological context) by S. Hacyan, Astrophys. J. 229, 42 (1979) and (more generally) by B. C. Reed and R. N. Henriksen, *ibid.* 236, 338 (1980). See also, B. D. Miller, *ibid* 208, 275 (1976), and Ref. 6. Note that for timelike Σ, and nonconstant m, $1>\epsilon>0$.

[9]We have, without loss of generality, set $v(t=0)=0$. Note that for the nonsingular history of Σ, $t>0$ and $v<0$.

[10]The boundary surface junction conditions are now rigorously completed by noting that $n_\alpha\delta u^\alpha/\delta\tau\,|_{\Sigma}^{\pm}=2(1+3\epsilon)\epsilon/3(1+\epsilon)(1-\epsilon^2)^{1/2}v$.

[11]An integration, equivalent to Eq. (12), has been given previously by B. C. Reed, M. Sc. thesis, Queen's University, 1979 (unpublished).

[12]That is, for w^α the unit tangent to the $u=$const trajectories, $\nabla_\beta\xi_\alpha+\nabla_\alpha\xi_\beta=2g_{\alpha\beta}$ for $\xi^\alpha\equiv w^\alpha(\,|\xi^\beta\xi_\beta|\,)^{1/2}$. The associated renormalized acceleration,

$\kappa^2 \equiv -(\xi^\alpha \xi_\alpha)\dot{\omega}^\beta \dot{\omega}_\beta$, takes on the constant values $\kappa_\pm = 6\epsilon^3(1+\epsilon)/(1+3\epsilon)^4 \alpha_\pm{}^2$ along the null surfaces $u = \alpha_\pm$. See C. C. Dyer and E. Honig, J. Math. Phys. 20, 1 (1979), and G. V. Bicknell and R. N. Henriksen, Astrophys. J. 219, 1043 (1978).

[13]See, for example, Tipler, Clarke, and Ellis, Ref. 1. For example, the Kretschmann scalar ($=48m^2/\mathfrak{r}^6$ here) diverges along all timelike homothetic Killing trajectories as $v \to 0$.

[14]The example given by Demianski and Lasota (Ref. 4) is only instantaneously singular since Minkowski space represents a C^2 continuation of their Vaidya field.

[15]It is true that $\mathfrak{r}_\Sigma$ also diverges initially. This, however, also holds in the Oppenheimer-Snyder case and is due simply to the choice of flat interior spatial sections.

[16]See, for example, E. R. Harrison, Mon. Not. R. Astron. Soc. 140, 281 (1968).

[17]If we treat the photons as blackbody, and the (neutral) matter component as protons (p) and electrons (e), it follows that (in geometrical units) $\nu = 64\pi^6 k^4/45h^3$ and $\mu = 128\pi^2 k^3(m_e + m_p)\zeta(3)/3h^3 f$, where m is the proper mass, k is Boltzmann's constant, h is Planck's constant, ζ is the Riemann zeta function, and f is the photon to baryon number density ratio.

[18]The boundary surface junction conditions can also be rigorously satisfied in this case. Moreover, a numerical investigation has shown that the "dust + radiation" case is indistinguishable (throughout the history of Σ) from a more realistic equilibrium relativistic Maxwell-Boltzmann gas + radiation treatment as long as $f(t=0) \gtrsim 100$. A detailed justification of these statements is somewhat involved, and will be published subsequently.

[19]W. A. Hiscock, Phys. Rev. D 23, 2813 (1981); 23, 2823 (1981).

[20]The inclusion of viscosity remains to be examined.

[21]See, for example, R. Hagedorn in *Cargèse Lectures in Physics, Vol. 6*, edited by E. Schatzman (Gordon and Breach, New York, 1973). For a comment on this, see R. Penrose in *Gravitational Radiation and Gravitational Collapse*, edited by C. DeWitt-Morette (Reidel, Dordrecht, 1974).