Sum rules for partial waves in production processes

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Using general crossing symmetry a complete set of sum rules for partial waves in production processes is derived. These sum rules are derived explicitly for the case of pion production and are particularly suited to provide constraints on model (e.g., isobar) amplitudes.

I. INTRODUCTION

In particle physics two-body reactions have attracted most attention due to their relative simplicity, both theoretical and experimental, compared with production processes where the final state consists of many particles. Production processes are important experimentally because at high energies they account for most of the total cross section. They are also of great theoretical interest; for example much work has been done applying quantum chromodynamics to the study of jets.

However, the increased number of particles results in considerable kinematic complexity; each production event is characterized by $3n-4$ independent variables, where n is the number of final-state particles. Thus for $n=3$ we need five independent variables and for $n=4$ there are eight, and so on. Moreover the spin dependence becomes more involved; ignoring possible restrictions due to symmetries we need

$$\prod_{\substack{i=a,b\\ f_1,\ldots,f_n}} (2s_i+1)$$

independent helicity amplitudes to describe the production process $a+b\to f_1+f_2+\cdots+f_n$. In fact there are more degrees of freedom than can be determined experimentally; this arises from the fact that it is not yet possible to measure the spin states of all the final-state particles in a production event. So in order to completely determine the individual amplitudes corresponding to the different spin states of the particles, it is necessary to make some assumptions about the functional dependence of the amplitudes on the $3n-4$ independent variables. In other words a model is adopted for the process. We mention, for example, the Veneziano model and the isobar model. Of course it is inherent in the nature of these model amplitudes that they are not unique, so it is important that any general symmetry or invariance principles available should be either incorporated in the model or imposed as a constraint.

Our purpose in this work, therefore, is to derive from general crossing symmetry relations between partial-wave amplitudes for production processes which we believe will provide useful constraints on model amplitudes. We derive sum rules for partial waves analogous to those derived for two-body reactions by Balachandran and coworkers 1(a)–1(c) and for multiparticle spinless reactions by Modjtehedzadeh.[2] Sum rules for multiparticle processes using crossing were also derived by Johannesson[3] using techniques different from those of this paper. The partial-wave amplitudes he uses are (like those of Modjtehedzadeh) diagonal only in the total angular momentum of the system. His results therefore, though elegant, are not readily generalized to cover final-state interactions which can be analyzed using the work of this paper. Section II contains kinematic definitions together with the helicity crossing matrix and kinematic singularities associated with the helicity amplitudes for the two crossing-isolated processes $N\pi\to N\pi\pi$ and $N\bar{N}\to\bar{\pi}\pi\pi$. We also summarize here the analytic behavior of the crossing matrix between kinematic-singularity-free helicity amplitudes. (This material is discussed in greater detail in our previous paper.[4]) In Sec. III, after giving the partial-wave expansions both for these nonsingular amplitudes and for arbitrary polynomials in the independent scalars, we use the results of Sec. II to obtain sum rules for partial-wave amplitudes in the two channels. An appendix contains definitions of angles and coefficient functions used, together with limits of phase-space integrals over the Euclidean region.

II. PRELIMINARIES

In this section we summarize the necessary preliminaries to calculating the sum rules of Sec. III. We refer the reader to our paper[4] for further details.

A. Kinematics

We consider the process $a+b\to 1+2+3$. A one-particle helicity state $|m,s,\eta;\vec{P},\lambda\rangle$ or $|\vec{P},\lambda\rangle$ is defined by

$$|\vec{P},\lambda\rangle = e^{-i\phi J_z} e^{-i\theta J_y} e^{-i\xi K_z} |\vec{0},\lambda\rangle . \quad (2.1)$$

where ξ, θ, ϕ are defined by

$$E = m \cosh\xi , \quad (2.2a)$$

$$|\vec{P}| = m \sinh\xi , \quad (2.2b)$$

and

$$(P_x, P_y, P_z) = (|\vec{P}| \sin\theta \cos\phi, \ |\vec{P}| \sin\theta \sin\phi, \ |\vec{P}| \cos\theta). \quad (2.3)$$

The helicity amplitudes $T_{\lambda_1,\lambda_2,\lambda_3;\lambda_a,\lambda_b}$ are taken to be

$$\langle \vec{P}_1\lambda_1; \vec{P}_2\lambda_2; \vec{P}_3\lambda_3 | T | \vec{P}_a\lambda_a; \vec{P}_b\lambda_b\rangle$$

normalized as in Ref. 4. Setting $P = P_a + P_b$ we take $Z = \{P^2, P\cdot P_1, P\cdot P_2, P_a\cdot P_1, P_a\cdot P_2\}$ as the set of linearly independent scalar variables on which $T_{\{\lambda\}}$ depends. There also exists one pseudoscalar $\epsilon_2 = \epsilon\mu\nu\rho\sigma P^\mu P_a^\nu P_1^\rho P_2^\sigma$ on which $T_{\{\lambda\}}$ may depend. Parity invariance gives the relation

$$T_{\{\lambda\}}(Z, -\epsilon_2) = \pi_{\{\lambda\}} T_{-\{\lambda\}}(Z, \epsilon_2) , \quad (2.4)$$

where $\pi_{\{\lambda\}} = \prod_{i=a}^{3} \eta_i (-1)^{s_i \lambda_i}$. The linear combinations

$$T_{\{\lambda\}}^{(1,2)} = \begin{pmatrix} 1 \\ \epsilon_2 \end{pmatrix} (T_{\{\lambda\}} \pm \pi_{\{\lambda\}} T_{-\{\lambda\}}) \quad (2.5)$$

are therefore even in ϵ_2, and thus regular at $\epsilon_2 = 0$. We define all angles and rapidities used in the paper in the Appendix.

We are interested in evaluating integrals of the form $\int f(Z)\, d\omega$, where

$$d\omega = \prod_{i=a,b}^{1,2,3} \delta(P_i^{\,2} - M_i^{\,2})\, d^4P_i\, \delta^4(P_a + P_b - P_1 - P_2 - P_3) \quad (2.6)$$

is the product of the initial- and final-state phase-space factors. It is a straightforward matter to show that

$$\int f(Z)\, d\omega = -16\pi^4 \int ds\, ds_{23}\, d\cos\theta_1\, d\cos\theta_1^c\, d\psi_1 f(Z) \times A^2 [\Delta_2(P_2, P_3)]^{1/2} / s_{23} , \quad (2.7)$$

where s, s_{ij}, and t_{ia} are given by $s = P^2$, $s_{ij} = (P_i + P_j)^2$, and $t_{ia} = (P_a - P_i)^2$. Also, for convenience, we write $A = [\Delta_2(P, P_a)\Delta_2(P, P_1)]^{1/4}$. The special Gram determinants Δ_i are defined in the Appendix. Following Balachandran *et al.*[1(b)] and Modjtehedzadeh[2] we choose to perform this investigation not over the physical region of the process $a + b \to 1 + 2 + 3$ but over the Euclidean region, where the space parts of the particle momenta are imaginary or zero. It is easily checked that the Euclidean region is exactly that region which corresponds to physical values for the angles θ, θ^c, and ψ in each channel, together with the appropriate range for the energy and subenergy in that channel. We give the limits on the variables s, s_{23}, θ_1, θ_1^c, and ψ_1 corresponding to this choice in the Appendix. Similarly the integral over the Euclidean region in terms of the crossed-channel variables is

$$\int \bar{f}(\bar{Z})\, d\bar{\omega} \equiv \int \bar{f}(\bar{Z}) \prod_{i=a,1}^{b,2,3} \delta(q_i^{\,2} - m_i^{\,2})\, d^4q_i\, \delta(q_a + q_1 - q_b - q_2 - q_3)$$
$$= \int dt_{1a}\, ds_{23}\, d\cos\bar{\theta}_1\, d\cos\bar{\theta}_1^c\, d\bar{\psi}_1 \bar{f}(\bar{Z}) [\Delta_2(P_2, P_3)\, \Delta_2(P_a, -P_1)\, \Delta_2(P_a - P_1, P_a - P)]^{1/2} / s_{23} . \quad (2.8)$$

B. Kinematic singularities and the crossing matrix

Again referring the reader to our paper[4] and to Svensson[5] for details, we give the regularized helicity amplitudes (RHA's) $\bar{T}_{\lambda_1\lambda_a}^{(1,2)}$, i.e., those linear combinations of helicity amplitudes free of the kinematic singularities which arise from the singular behavior of the helicity states on certain hypersurfaces in the space of the scalar variables Z. We now restrict ourselves to the case $s_a = s_1 = \frac{1}{2}$, $s_b = s_2 = s_3 = 0$, e.g., pion production $N\pi \to N\pi\pi$ and the corresponding crossed process $N\bar{N} \to \bar{\pi}\pi\pi$.

The RHA's for the subchannel process $a + b \to 1 + 2 + 3$ are

$$\bar{T}_{\lambda_1\lambda_a}^{(1,2)} = [\Delta_2(P, P_a)\, \Delta_2(P, P_1)]^{1/4} \{[D_3^+(P, P_a, P_1)]^{1/2}\}^{|\lambda_a - \lambda_1|/2} \{[D_3^-(P, P_a, P_1)]^{1/2}\}^{|\lambda_a + \lambda_1|/2} T_{\lambda_1\lambda_a}^{(1,2)} . \quad (2.9)$$

We also found the RHA's for the continued t_{1a}-channel $(a + \bar{1} \to \bar{b} + 2 + 3)$ helicity amplitudes as

$$\bar{T}_{\lambda_a\lambda_1}^{(1)} = [D_2^-(P_a, -P_1)]^{1/2} [D_3^+(P, P_a, P_1)\, D_3^-(P, P_a, P_1)]^{|\lambda_a - \lambda_1|/2}\, T_{\lambda_a\lambda_1}^{(1)} ,$$
$$\bar{T}_{\lambda_a\lambda_1}^{(2)} = [D_3^+(P_a, -P_1)]^{1/2} [D_3^+(P, P_a, P_1)\, D_3^-(P, P_a, P_1)]^{|\lambda_a - \lambda_1|/2}\, T_{\lambda_a\lambda_1}^{(2)} . \quad (2.10)$$

The helicity crossing matrix connecting the linear combinations $T_{\lambda_1\lambda_a}^{(1,2)(s_{ab})}$ and $T_{\lambda_a\lambda_1}^{(1,2)(t_{1a})}$ is shown in Ref. 4 to be

$$\begin{bmatrix} T^{(1)}_{++} \\ T^{(1)}_{+-} \\ T^{(2)}_{++} \\ T^{(2)}_{+-} \end{bmatrix}^{(s_{ab})} = -i \begin{bmatrix} \sin\left(\frac{\psi_a-\psi_1}{2}\right) & -\cos\left(\frac{\psi_a-\psi_1}{2}\right) & 0 & 0 \\ \cos\left(\frac{\psi_a-\psi_1}{2}\right) & \sin\left(\frac{\psi_a-\psi_1}{2}\right) & 0 & 0 \\ 0 & 0 & \sin\left(\frac{\psi_a+\psi_1}{2}\right) & \cos\left(\frac{\psi_a+\psi_1}{2}\right) \\ 0 & 0 & \cos\left(\frac{\psi_a+\psi_1}{2}\right) & -\sin\left(\frac{\psi_a+\psi_1}{2}\right) \end{bmatrix} \begin{bmatrix} T^{(1)}_{++} \\ T^{(1)}_{+-} \\ T^{(2)}_{++} \\ T^{(2)}_{+-} \end{bmatrix}^{(t_{1a})} . \tag{2.11}$$

From (2.11) for the crossing matrix together with (2.9) and (2.10) for the RHA's it follows after a certain amount of manipulation that the regularized crossing matrix (RCM)—the crossing matrix between the RHA's—can be written in the simple form

$$\overline{X}^{(1)} = -i \begin{bmatrix} D_3^-(P,P_a,P_1)f_1 & -f_2 \\ D_3^+(P,P_a,P_1)f_2 & f_1 \end{bmatrix} \Big/ D_2^+(P_a, -P_1)(2s)^{1/2} \tag{2.12}$$

and

$$\overline{X}^{(2)} = -i \begin{bmatrix} D_3^-(P,P_a,P_1)f_3 & f_4 \\ D_3^+(P,P_a,P_1)f_4 & -f_3 \end{bmatrix} \Big/ D_2^-(P_a, -P_1)(2s)^{1/2} , \tag{2.13}$$

where

$$f_1 = (sm_a{}^2 + A^2 - P\cdot P_a P\cdot P_1)^{1/2} , \tag{2.14a}$$

$$f_2 = (sm_a{}^2 - A^2 - P\cdot P_a P\cdot P_1)^{1/2} , \tag{2.14b}$$

$$f_3 = (-sm_a{}^2 + A^2 - P\cdot P_a P\cdot P_1)^{1/2} , \tag{2.14c}$$

and

$$f_4 = (-sm_a{}^2 - A^2 - P\cdot P_a P\cdot P_1)^{1/2} . \tag{2.14d}$$

The 2×2 matrices $\overline{X}^{(1)}$ and $\overline{X}^{(2)}$ correspond to the upper-left and lower-right blocks in $X^{(1,2)\lambda_1'}_{\lambda_a}$, respectively.

It is also useful to calculate the inverse RCM:

$$\overline{X}^{(1)-1} = i \begin{bmatrix} f_1 & f_2 \\ -f_2 D_3^+(P,P_a,P_1) & f_1 D_3^-(P,P_a,P_1) \end{bmatrix} \Big/ A^2(2s)^{1/2} \tag{2.15}$$

and

$$\overline{X}^{(2)-1} = i \begin{bmatrix} f_3 & f_4 \\ f_4 D_3^+(P,P_a,P_1) & -f_3 D_3^-(P,P_a,P_1) \end{bmatrix} \Big/ A^2(2s)^{1/2} . \tag{2.16}$$

It is immediately apparent that $\overline{X}^{-1}$ is not only finite at $\Delta_3(P,P_a,P_1)=0$ but in fact it is linear in $D_3^\pm(P,P_a,P_1)$ and thus, as

$$D_3^\pm(P,P_a,P_1) \equiv A^2(1\pm\cos\theta_1^c) , \tag{2.17}$$

it follows that $\overline{X}^{-1}$ is linear in $\cos\theta_1^c$.

We observe in passing that this inverse crossing matrix still exhibits singular behavior at $\Delta_2(P,P_a)=0$, $\Delta_2(P,P_1)=0$ and $s=0$. The branch point at $s=0$ can, in fact, be removed by redefining the t_{1a}-channel RHA's while those at $\Delta_2(P,P_a)=0$ and $\Delta_2(P,P_1)=0$ correspond to the existence of constraints on the s_{ab}-channel RHA's on these submanifolds. However, as we are not primarily concerned with the dependence of the amplitudes on s and s_{23} we do not pursue this further, merely noting that these singularities in the inverse RCM do not affect the derivation of the sum rules in the next section.

III. SUM RULES FOR PARTIAL WAVES

In this section we derive sum rules relating partial waves for the direct and crossed-channel processes. Our techniques are based on the work of Modjtehedzadeh[2] on sum rules for processes of type $a+b\to 1+2+3$, where all particles have zero spin. The introduction of spin, of course, introduces several complications, namely, kinematic singularities and the helicity crossing matrix, both discussed in Sec. II. We begin by considering partial-wave expansions for production processes.

A. Partial-wave expansions

Namyslowski *et al.*[6] use the three-particle angular-momentum states of Wick[7] to write a partial-wave expansion for the helicity amplitudes $T_{\lambda_1\lambda_a}$ as

$$T_{\lambda_1\lambda_a} = \sum_{J,j_1,m_1} \left(\frac{s^2 s_{23}}{[\Delta_2(P,P_a)\Delta_2(P,P_1)]^{1/2}}\right)^{1/2} N_J{}^2 N_{j_1} d^{j_1}_{m_1 0}(\theta_1) D^{J\,*}_{\lambda_a, m_1-\lambda_1}(\phi_1^c, \theta_1^c, \psi_1) T^{Jj_1m_1\lambda_1}_{\lambda_a}(s, s_{23}) , \tag{3.1}$$

where $N_J = (2/2J+1)^{1/2}$. Again we define the angles in the Appendix. For an unpolarized target, ϕ_1^c is of course not measured, moreover Wick's three-particle helicity states contain an overall factor $e^{-i\lambda_1\phi_1}$ which does not appear in the states $|P_{\bar{1}}\lambda_1; P_{\bar{2}}\lambda_2; P_{\bar{3}}\lambda_3\rangle$ we choose to work with. Thus, using (2.9) for the RHA's we find

$$\overline{T}^{(1,2)}_{\lambda_1\lambda_a} = \begin{pmatrix} 1 \\ \epsilon_2 \end{pmatrix} \sum_{J, j_1, m_1} \left(\frac{s^2 s_{23}}{[\Delta_2(P_2, P_3)]^{1/2}}\right)^{1/2} N_J{}^2 N_{j_1} (\cos\theta_1^c/2)^{|\lambda_a-\lambda_1|}(\sin\theta_1^c/2)^{|\lambda_a+\lambda_1|} A\, d^{j_1}_{m_1 0}(\theta_1) d^J_{\lambda_a, m_1-\lambda_1} \times (e^{im_1\phi_1} \pm e^{-im_1\phi_1}) \mathcal{T}^{J j_1 m_1 \lambda_1}_{\lambda_a}(s, s_{23}) . \quad (3.2)$$

Similarly we can write the partial-wave expansion for the t_{1a}-channel RHA's as

$$\overline{T}^{(t_{1a})(1,2)}_{\lambda_a\lambda_1} = \begin{pmatrix} 1 \\ \epsilon_2 \end{pmatrix} \sum_{J, j_1, m_1} \left(\frac{t_{1a}{}^2 s_{23}}{[\Delta_2(P_a, -P_1)\Delta_2(P_a-P_1, P_a-P)\Delta_2(P_2, P_3)]^{1/2}}\right)^{1/2} N_J{}^2 N_{j_1} [D_3^\pm(P_a, -P_1)]^{1/2} \times [D_3^+(P, P_a, P_1) D_3^-(P, P_a, P_1)]^{|\lambda_a-\lambda_1|/2} d^{j_1}_{m_1 0}(\overline{\theta}_1) d^J_{\lambda_a-\lambda_1, m_1}(\overline{\theta}_1^c)(e^{im_1\overline{\phi}_1} \pm e^{-im_1\overline{\phi}_1}) \mathcal{T}^{J j_1 m_1}_{\lambda_a, \lambda_1} . \quad (3.3)$$

In his paper[2] on sum rules for processes of type $2 \to N$ where all particles have spin-zero, Modjtehedzadeh gives a sketch proof that any polynomial in the five scalars Z has a finite partial-wave expansion. We carried out a similar check for the more complicated cases $N\pi \to N\pi\pi$ and $N\overline{N} \to \overline{\pi}\pi\pi$ with the same conclusion. We find, for an arbitrary polynomial $\mathcal{P}_5(Z)$, that

$$(1+Z_1^c)^{|\lambda_a-\lambda_1|}(1-Z_1^c)^{|\lambda_a+\lambda_1|}\mathcal{P}_5(Z) = \sum_{J, j_1, m_1} \left(\frac{s^2 s_{23}}{[\Delta_2(P_2, P_3)]^{1/2}}\right)^{1/2} N_J{}^2 N_{j_1} d^J_{\lambda_a, m_1-\lambda_1}(\theta_1^c) \times (1+Z_1^c)^{|\lambda_a-\lambda_1|/2}(1-Z_1^c)^{|\lambda_a+\lambda_1|/2} d^{j_1}_{m_1 0}(\theta_1) e^{im_1\phi_1}\, C^{J j_1 m_1 \lambda_1}_{\lambda_a}(s, s_{23}) , \quad (3.4)$$

where $Z_1^c \equiv \cos\theta_1^c$ and the limits of the summation are given by

$$|m_1| \le \epsilon , \quad (3.5a)$$

$$K \equiv \max\{|\lambda_a|, |m_1-\lambda_1|\} \le J \le K + \delta + \epsilon - \tfrac{1}{2}(|\lambda_a + m_1 - \lambda_1| - |\lambda_a - \lambda_1| + |\lambda_a - m_1 + \lambda_1| - |\lambda_a + \lambda_1|) , \quad (3.5b)$$

$$|m_1| \le j_1 \le \gamma + \epsilon , \quad (3.5c)$$

and γ, δ, and ϵ are the highest powers of s_{12}, t_{1a}, and t_{2a}, respectively, in $\mathcal{P}_5$. A sketch proof of (3.4) appears in the Appendix.

Similarly we have checked that

$$(1-\overline{Z}_1^{c2})^{|\lambda_a-\lambda_1|}\overline{\mathcal{P}}_5(\overline{Z}) = \sum_{J, j_1, m_1} \left(\frac{t_{1a}{}^2 s_{23}}{[\Delta_2(P_a, -P_1)\Delta_2(P_a-P_1, P_a-P)\Delta_2(P_2, P_3)]^{1/2}}\right)^{1/2} \times (1-\overline{Z}_1^{c2})^{|\lambda_a-\lambda_1|/2} d^J_{\lambda_a-\lambda_1, m_1}(\overline{\theta}_1^c) d^{j_1}_{m_1 0}(\overline{\theta}_1) e^{im_1\overline{\phi}_1} C^{J j_1 m_1}_{\lambda_a\lambda_1}(t_{1a}, s_{23}) , \quad (3.6)$$

where now the limits of the summations are given by

$$|m_1| \le \epsilon , \quad (3.7a)$$

$$K' \equiv \max\{|\lambda_a - \lambda_1|, |m_1|\} \le J \le K' + \delta + \epsilon - \tfrac{1}{2}(|\lambda_a - \lambda_1 - m_1| + |\lambda_a - \lambda_1 + m_1| - 2|\lambda_a - \lambda_1|) , \quad (3.7b)$$

$$|m_1| \le j_1 \le |m_1| + \gamma + \epsilon , \quad (3.7c)$$

and γ, δ, and ϵ are the highest powers of t_{2b}, s, and t_{2a}, respectively, in $\mathcal{P}_5(\overline{Z})$.

In Sec. IIIB, when using (3.2) and (3.3), we ignore the factors 1, ϵ_2, corresponding to $\overline{T}^{(1)}$ and $\overline{T}^{(2)}$. This leads to no error for, as may be seen from (2.11), the RCM only relates amplitudes of the same signature in different channels. The omission of these factors is made for reasons of simplicity—the square-root singularities at $\epsilon_2 = 0$ thus reintroduced are, of course, integrable and so the integrals over the Euclidean region performed in the next section are well defined.

B. Sum rules

The crossing relation for RHA's, expressed in terms of the inverse RCM [(2.15) and (2.16)], is

$$\overline{X}^{(1,2)-1}_{\lambda_1'\lambda_a}\overline{T}^{(1,2)(s_{ab})}_{\lambda_1\lambda_a} = \overline{T}^{(1,2)(t_{1a})}_{\lambda_a'\lambda_1'} , \quad (3.8)$$

where parity conservation allows us to set $\lambda_1 = \lambda_a' = +\frac{1}{2}$. We obtain our sum rules from the resulting identity

$$\Delta_3(P, P_a, P_1)\mathcal{P}_5^*(Z)\overline{X}^{(1,2)^{-1}}_{\lambda_1^*\lambda_a'}\overline{T}^{(1,2)(s_{ab})}_{\lambda_1'\lambda_a'} = \Delta_3(Q, P_a, P_a - P)\overline{\mathcal{P}}_5^*(\overline{Z})\overline{T}^{(1,2)(t_{1a})}_{\lambda_a^*\lambda_1^*} , \quad (3.9)$$

where again $\mathcal{P}_5$ is an arbitrary polynomial in the set Z and $\overline{\mathcal{P}}_5$ is the same polynomial expressed in terms of $\overline{Z}$, the crossed-channel version of Z [of course, $\Delta_3(P,P_a,P_1)\equiv\Delta_3(Q,P_a,P_a-P)$ for $Q=P_a-P_1$]. Next we integrate (3.9) over the Euclidean region:

$$\int d\omega\,\Delta_3(P,P_a,P_1)\mathcal{P}_5^*(Z)\overline{X}^{(1,2)^{-1}}_{\lambda_1^*\lambda_a'}\overline{T}^{(1,2)(s_{ab})}_{\lambda_1'\lambda_a'}=\int d\omega'\Delta_3(Q,P_a,P_a-P)\,\overline{\mathcal{P}}_5^*(\overline{Z})\overline{T}^{(1,2)(t_{1a})}_{\lambda_a^*\lambda_1^*}, \quad (3.10)$$

projecting out a finite number of partial waves in each channel.

Consider first the left-hand side of (3.10), $\mathcal{I}_s^{(1,2)}$ say. On substituting into $\mathcal{I}_s^{(1,2)}$ the partial-wave expansions (3.2) and (3.4), together with the expression (2.7) for the phase-space factor $d\omega$ and using the linearity of $\overline{X}^{(1,2)-1}$ in $\cos\theta_1^c$ we find after integrating over θ_1, θ_1^c, and ψ_1—and a great deal of algebra—that

$$\mathcal{I}_s^{(1,2)}=-\pi^2\int ds\,ds_{23}A^7s\sum_{J\geq K}^{J_+}\sum_{j_1\geq|m_1|}^{j_+}(2J+1)\sum_{n=0}^{3}\alpha_n^{(\lambda_1^*,\lambda_a')(1,2)}(s,s_{23})\sum_{i=-n}^{n}\gamma_n^{(i)(\lambda_1',\lambda_a',m_1)}$$
$$\times[C^{Jj_1m_1\lambda_1^*}_{\lambda_a'}(s,s_{23})\mathcal{T}^{J-i,j_1,m_1,\lambda_1'}_{\lambda_a'}(s,s_{23})\pm(-1)^{\lambda_1'+\lambda_a'}C^{J,j_1,m_1,-\lambda_1^*}_{-\lambda_a'}(s,s_{23})\mathcal{T}^{J-i,j_1,-m_1,\lambda_1'}_{\lambda_a'}(s,s_{23})], \quad (3.11)$$

where J_+ and j_+ are given by (3.5b) and (3.5c) and we define the known quantities $\alpha_n^{(\lambda_1^*,\lambda_a')(1,2)}(s,s_{23})$ and $\gamma_n^{(i)(\lambda_1',\lambda_a',m_1)}$ in the Appendix.

Similarly, calling the right-hand side of (3.10) $\mathcal{T}_t^{(1,2)}$, we find after integrating over the angular dependence that

$$\mathcal{I}_t^{(1,2)}=-\pi^2\int dt_{1a}ds_{23}s_{23}^{-1/2}[\Delta_2(P_a,-P_1)\Delta_2(Q,P_b)]^{3/2}[D_2^{\pm}(P_a,-P_1)]^{1/2}$$
$$\times\sum_{J\geq K'}^{J_+}\sum_{j_1\geq|m_1|}^{j_+}\left\{C^{Jj_1m_1*}_{\lambda_a^*\lambda_1^*}(t_{1a},s_{23})\left[\mathcal{T}^{Jj_1m_1}_{\lambda_a^*\lambda_1^*}(t_{1a},s_{23})-\sum_{i=-2}^{2}\delta^{(i)(\lambda_a^*,\lambda_1^*,m_1)}\mathcal{T}^{J-i,j_1,m_1}_{\lambda_a^*\lambda_1^*}(t_{1a},s_{23})\right]\right.$$
$$\left.\pm(-1)^{\lambda_a^*-\lambda_1^*}C^{Jj_1m_1*}_{-\lambda_a^*,-\lambda_1^*}(t_{1a},s_{23})\left[\mathcal{T}^{Jj_1-m_1}_{\lambda_a^*,\lambda_1^*}(t_{1a},s_{23})-\sum_{i=-2}^{2}\delta^{(i)(\lambda_a^*,\lambda_1^*,-m_1)}\mathcal{T}^{J-i,j_1,-m_1}_{\lambda_a^*,\lambda_1^*}(t_{1a},s_{23})\right]\right\}. \quad (3.12)$$

The variables $\delta^{(i)(\lambda_a^*,\lambda_1^*,m_1)}$ are defined in the Appendix, and are analogous to the $\alpha_n^{(\lambda_a^*,\lambda_1)}$ mentioned above.

Thus, equating

$$\mathcal{I}_s^{(1)}=\mathcal{I}_t^{(1)} \quad (3.13)$$

and

$$\mathcal{I}_s^{(2)}=\mathcal{I}_t^{(2)} \quad (3.14)$$

for $\lambda_1^*=-\frac{1}{2},\ \frac{1}{2}$, we obtain the required sum rules relating only a finite number of partial waves in each channel, where, of course, the integrals are performed over the ranges specified in the Appendix.

IV. CONCLUSIONS

In this work our objective was to obtain sum rules—constraints on partial waves—for production processes like those obtained by Balachandran *et al.* for four-particle scattering and by Modjtehedzadeh for five-particle spinless processes. We began by summarizing the contents of our previous paper[4] in which we considered in detail the analytic behavior of the RCM in order to check that the integrals over the matrix's angular dependence could be performed. Next, after checking that an arbitrary polynomial in Z had a finite partial-wave expansion it was a straightforward, though algebraically involved, task to derive sum rules connecting partial-wave amplitudes for crossing-related processes.

We have considered[8] one obvious application of these sum rules, i.e., isobar-model fits to production and (the crossing-related) annihilation processes. In complete analogy to the work of Roskies[9] on the application of sum rules to π-π elastic scattering, on substituting the isobar-model partial-wave amplitudes into the sum rules (3.13) and (3.14) we obtain a set of constraints on the parameter-dependent amplitudes. Thus the imposition of the sum rules reduces the number of degrees of freedom in a fit to experimental data. We consider this to be the most immediate use of our work.

For reasons of space we do not give these isobar-model constraints here. They will be included in a paper on more general applications, now being prepared.

APPENDIX

1. Angles

We begin by defining the special Gram determinants $\Delta_2(P,q)$, $\Delta_3(P,q,R)$, and $\Delta_4(P,q,R,s)$ using the notation of Ref. 3:

$$\Delta_2(P,q) = -\begin{pmatrix} P & q \\ P & q \end{pmatrix},$$

where

$$\begin{pmatrix} q_1 & q_2 & \cdots & q_n \\ R_1 & R_2 & \cdots & R_n \end{pmatrix} = \det(q_i \cdot R_k)$$

and $(q_i \cdot R_k)$ is the $n \times n$ matrix whose (i,k) entry is $q_i \cdot R_k$. Similarly

$$\Delta_3(P,q,R) = \begin{pmatrix} P & q & R \\ P & q & R \end{pmatrix}$$

and

$$\Delta_4(P,q,R,s) = \begin{pmatrix} P & q & R & s \\ P & q & R & s \end{pmatrix}.$$

Then

$$D_n^{\pm}(P_1, P_2, \ldots, P_{n-2}; P_{n-1}, P_n)$$

are defined by

$$\Delta_{n-2}(P_1, \ldots, P_{n-2})\, \Delta_n(P_1, \ldots, P_n) = D_n^{+}(P_1, P_2, \ldots, P_{n-2}; P_{n-1}, P_n) \times D_n^{-}(P_1, P_2, \ldots, P_{n-2}; P_{n-1}, P_n),$$

where

$$D_n^{\pm}(P_1, P_2, \ldots, P_{n-2}; P_{n-1}, P_n) = [\Delta_{n-1}(P_1, P_2, \ldots, P_{n-2}, P_{n-1})]^{1/2} \times [\Delta_{n-1}(P_1, P_2, \ldots, P_{n-2}, P_n)] \pm \begin{pmatrix} P_1 & P_2 & \cdots & P_{n-2} & P_{n-1} \\ P_1 & P_2 & \cdots & P_{n-2} & P_n \end{pmatrix}.$$

Thus

$$D_2^{\pm}(P,q) = P \cdot q \pm (P^2)^{1/2}(q^2)^{1/2}$$

and

$$D_3^{\pm}(P,q,R) = [\Delta_2(P,q)]^{1/2}[\Delta_2(P,R)]^{1/2} \pm \begin{pmatrix} P & q \\ P & R \end{pmatrix}.$$

Next we define the angles and rapidities used in this paper. In Table I we give the polar and azimuthal angles and the rapidities, while in Table II we give the crossing angles. The crossed-channel versions of the angles in Table I are simply obtained by the substitutions ($P_a \to q_a$; $P_b \to q_1$; $P_1 \to q_b$; $P_2 \to q_2$; $P_3 \to q_3$).

TABLE I. Polar and azimuthal angles and rapidities.

	Initial-state particle ($i=a,b$)	Final-state particle ($k=1,2,3$).
$\cosh\xi$	$\dfrac{P \cdot P_i}{m_i\sqrt{s}}$	$\dfrac{P \cdot P_k}{m_k\sqrt{s}}$
$\sinh\xi$	$\dfrac{[\Delta_2(P,P_i)]^{1/2}}{m_i\sqrt{s}}$	$\dfrac{[\Delta_2(P,P_k)]^{1/2}}{m_k\sqrt{s}}$
$\cos\theta^c$	$+1$ for $i=a$ -1 for $i=b$	$\dfrac{-\begin{bmatrix} P & P_a \\ P & P_k \end{bmatrix}}{[\Delta_2(P,P_a)]^{1/2}[\Delta_2(P,P_k)]^{1/2}}$
$\sin\theta^c$	0	$\dfrac{[s\Delta_3(P,P_a,P_k)]^{1/2}}{[\Delta_2(P,P_a)]^{1/2}[\Delta_2(P,P_k)]^{1/2}}$
$\cos\theta$		$\dfrac{\begin{bmatrix} P-P_k & P_i \\ P-P_k & P_j \end{bmatrix}}{[\Delta_2(P,P_k)]^{1/2}[\Delta_2(P_i,P_j)]^{1/2}}$
$\sin\theta$		$\dfrac{[s_{ij}\Delta_3(P_1,P_2,P_3)]^{1/2}}{[\Delta_2(P,P_k)]^{1/2}[\Delta_2(P_i,P_j)]^{1/2}}$
$\cos\phi$	1	$\dfrac{\begin{bmatrix} P & P_a & P_1 \\ P & P_a & P_k \end{bmatrix}}{[\Delta_3(P,P_a,P_1)]^{1/2}[\Delta_3(P,P_a,P_k)]^{1/2}}$
$\sin\phi$	0	$\dfrac{\epsilon(P,P_a,P_1,P_k)[\Delta_2(P,P_a)]^{1/2}}{[\Delta_3(P,P_a,P_1)]^{1/2}[\Delta_3(P,P_a,P_k)]^{1/2}}$
$\cos\psi$	1	$\dfrac{\begin{bmatrix} P & P_1 & P_a \\ P & P_1 & P_k \end{bmatrix}}{[\Delta_3(P,P_a,P_1)]^{1/2}[\Delta_3(P,P_1,P_k)]^{1/2}}$
$\sin\psi$	0	$\dfrac{\epsilon(P,P_1,P_a,P_k)[\Delta_2(P,P_1)]}{[\Delta_3(P,P_a,P_1)]^{1/2}[\Delta_3(P,P_1,P_k)]^{1/2}}$

2. Limits of integrations

The limits of the integration over the Euclidean region in (2.7) are given by

$$-1 \leq \cos\theta_1^c \leq 1,$$

$$-1 \leq \cos\theta_1 \leq 1,$$

$$0 \leq \psi_1 \leq 2\pi,$$

$$(m_a - m_b)^2 \leq s \leq (m_a + m_b)^2,$$

and

$$s_{23}^{-} \leq s_{23} \leq s_{23}^{+},$$

TABLE II. Crossing angles.

	$i=a$	$i=1$
$\cos\psi$	$\dfrac{\begin{bmatrix} P_a & P \\ P_a & P_1 \end{bmatrix}}{[\Delta_2(P,P_a)\Delta_2(P_a-P_1)]^{1/2}}$	$\dfrac{\begin{bmatrix} P_1 & P \\ P_1 & P_a \end{bmatrix}}{[\Delta_2(P,P_1)\Delta_2(P_a-P_1)]^{1/2}}$
$\sin\psi$	$\left[\dfrac{m_a{}^2\Delta_3(P,P_a,P_1)}{[\Delta_2(P,P_a)\Delta_2(P_a-P_1)}\right]^{1/2}$	$\left[\dfrac{m_1{}^2\Delta_3(P,P_a,P_1)}{\Delta_2(P,P_1)\Delta_2(P_a-P_1)}\right]^{1/2}$

where

$$s_{23}^{+}=\min\{(m_2+m_3)^2,\,(m_1+\sqrt{s})^2\},$$

$$s_{23}^{-}=\max\{(m_2-m_3)^2,\,(m_1-\sqrt{s})^2\}.$$

The crossed-channel angles $\bar\theta_1^c$, $\bar\theta_1$, and $\bar\psi_1$ have the same range of values, while the limits on t_{1a} and s_{23} in (2.8) are given by

$$0\le t_{1a}\le 4m_a{}^2\quad (m_a=m_1)$$

and

$$s_{23}'^{-}\le s_{23}\le s_{23}'^{+},$$

where

$$s_{23}'^{+}=\min\{(m_2+m_3)^2,\,(m_1+\sqrt{t_{1a}})^2\},$$

$$s_{23}'^{-}=\max\{(m_2-m_3)^2,\,(m_1-\sqrt{t_{1a}})^2\}.$$

3. Coefficient functions used in sum rules

The coefficient function $\alpha_n^{(\lambda_1^*,\lambda_a^1)(1,2)}(s,s_{23})$ which appears in (3.11) is simply defined by

$$[1-(Z_1^c)^2]\bar X_{\lambda_1^*,\lambda_a'}^{(1,2)\ -1}=\sum_{n=0}^{3}\alpha_n^{(\lambda_1^*,\lambda_a')(1,2)}(s,s_{23})(Z_1^c)^n .$$

That the left-hand side of the above equation is a cubic in Z_1^c follows from (2.15), (2.16), where $\bar X^{(1,2)-1}$ is shown to be linear in Z_1^c.

The coefficients $\gamma_n^{(i)(\lambda_a',\lambda_1',m_1)}$ are given by the relation

$$(Z_1^c)^n d^{J'}_{\lambda_a',m_1-\lambda_1'}(\theta_1^c)=\sum_{i=-n}^{n}\gamma_n^{(i)(\lambda_a',\lambda_1',m_1)}d^{J'+i}_{\lambda_a',m_1-\lambda_1'}(\theta_1^c).$$

For any given value of n, the explicit form of these coefficients can be found by repeated use of the recursion relation

$$\begin{aligned}ZP_n^{(\alpha,\beta)}(Z)=&[(2n+\alpha+\beta)(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)]^{-1}\\&\times[2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)P_{n-1}^{(\alpha,\beta)}(Z)+(\beta^2-\alpha^2)P_n^{(\alpha,\beta)}(Z)\\&+2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)P_{n+1}^{(\alpha,\beta)}(Z)],\end{aligned}$$

where $P_n^{(\alpha,\beta)}(Z)$ is the Jacobi polynomial of degree η. Finally, $\delta^{(i)(\lambda_a^*,\lambda_1^*,m_1)}$, which appears in (3.12) is defined by

$$(\bar Z_1^c)^2 d^J_{\lambda_a^*-\lambda_1^*,m_1}(\bar\theta_1^c)=\sum_{i=-2}^{+2}\delta^{(i)(\lambda_a^*,\lambda_1^*,m_1)}d^{J+i}_{\lambda_a^*-\lambda_1^*,m_1}(\bar\theta_1^c).$$

Again, the coefficients can be found explicitly using the above recursion relation for the Jacobi polynomials.

4. Sketch proof of (3.4)

It suffices to prove the result for the case $\mathcal{P}_5(Z)=s^\alpha s_{23}{}^\beta s_{12}{}^\gamma t_{1a}{}^\delta t_{2a}{}^\epsilon$. We obtain the proof by showing that the expansion in the total angular momentum J and the dimeson angular momentum j_1 terminate when considered separately. Then, by orthogonality arguments, it follows that the full partial-wave expansion (3.4) terminates.

(a) The expansion in J. Here let the quantities A, B, C, etc. represent functions of s, s_{23}, s_{31} only. It is straightforward to show that

$$t_{1a}{}^\delta t_{2a}{}^\epsilon=\sum_{n=0}^{\epsilon}\ \sum_{l=0}^{\delta+\epsilon-n}\ \sum_{k=-n}^{+n}C_{\delta n l k}[1-(Z_1^c)^2]^{n/2}(Z_1^c)^l e^{ik\psi_1}.$$

Now $(1+Z_1^c)^{|\lambda_a-\lambda_1|/2}(1-Z_1^c)^{|\lambda_a+\lambda_1|/2}$ is a sum of terms of the form

$$[(Z_1^c)^l(1+Z_1^c)^{(n-|\lambda_a+k-\lambda_1|+|\lambda_a-\lambda_1|)/2}(1-Z_1^c)^{(n-|\lambda_a-k+\lambda_1|+|\lambda_a+\lambda_1|)/2}](1+Z_1^c)^{|\lambda_a+k-\lambda_1|/2}(1-Z_1^c)^{|\lambda_a-k+\lambda_1|/2}e^{ik\psi_1}.$$

The factor in square brackets, $\mathcal{K}(Z_1^c)$, is a polynomial of degree

$$\tfrac12(2l+2n-|\lambda_a+k-\lambda_1|+|\lambda_a-\lambda_1|-|\lambda_a-k+\lambda_1|+|\lambda_a+\lambda_1|)$$

and can therefore be written

$$\mathcal{K}(Z_1^c)=\sum_J D_J^{k\lambda_1\lambda_a}P_{J-K}^{(|\lambda_a+k-\lambda_1|,\,|\lambda_a-k+\lambda_1|)}(Z_1^c),$$

where the summation runs from

$$J=K\equiv\max\{|\lambda_a|,\,|k-\lambda_1|\}$$

to

$$J=K+l+\tfrac12(2n-|\lambda_a+k-\lambda_1|+|\lambda_a-\lambda_1|-|\lambda_a-k+\lambda_1|+|\lambda_a+\lambda_1|)$$

and again $P_n^{(\alpha,\beta)}$ is the Jacobi polynomial of degree n. Using the well-known relation between the Wigner rotation coefficients $d^J_{\lambda\lambda'}$ and the Jacobi polynomials, we find, after summing over n and l that

$$(1+Z_1^c)^{|\lambda_a-\lambda_1|/2}(1-Z_1^c)^{|\lambda_a+\lambda_1|/2} t_{1a}^{\delta}\, t_{2a}^{\epsilon} = \sum_{\substack{J-K=0 \\ |k|\le\epsilon}}^{\delta+\epsilon-1/2(|\lambda_a+k-\lambda_1|-|\lambda_a-\lambda_1|+|\lambda_a-k+\lambda_1|-|\lambda_a+\lambda_1|)} C^{Jk\lambda_1\lambda_a} d^J_{\lambda_a,\,k-\lambda_1}(\theta_1^c)\, e^{ik\psi_1},$$

i.e., the required finite expansion in J.

(b) The expansion in j_1. Similarly it can be shown that $s_{12}^{\gamma} t_{2a}^{\epsilon}$ (and therefore $\mathcal{P}_5$) has a finite expansion of j_1 of the form

$$s_{12}^{\gamma} t_{2a}^{\epsilon} = \sum_{j_1-|k|=0}^{\gamma+\epsilon} \sum_{k=-\epsilon}^{\epsilon} C_k^j d_{k0}^j(\theta_1)\, e^{ik\psi_1},$$

where C_K^J is a function of s, s_{23} and t_{1a} only.

(c) The complete expansion. Finally, because any function $f_5(Z)$ has a (in general nonterminating) partial-wave expansion similar to (3.2), i.e.,

$$f_5(Z) = \sum_{J,j_1,m_1} \left(\frac{s^2 s_{23}}{[\Delta_2(P_2,P_3)]^{1/2}}\right)^{1/2} N_J{}^2 N_{j_1} \cos(\theta^c_{1/2})^{|\lambda_a-\lambda_1|} \times \sin(\theta^c_{1/2})^{|\lambda_a+\lambda_1|} d^{j_1}_{m_1 0}(\theta_1)\, d^J_{\lambda_a,\,m_1-\lambda_1}(\theta_1^c)\, e^{im_1\psi_1},$$

it is straightforward, though tedious, to show that, on integrating over the angular dependence $(\theta_1^c, \theta_1, \psi_1)$ having first multiplied by the appropriate Wigner rotation, the orthogonality of the d functions, together with the results of (a) and (b) above, implies that the complete partial-wave expansion (3.4) terminates in both J, j_1, and m_1 [where the limits are just those of (a) and (b)].

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