New narrow-resonance models with nonlinear Regge trajectories

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New crossing-symmetric narrow-resonance models with nonlinear Regge trajectories are presented. The N-point amplitudes have cyclic crossing symmetry and are meromorphic in all channel invariants. The amplitudes are shown to factorize on the leading trajectory.

It is a very appealing idea to attempt a description of hadrons and strong-interaction phenomena based on a narrow-resonance approximation.[1] Within the context of quantum chromodynamics (QCD), such an approximation is probably provided by 't Hooft's large-number-of-colors limit, $N_c \to \infty$ with $N_c g_s^2$ fixed.[2]

Historically, a tremendous amount of work has been devoted to a particular class of narrow-resonance models, the so-called dual resonance models (DRM[25]). These models are all based on a fundamental hypothesis that all the narrow resonances lie on exactly linear Regge trajectories. Except for a very interesting model invented by Coon[3] and developed by Baker and Coon,[4] little thought has been devoted to narrow-resonance models with nonlinear Regge trajectories. There appears to be no good reason that any of the Regge trajectories of infinite-N_c QCD should be exactly linear, and, moreover, some of the trajectories, notably those on which the states of heavy quarkonia lie, are certainly nonlinear.[5] We therefore believe it is worthwhile to give serious consideration to the construction of self-consistent narrow-resonance models with nonlinear trajectories.

In this paper a large class of candidates for such models is presented. So far only the simplest self-consistency checks have been made on these models. For example, we have not yet established full factorizability of the N-point amplitudes, and, indeed, it is doubtful that the models in their present form do possess full factorizability. Nonetheless, these models seem to possess enough nice features, both theoretically and phenomenogically, to merit publication at this preliminary stage of their development. It is hoped that, in spite of flaws these models may possess, they will represent a useful step toward a satisfactory narrow-resonance approximation for the strong interactions.

In the models constructed below, the trajectory functions $\alpha_a(s)$ are roots of an nth-order polynomial with coefficients which are polynomials in s:

$$P(z, s) = z^n + c_1(s)z^{n-1} + c_2(s)z^{n-2} + \cdots + c_n(s)z^0 = \sum_{i=0}^{n} c_i z^{n-i}, \qquad (1)$$

where $c_i(s)$ is a polynomial in s. The functions $\alpha_a(s)$ are then determined from the equation

$$P(\alpha_a(s), s) = 0 .$$

In general there will be n such roots: $a = 1, 2, \ldots, n$. Physical requirements will place restrictions on the nature of the polynomials $c_i(s)$.

The trajectory functions $\alpha_a(s)$ will typically have branch cuts in s. For example, in the case $n = 2$ and

$$P(z, s) = z^2 - (\alpha' s + b)z + (cs + d) \qquad (2)$$

we have the two roots

$$\alpha_\pm(s) = +\tfrac{1}{2}(\alpha' s + b) \pm \left[\frac{(\alpha' s + b)^2}{4} - cs - d\right]^{1/2}$$
$$\equiv \frac{\alpha'(s + s_0)}{2} + \gamma \pm \left[\frac{[\alpha'(s + s_0)]^2}{4} + \beta\right]^{1/2} . \qquad (3)$$

Since scattering amplitudes in the narrow-resonance approximation must be meromophic in all invariants, the trajectory functions must enter in such a way that all these cuts are absent. A way to arrange this is to make use of the fact that

$$\sum_{a=1}^{n} f(\alpha_a(s)) \qquad (4)$$

is a meromorphic function of s if $f(z)$ is a meromorphic function of z. One can then immediately write down meromorphic amplitudes with crossing symmetry by evaluating the old dual resonance amplitudes expressed as a function of the trajectory functions, $A_N(\alpha(s_{ij}))$, with each $\alpha(s_{ij})$ replaced by $\alpha_{a_{ij}}(s_{ij})$, and then summing independently over each a_{ij}.

To see what this prescription yields, consider the four-point function for scalar particles:

$$A_4(s,t)=\sum_{a,b=1}^{n} B(-\alpha_a(s),-\alpha_b(t)), \tag{5}$$

where

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

is the Euler beta function. A_4 is clearly meromorphic and symmetric under $s\leftrightarrow t$. A_4 has poles in s whenever

$$\alpha_a(s)=k,\quad k=0,1,2,\ldots \tag{6}$$

for some a. Because of the sum over b, the residue of each of these poles is a polynomial in t. In fact, the residue is proportional to

$$\sum_{b=1}^{n}\frac{\Gamma(-\alpha_b(t))}{\Gamma(-k-\alpha_b(t))}=\sum_{b=1}^{n}[-k-\alpha_b(t)][1-k-\alpha_b(t)]\cdots[-1-\alpha_b(t)]=\sum_{b=1}^{n}P_k(-\alpha_b(t)), \tag{7}$$

with $P_k(z)$ a kth-order polynomial in z. The order of the polynomial as a function of t will depend on the detailed nature of the polynomial which determines the α_a, i.e., upon the polynomials c_i.

In the following, the $c_i(s)$ are restricted to be *linear functions* of s, and, in particular, we demand that

$$c_1(s)=-\alpha' s-b \tag{8}$$

with $\alpha'>0$. With these restrictions, the location s_k of the resonances corresponding to

$$\alpha_a(s_k)=k,\quad k=0,1,2,\ldots \tag{9}$$

is a unique function of k, and only one of the n trajectory functions, $\alpha_a(s)$, will pass through large positive integral values. We shall label this trajectory $\alpha_1(s)$, which accordingly has the asymptotic behavior

$$\alpha_1(s)\underset{s\to+\infty}{\sim}\alpha' s. \tag{10}$$

Such an asymptotically linear trajectory is compatible with a quark confining force which approaches a constant $T_0=1/2\pi\alpha'$ at large distances. In the $n=2$ example mentioned above, $\alpha_1(s)=\alpha_+(s)$. For the range of parameters $\gamma<0$, $\beta>0$, only $\alpha_+(s)$ achieves integral values ≥ 0.

For application to heavy quarkonium systems we mention the trajectories determined by the polynomial

$$P(z)=(z+1)^3-(\alpha' s+b)(z+1)^2-\alpha' m_q{}^2 c. \tag{11}$$

For this case, the mass level $\alpha_1(s_l)=l$ is

$$s_l=\frac{1}{\alpha'}(l+1)-\frac{b}{\alpha'}-\frac{m_q{}^2 c}{(l+1)^2}, \tag{12}$$

i.e., a linear superposition of a linear plus Coulomb mass spectrum.

Let us now return to Eq. (7) which contains information about the angular momentum of the resonances. To write (7) as a polynomial in t, we need the formula

$$\sum_{a=1}^{n}\alpha_a(s)^p=n\delta_{p0}-p\sum_{m=\max(p/n,1)}^{p}\ \sum_{i_1+i_2+\cdots+i_m=p}\frac{(-1)^{m+1}}{m}c_{i_1}c_{i_2}\cdots c_{i_m}. \tag{13}$$

With our restrictions on the $c_i(t)$, the maximum power of t which occurs on the right-hand side is in the term with $m=p$, $i_1=i_2=\cdots=i_p=1$, i.e., the power t^p. Thus, the right-hand side of (z) is a kth-order polynomial in t, so that the resonances at $\alpha=k$ have angular momentum $\leq k$.

So far we have seen that the four-point function of Eq. (5) is crossing symmetric, meromorphic in s with poles at $\alpha(s)=k$ with residues which are kth-order polynomials in t and similarly for the poles in t. Next consider the Regge limit $s\to\infty$ at fixed t. For definiteness consider $n=2$. When $s\to+\infty$, $\alpha_+(s)\sim\alpha' s$ whereas $\alpha_-(s)\to\gamma$. Thus the terms in (5) containing $\alpha_+(s)$ will be Regge behaved, but the terms containing $\alpha_-(s)$ will approach a constant:

$$A_4(s,t)\underset{s\to+\infty}{\sim}\sum_{b=1}^{2}s^{\alpha_b(t)}\Gamma(-\alpha_b(t))+\sum_{b=1}^{2}B(-\gamma,-\alpha_b(t))+O\left(s^{\alpha_b(t)-1},\frac{1}{s}\right). \tag{14}$$

In complex angular momentum plane parlance, the constant piece corresponds to a nonanalytic "Kronecker δ" singularity δ_{J0}. When we consider factorizability of the N-point function at resonances on the leading trajectory, we shall find that A_4 must be modified in such a way that when $\gamma=-1$, the Kronecker δ piece is absent. Even so, there will still be fixed powers like $1/s$ in the asymptotic behavior.: these correspond to fixed poles at nonsense values of J and are not ruled out by unitarity in a narrow-resonance approximation. However, these poles must move with energy as soon as unitarity corrections are turned on.

Now let us consider the high-energy fixed-angle behavior of A_4. In this limit $s\to+\infty$ and $t\to-\frac{1}{2}s(1-\cos\theta)\to-\infty$. So $\alpha_+(s)\to\alpha's$, $\alpha_-(s)\to\gamma$, $\alpha_+(t)\to\gamma$, and $\alpha_-(t)\to\alpha't$. The term containing $\alpha_+(s)$ and $\alpha_-(t)$ will fall off exponentially. The term containing $\alpha_+(s)$ and $\alpha_+(t)$ will behave as s^γ, as will the term with $\alpha_-(s)$ and $\alpha_-(t)$. Finally the term with $\alpha_-(s)$ and $\alpha_+(t)$ will approach the constant

$$B(-\gamma,-\gamma)\,. \tag{15}$$

Thus, our amplitudes incorporate power behavior at large momentum transfer, certainly a desirable feature from the point of view of phenomenology.

As a final self-consistency check on our models, let us consider the factorizability of the N-point function. The obvious guess for the N-point function would be, using Koba-Nielsen variables z_i,[1]

$$A_N(s_{ij})=\int\frac{dz_1\cdots[dz_a]\cdots[dz_b]\cdots[dz_c]\cdots dz_N(z_b-z_a)(z_c-z_b)(z_a-z_c)}{(z_3-z_1)(z_4-z_2)\cdots(z_N-z_{N-2})(z_1-z_{N-1})(z_2-z_N)}\times\prod_{ij}\left\{\sum_{a=1}^{n}\left[\frac{(z_j-z_i)(z_{j+1}-z_{i-1})}{(z_j-z_{i-1})(z_{j+1}-z_i)}\right]^{-\alpha_a(s_{ij})-1}\right\}, \tag{16}$$

where z_a, z_b, z_c are fixed and the range of integration is $z_1\le z_2\le\cdots\le z_N$; $s_{ij}=-(p_i+p_{i+1}+\cdots+p_j)^2$. This amplitude has cyclic crossing symmetry and meromorphy as required. However, the residues of the poles are not necessarily factorizable. Let us consider factorizability in the $1,\ldots,k$ channel, for which it is convenient to choose $z_1=0$, $z_{k+1}=1$, and $z_N=\infty$.[1] Further call $z_k=z$ and change variables to

$$z_i=\begin{cases} zy_i\,, & i\le k\\ \dfrac{1}{\bar{y}_j}\,, & i>k\end{cases} \tag{17}$$

so that A_N can be written

$$A_N=\int\{dy_i\}\{d\bar{y}_j\}I(y_i)I(\bar{y}_j)\int_0^1 dz\sum_{a=1}^{n}z^{-\alpha_a(s_{1k})-1}\frac{1}{(1-zy_{n-1})(1-z\bar{y}_{k+2})}\times\prod_{i=2}^{k}\prod_{j=k+1}^{N-1}\left\{\sum_{b=1}^{n}\left[\frac{(1-zy_i\bar{y}_j)(1-zy_{i-1}\bar{y}_{j+1})}{(1-zy_{i-1}\bar{y}_j)(1-zy_i\bar{y}_{j+1})}\right]^{-\alpha_b(s_{ij})-1}\right\}. \tag{18}$$

The pole at $\alpha_1(s_{1k})=l$ will come from integrating

$$\int_0^1 dz\,z^{-\alpha_1(s_{1k})-1+l}=\frac{1}{l-\alpha_1(s_{1k})}\,,$$

i.e., we must expand the factors multiplying $z^{-\alpha_1(s_{1k})-1}$ in a power series in z and isolate the term z^l. Consider the terms corresponding to a resonance of spin l, the maximum spin at this level. These are the terms with l factors of the various s_{ij}. Inspection of Eq. (13), tells us that

$$\sum_{a=1}^{n}\alpha_a(s)^p=n\delta_{p0}+(c_1)^p+\text{lower powers of } s=n\delta_{p0}+(\alpha's+b)^p+\text{lower powers of } s\,.$$

Thus for factorizing the spin-l resonances, one can replace

$$\sum_{a=1}^{n}\left[\frac{(1-zy_i\bar{y}_j)(1-zy_{i-1}\bar{y}_{j+1})}{(1-zy_{i-1}\bar{y}_j)(1-zy_i\bar{y}_{j+1})}\right]^{-\alpha_a(s_{ij})-1}\to n-1+\left[\frac{(1-zy_i\bar{y}_j)(1-zy_{i-1}\bar{y}_{j+1})}{(1-zy_{i-1}\bar{y}_j)(1-zy_i\bar{y}_{j+1})}\right]^{-(\alpha's_{ij}+b)-1}\,.$$

All resonances on the leading Regge trajectory will factorize just as in the ordinary dual model, provided Eq. (16) is modified to read

$$A_N=\int\frac{\{dz\}}{(z_3-z_1)\cdots(z_2-z_N)}\prod_{ij}\left\{1-n+\sum_{a=1}^{n}\left[\frac{(z_j-z_i)(z_{j+1}-z_{i-1})}{(z_j-z_{i-1})(z_{j+1}-z_i)}\right]^{-\alpha_a(s_{ij})-1}\right\}. \tag{19}$$

Consider now what the modification in Eq. (19) does to the four-point function. A_4 changes to

$$A_4(s,t)=(1-n)^2+(1-n)\sum_{a=1}^{n}\left[\frac{1}{-\alpha_a(s)}+\frac{1}{-\alpha_a(t)}\right]+\sum_{a,b}B(-\alpha_a(s),-\alpha_b(t))\,. \tag{20}$$

The resonance structure is not modified except that the residue of the pole at $\alpha=0$ changed from n to 1. There are new fixed power pieces in the asymptotic behavior, however. We notice that the constant piece of the asymptotic behavior is now

$$1-\left[\frac{1}{-\gamma}+\sum_{a=1}^{n}\frac{1}{-\alpha_a(t)}\right]+\sum_b B(-\gamma,\ -\alpha_b(t))=1+\frac{1}{\gamma}+\sum_b\left[\frac{1}{\alpha_b(t)}+\frac{\Gamma(-\gamma)\Gamma(-\alpha_b(t))}{\Gamma(-\gamma-\alpha_b(t))}\right]$$

and that this $=0$ if $\gamma=-1$. The remaining fixed powers in the asymtotic behavior correspond to fixed poles at nonsense values of the angular momentum. The constant piece in the fixed-angle asymptotic behavior also disappears for this value of γ.

We have not yet succeeded in showing that Eq. (19) factorizes for resonances on the nonleading trajectories. It would not be surprising if further modifications are required to achieve full factorizability, to say nothing of the absence of ghosts. Nonetheless, we think it is remarkable that such a simple-minded extension of the old dual models possesses the degree of self-consistency already established. In view of the fact that infinite-N_c QCD will predict at least some nonlinear trajectories, we feel that further development of these types of models will be worthwhile.

We conclude this paper with some remarks on phenomenology and future directions. Recent data from inclusive production of π and η mesons indicate that the ρ and A_2 trajectories tend to flatten off for $-t \gtrsim 1.5$ GeV^2.[6] Our $n=2$ model has trajectories with this behavior. In this regard it would be interesting to extend the Lovelace-Shapiro model[7] for pion scattering to incorporate nonlinear trajectories. One's first guess might be

$$\sum_{a,b}\frac{\Gamma(1-\alpha_a(s))\Gamma(1-\alpha_b(t))}{\Gamma(1-\alpha_a(s)-\alpha_b(t))}\ .$$

However, the need for modifications as in Eq. (20) is likely. These modifications were required by factorization of the leading trajectory resonances in the N-point function, so one has to look at N-point extensions[8] of the Lovelace-Shapiro model.

Where does one go from here? We presented these models as a step toward guessing a set of scattering amplitudes for large-N_c QCD. It seems that factorizability of the N-point function is indeed very restrictive and that there may be only a limited number of possible narrow-resonance models which can be taken as consistent Born approximations. We doubt that any of the models we have presented here are completely consistent—although this has not yet been ruled out. We hope that they do indicate a useful direction for further research.

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[1]For background on the early development of this idea, see the lectures by S. Mandelstam, in *Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser, M. Grisaru, and H. Pendleton, (MIT, Cambridge, Mass., 1970).

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