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Comment on connections between nonlinear evolution equations

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An open problem raised in a recent paper by Chodos is treated. We explain the reason for the interrelation between the conservation laws of the Korteweg-de Vries (KdV) and sine-Gordon equations. We point out that it is due to a corresponding connection between the infinite-dimensional Abelian symmetry groups of these equations. While it has been known for a long time that a Bäcklund transformation (in this case the Miura transformation) connects corresponding members of the KdV and the sine-Gordon families, it is quite obvious that no Bäcklund transformation can exist between different members of these families. And since the KdV and sine-Gordon equations do not correspond to each other, one cannot expect a Bäcklund transformation between them; nevertheless we can give explicit relations between their two-soliton solutions. No inverse scattering techniques are used in this paper.

In recent independent contributions,[1] it has been demonstrated that the geometric nature of the Korteweg-de Vries (KdV) equation and the sine-Gordon equation (SGE) is similar insofar as they both describe pseudospherical surfaces, i.e., surfaces of constant negative Gaussian curvature.

A connection between the KdV and the modified KdV (mKdV) equations [cf. Eq. (9) below] has been established by the Miura transformation.[2] Since the mKdV equation and the SGE lead to the same inverse scattering problem[3] the existence of further relations between these two equations is suggested. But unfortunately one never finds Bäcklund transformations between different equations which are solvable by the *same* inverse scattering transform.

However, Chodos[4] confirmed the impression that there *are* more intimate connections between the two equations. He managed to do so by finding a very ingenious way of relating the conservation laws of these equations with each other. He gave an explicit relation between the corresponding Lax operators and then interpreted the coefficients of the derivatives $\delta^{(n)}(0)$ (i.e., the nth derivative of the δ distribution) in the traces of the powers of the Lax operator as conservation laws of the evolution equations under consideration. At first sight expressions like $\delta^{(n)}(0)$ appear to be rather strange, but in fact the author gave a rigorous approach in terms of pseudodifferential operators. This procedure yields finally a direct correspondence between the respective conservation laws. At the end of his paper the author states "Whether this (correspondence) is merely fortuitous, or whether it bespeaks a more profound connection between the two theories remains an open question." We would like to point out that in fact the latter is true. There *is* a profound connection between the two evolution equations. A detailed analysis[5] shows that the mKdV equation is just the second generalization of the potential form of the SGE (exactly in the same sense in which the KdV equation is related to its generalizations). Furthermore, there is a complex Bäcklund transformation between the KdV and the mKdV equations. Now, if two evolution equations are related by a Bäcklund transformation, then there are explicit correspondences between their symmetries, their conservation laws, and their soliton solutions.[6] Furthermore, if two equations belong to the same hierarchy (i.e., one is a generalization of the other), then they have the same symmetries and conservation laws.[5] This certainly explains the relation between the conservation laws.

The relations between the corresponding soliton solutions are a bit more sophisticated. There is unfortunately no direct way to interrelate the solutions of different members of the same hierarchy. But the following statement holds: The finite-dimensional manifold given by all N-soliton solutions is the same for all members of the hierarchy; only the orbits on these manifolds (with respect to the time evolution) and the asymptotic speeds depend on the special member of the hierarchy to which the solution belongs. Indeed, the fact that the time history of the multisoliton orbits on the corresponding finite-dimensional manifold differ for different members of the hierarchy is responsible for the observation that there is no Bäcklund transformation between these different members.

In the case of one-soliton solutions the orbits are easily related to each other since the corresponding manifold is one-dimensional (and generated by a translation in the x space). As far as the two-soliton solutions are concerned, the two-dimensional manifolds are generated by space (x) and time (t) translations. Hence, there also exists a transformation between the corresponding independent variables (x, t) connecting the two-soliton solutions of different members of the hierarchy with each other.[7]

To make the preceding a bit more transparent we have to emphasize the Lie-algebra aspects of nonlinear evolution equations. Let us study the evolution equation

$$u_t = K_1(u), \tag{1}$$

where K_1 is a nonlinear map, e.g., the right-hand side of the KdV equation, in a suitable space of C^∞ functions. This equation may be considered as the equation for the infinitesimal generator of the one-parameter resolvent group $R_K(t)$, where $R_K(t)$ maps the initial condition $u(0)$ onto $u(t)$. These resolvent groups generate in a canonical way the Lie algebra

$$[G, H]: [G, H](u) = \frac{\partial}{\partial\epsilon}[G(u + \epsilon H(u)) - H(u + \epsilon G(u))]\Big|_{\epsilon=0}. \tag{2}$$

Hence, R_G is a one-parameter symmetry group for (1) if and only if the flow $u_t = G(u)$ commutes with (1), which is equivalent to $[K_1, G] = 0$ for the infinitesimal generator G. Of course, Bäcklund transformations between different equations also automatically provide isomorphisms for the respective Lie algebras.

Let us now consider linear operators Φ in the Lie algebra (given by operator-valued functions in the solution space) which have the property

$$\Phi^2[G, H] + [\Phi G, \Phi H] = \Phi([G, \Phi H] + [\Phi G, H]) \tag{3}$$

for all G and H. These operators may be derived from very special deformations of the Lie algebra (cf. Ref. 8). In Ref. 5 the name *hereditary symmetries* has been assigned to them. If such an operator commutes with K (i.e., $\Phi[K, G] = [K, \Phi G]$ for all G), then $\{\Phi^n K \mid n \in N$ or even Z if that makes sense$\}$ is an Abelian subalgebra of our Lie algebra. In other words, every equation of the form $u_t = K_n(u)$, $K_n = \Phi^n K$, admits a (usually infinite-dimensional) Abelian group of symmetries.

In this paper we deal only with hereditary Φ which commute with K_0 (where $K_0(u) = u_x$) and which are invertible in a weak sense. Then the class of equations $u_t = K_n(u)$ $(n \in Z, K_n = \Phi^n K_0)$ is called a hierarchy and the equation with the right-hand side K_{n+l} is said to be the lth generalization of the equation with right-hand side K_n. All the K_n are infinitesimal generators of symmetries for the whole hierarchy and they are all generated by Φ. But Φ has further (and very surprising) properties,[5] namely,

(i) If we denote by $\Phi^\dagger(u)$ the adjoint of $\Phi(u)$, then for all equations in the hierarchy, gradients of conservation laws are mapped by $\Phi^\dagger(u)$ onto gradients of conservation laws.

(ii) The N-soliton solutions of $u_t = K_n(u)$ are completely described by the decomposition $u_x = \sum_{i=1}^{N} \omega_i$, where the ω_i are eigenvectors of $\Phi(u)$ with eigenvalues given by $-(c_i)^{1/n}$. The c_i are the asymptotic speeds of the solitons supported by the system.

It is of interest to note that the property (i) is a weak form of Noether's theorem (an analysis of this point can be found in[11] and a detailed version[9] is to appear soon). Summarizing, it may be stated that the properties of the system with respect to symmetries, conservation laws, and solitons of the whole hierarchy are completely described by the operator Φ.

After these rather general remarks we would like to return to the case of the KdV equation. It has been shown[5] that the operator-valued function

$$\Phi(u) = D^2 + 4u_x D^{-1} u + 4u^2 \tag{4}$$

(D denotes the differential operator and D^{-1} its inverse) is hereditary. Using the Bäcklund transformation (modified Miura transformation) given by

$$\tilde{u} = u^2 + iu_x \tag{5}$$

to define $\tilde{u}$, we obtain by virtue of the canonical Lie-algebra isomorphism another hereditary symmetry,[6] namely,

$$\Psi(\tilde{u}) = D^2 + 4u + 2u_x D^{-1}. \tag{6}$$

Hence all equations of the form

$$\tilde{u}_t = \psi(\tilde{u})^n \tilde{u}_x$$

and

$$u_t = \Phi(u)^n u_x \quad \text{with } n \in Z \tag{7}$$

are completely integrable and, for the same n, interrelated by the Bäcklund transformation (5). Evaluation of the right-hand sides of these equations yields

$$u_t = \Phi(u)^{-1} u_x = \tfrac{1}{2}\sin\left(2\int_{-\infty}^{x} u(\xi)\, d\xi\right) \quad \text{(the potential form of the sine-Gordon equation)} \tag{8}$$

and

$$u_t = \Phi(u) u_x = u_{xxx} + 6u^2 u_x \quad \text{(the modified KdV equation, i.e., the mKdV equation)} \tag{9}$$

and

$$\tilde{u}_t = \psi(\tilde{u})\tilde{u}_x = \tilde{u}_{xxx} + 6\tilde{u}\tilde{u}_x, \tag{10}$$

which we recognize as the KdV equation itself. The substitution $u_x = v$ reduces Eq. (8) to the usual form of the SGE in light-cone variables.

It should be added that the transformation (5) relates real-valued solutions of the mKdV with complex solutions of the KdV equation. This has already been observed before, but most authors must have surmised complex solutions of the KdV equation to be useless. This is connected with the fact that in applications of the KdV equation to fluid dynamics, plasma physics, and the like, it is indeed somewhat unusual to use imaginary terms for the description of waves, etc. However, we appreciated the existence of a relation such as (5) since it provided us with some support for associating part of the general solution of the KdV equation (i.e., the shelf or phonon part) with imaginary contributions to the potential of a related Schrödinger problem. This has been done in successful attempts to model real and imaginary parts of the nuclear interaction by aid of the KdV equation.[10] But, of course, even if one is only interested in real solutions, the transformation (5) connects the infinitesimal generators of symmetries for the KdV equation and the SGE with each other. But what about the conservation laws? One easily observes[11] that for all these equations G is a gradient of a conservation law if and only if G_x is the infinitesimal generator of a symmetry (Noether's theorem). Hence, we recover also the relations between the conservation laws of the KdV equation and the SGE as given by Chodos.[1]

Finally, let us consider as a specific example the two-soliton solutions of these equations. Observing that the solution manifolds are two-dimensional ones generated by time and space translations, one obtains via some simple calculations[7] from (ii) that

$$u(x,t) = \int_{-\infty}^{x} v(\xi,t)\,d\xi \tag{11}$$

is a two-soliton solution of the sine-Gordon equation (with asymptotic speeds $1/c_1$, $1/c_2$) if and only if

$$u(x,t) = v(x-(c_1+c_2)t, -c_1c_2)^2 + iv_x(x-(c_1+c_2)t, -c_1c_2t) \tag{12}$$

is a (complex) two-soliton solution of the KdV equation with asymptotic speeds c_1, c_2. Again, the fact that the two-soliton solutions of the two equations are obtained by a variable transformation which is not independent of t shows that we cannot expect to find a Bäcklund transformation between them.

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