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Comment on stability properties of degenerate systems

W. Mecklenburg*
Department of Physics, The University, Southampton SO9 5NH
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Stability properties of degenerate systems are discussed. It is argued that degeneracies corresponding to zero-frequency modes may in general indicate instabilities of the system under consideration. It is pointed out that a stability analysis of the classical Lagrangian is particularly relevant when, due to some symmetry, higher-order corrections do not alter the form of the potential.

Degenerate systems occur on various occasions in field theory (compare Refs. 1–3 and references therein). A system with Lagrangian $\mathcal{L}(\phi)$ will be called degenerate in this comment if it has a constant solution ϕ_0 of the field equations that minimizes the energy (i.e., a ground-state solution) and that can be continuously deformed into other constant solutions without changing the energy. The latter statement means that in fact a continuous set of constant ground-state solutions exists that all have the same energy (each member of the set may be continuously transformed into any other).

Goldstone-type potentials are typically of this form; for instance,

$$\mathcal{L}_{\rm I}=\tfrac{1}{2}(\dot\phi^2+\dot\psi^2)-\frac{\mu^2}{2}(\phi^2+\psi^2)+\frac{\lambda}{4}(\phi^2+\psi^2)^2 . \quad (1)$$

Here and in the following only the time dependence of the fields will be kept, since this is sufficient for our purposes. Any constants ϕ_0, ψ_0 comprise a ground-state solution of (1) if for ${\phi_0}^2+{\psi_0}^2=c^2$ one has

$$\mu^2=\lambda c^2 . \quad (2)$$

Any two such solutions may be continuously transformed into each other by a rotation in the (ϕ, ψ) plane. Note that such a rotation is a symmetry transformation of $\mathcal{L}_{\rm I}$.

Another example is given by

$$\mathcal{L}_{\rm II}=\tfrac{1}{2}(\dot\phi^2+\dot\psi^2)-\frac{\tau}{4}\phi^2\psi^2 . \quad (3)$$

Constant ground-state solutions are now of the form $(\phi_0, \psi_0)=(c, 0)$ and $(\phi_0, \psi_0)=(0, c)$ where c is an arbitrary constant. Again ground-state solutions corresponding to different values of c may be continuously transformed into each other. In this case, however, they are not related by an internal symmetry transformation of $\mathcal{L}_{\rm II}$.

Degenerate systems such as $\mathcal{L}_{\rm I}$ and $\mathcal{L}_{\rm II}$ exhibit characteristic difficulties if the stability properties of their ground-state solutions are analyzed. These difficulties are the objective of the present comment. They are related to occurrence of zero-frequency modes.[4] Technically, the difficulties arise because the usual Liapunov criterion does not decide stability in these cases. The stability question itself is important when one wants to construct Lagrangians displaying a Goldstone-Higgs-Kibble phenomenon.

In the following it will be pointed out that the existence of zero-frequency modes may in general indicate instability. It is important to keep in mind, however, that degeneracy does not always imply instability.[5,6]

In the following a short compilation of results on stability will be given. This is being done in order to have a definition of stability available that is independent of Liapunov's criterion (which is insufficient in the degenerate case).

Furthermore, it will be seen that the rate of growth in time of the fluctuations around a given ground-state solution is in general not linked with the stability of these fluctuations; in particular the fluctuations around an unstable ground-state solution may grow polynomially (rather than exponentially) in time.[7] (Compare the example given below.)

Stability in the sense of *Liapunov* is defined in the following way.[8] Let

$$\dot x_i=f_i(x_1\cdots x_n), \quad i=1, \ldots, n \quad (4)$$

be a system of differential equations of first order for the unknown functions $x_i(t)$. Let us choose a particular solution, called *reference solution*

$\bar{x}_i(t)$, by specifying initial values $\bar{x}_i(0)$ at the instant $t=0$. This reference solution is called *stable* if small variations of the initial values produce a variation of the solution of (4) which remains small for any value of $t>0$.[9] Of particular interest below will be the case of constant reference solutions, $\bar{x}_i(t)=a_i$, $i=1,\ldots,n$. Such solutions will be called *equilibrium solutions*; (a_i) will be an *equilibrium point*. Note that free motion with nonvanishing velocity does not constitute an equilibrium solution in the sense of this definition.

Consider now a small perturbation $\delta x_i = y_i$ around an equilibrium point (a_i) of (4). Neglecting terms other than linear in y_i we find

$$\dot{y}_i=\sum_{k=1}^{n} f_{i,k}(a_1\cdots a_n)y_k,\quad f_{i,k}=\frac{\partial f_i}{\partial x_k},\quad k=1,\ldots,n\,. \tag{5}$$

The general solution of this system will be of the form

$$y_i(t)=\sum_{k=1}^{n} b_{ik}e^{i\omega_k t},\quad i=1,\ldots,n \tag{6}$$

where b_{ik} and ω_k are constants. If all the frequencies ω_k, $k=1,\ldots,n$ are real and not zero then (a_i) is a stable solution of (4). The solution is unstable if some or all of the ω_k are complex ("exponential instability"). If the frequencies are real but some of them are zero ("zero-frequency modes") the solution may be stable or not. This latter situation is the degenerate case that is the topic of this comment. This criterion for stability is due to Liapunov.[10] Intuitively it is based on the observation that the harmonic-oscillator solution of $\ddot{x}+\omega^2x=0$ is indeed a stable solution in the sense of the definition of stability given above.[8]

The occurrence of degeneracies elucidates the statements of *Dirichlet's theorem*.[10] In fact this theorem states the following: "Assume that $f_i(x_1\cdots x_n)$ [see Eq. (4)] are a power series without a constant term in $x_1,\ldots,x_n$, convergent in some neighborhood of the origin. If the system (4) then has an integral $g(x_1\cdots x_n)$ that does not depend on t and has a relative minimum in the strong sense at $x_i=0$, then the equilibrium solution $x_i=0$ is stable." Clearly it does not matter whether one chooses the origin as the equilibrium point rather than the constant $x_i=a_i$ as above. The theorem gives a sufficient condition for stability. Note that the integral $g(x_1\cdots x_n)$ has to have a *strong* relative minimum at the origin, meaning that for sufficiently small x_i one requires $g(0)<g(x_1\cdots x_n)$ rather than just $g(0)\le g(x_1\cdots x_n)$. This explains from the point of view of Dirichlet's theorem why the existence of degeneracies as specified above may indicate instability of an equilibrium solution. Note that the formulation of Dirichlet's theorem does not refer to the situation of minimal energy alone but as well to the occurrence of a minimum for any constant of motion. Minimality of energy is therefore not a necessary condition for stability if, say, angular momentum is at a strong relative minimum. The relevance of this point in particular in the context of Yang-Mills theories has recently been emphasized by Jackiw and Rossi.[6]

Let us now discuss examples. Consider the Lagrangian $\mathcal{L}_{II}$. As remarked above it has degenerate ground-state equilibrium solutions $\phi=\phi_0$, $\psi=0$; $\dot{\phi}=0$, $\dot{\psi}=0$; and $\psi\leftrightarrow\phi$. It has been demonstrated in Ref. 1 that these solutions are not stable. The reason for this is that any solution of this system with vanishing initial velocities and boundary conditions $\phi(0)\neq0$, $\psi(0)\neq0$ will move towards the origin in the (ϕ,ψ) plane. The small perturbations around the ground-state solutions are exactly of this type. They will therefore not stay in the vicinity of the chosen ground-state solution. This is in disagreement with our definition of stability as given above.

Let us now study the growth in time of the fluctuations around the ground-state solutions. The field equations are

$$\ddot{\phi}=-\frac{\tau}{2}\phi\psi^2\,, \tag{7}$$

$$\ddot{\psi}=-\frac{\tau}{2}\psi\phi^2\,. \tag{8}$$

Linearizing around the ground-state solutions with $\phi=\phi_0+A$, $\psi=B$ gives

$$\ddot{A}=0\,, \tag{9}$$

$$\ddot{B}=-\frac{\tau}{2}B\phi_0^{\,2}\,. \tag{10}$$

Note that the zero-frequency mode is explicitly seen. From (10) with $\dot{B}(0)=0$,

$$B=b\sin(\tau\phi_0^{\,2}/2)^{1/2}t,\quad b=\text{const.} \tag{11}$$

Thus, in the (ϕ,ψ) plane we find oscillating behavior in the direction of ψ. In the direction of ϕ we include quadratic terms so that (7) becomes

$$\ddot{A}=-\frac{\tau}{2}B^2\phi_0\,. \tag{12}$$

Using (11) this may be explicitly integrated and it can be seen that the rate of growth for A for small t is polynomial.

We thus find that if zero-frequency modes indicate instabilities, the rate of growth of instability need not necessarily be exponential as it would be for the case of complex frequencies. This type of polynomial growth is not to be mistaken with the following one. If some of the frequencies ω_k

[see Eq. (6)] are identical (but real and nonvanishing), then for suitable boundary conditions superpositions of y_i may show polynomial growth with small t. Thus polynomial growth for small t alone does not necessarily indicate instability.[6,11]

A similar analysis may be carried out for $\mathcal{L}_I$. The zero-frequency mode can be explicitly seen if small fluctuations around $\phi=\phi_0$, $\psi=0$ are studied. However, taking into account quadratic corrections much like the above for $\mathcal{L}_{II}$, the stability question in this case can be answered in the affirmative.

In order to build a Lagrangian which exhibits a Goldstone-Higgs-Kibble phenomenon, not only the "classical" Lagrangians such as $\mathcal{L}_I$ and $\mathcal{L}_{II}$ have to display nonvanishing stable ground-state solutions, but also those Lagrangians where the potentials of $\mathcal{L}_I$ and $\mathcal{L}_{II}$ are replaced by the so-called effective potential, thus taking into account quantum corrections to the theory.

For Goldstone-type Lagrangians such as $\mathcal{L}_I$ the situation has been analyzed in a classic paper.[12] For $\mathcal{L}_{II}$, a renormalized effective potential has been given by Drummond.[13] Defining

$$\xi=\phi^2+\psi^2, \quad \zeta=\phi^2\psi^2 \tag{13}$$

and

$$U_{1,2}=\tfrac{1}{4}[\tau\xi\pm(\tau^2\xi^2+12\tau^2\zeta)^{1/2}], \tag{14}$$

it is given by ($M^2=\text{const}$)

$$V_{\text{eff}}=\frac{\tau}{4}\phi^2\psi^2+\frac{1}{64\pi^2}\left[U_1{}^2\left(\ln\frac{U_1}{M^2}-\frac{1}{2}\right)+U_2{}^2\left(\ln\frac{U_2}{M^2}-\frac{1}{2}\right)-\frac{\tau^2}{4}\left(\ln\frac{\tau}{2}+1\right)(\phi^2+\psi^2)^2-3\tau^2\left(\ln\frac{\tau}{2}\right)\phi^2\psi^2\right]. \tag{15}$$

One then finds by inspection that the degeneracy of the original theory has disappeared[14] and the situation with respect to stability is completely changed.

A still different situation is given for the supersymmetric Lagrangian discussed in Ref. 2. Here the potential contains a term

$$\frac{g^2}{2}(\epsilon^{ij}\phi_i\psi_j)^2, \quad i,j=1,2, \quad \epsilon^{ij}=-\epsilon^{ji}, \quad \epsilon^{12}=1, \tag{16}$$

and a degeneracy similar to that for $\mathcal{L}_{II}$ occurs. However, the one-loop corrections to the effective potential vanish in this case, due to the underlying supersymmetry of the Lagrangian. Thus in this case the stability problem remains the one of the classical Lagrangian. It is then the requirement of "classical" stability that prevents us in the given context from using (16) as a Goldstone-type potential.

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*Present address: International Centre for Theoretical Physics, I 34 100 Trieste, Italy.

[1]W. Mecklenburg and D. P. O'Brien, Lett. Nuovo Cimento 23, 566 (1978).

[2]Ashok Das, Trieste Report No. IC/80/73, 1980 (unpublished).

[3]P. P. Martin and W. Mecklenburg, Phys. Rev. D 22, 3088 (1980).

[4]For Goldstone-type potentials, these zero frequencies are just the vanishing masses of the Goldstone bosons.

[5]R. Jackiw, Rev. Mod. Phys. 49, 681 (1977).

[6]R. Jackiw and P. Rossi, Phys. Rev. D 21, 426 (1980).

[7]This seems to be in contradiction with a corresponding statement in Ref. 6. Compare, however, the discussion following Eq. (12) of this comment.

[8]E. L. Stiefel and G. Scheifele, *Linear and Regular Celestial Mechanics* (Springer, Berlin, 1971).

[9]More precisely this requirement has the following meaning. Let $\epsilon>0$ be any positive quantity given in advance. Then it is possible to construct n numbers $\delta_i>0$ such that the condition

$$|x_i(0)-\bar{x}_i(0)|<\delta_i, \quad i=1,\ldots,n$$

implies

$$|x_i(t)-\bar{x}_i(t)|<\epsilon, \quad i=1,\ldots,n$$

for any positive value of t.

[10]C. L. Siegel and J. K. Moser, *Lectures on Celestial Mechanics* (Springer, Berlin, 1971).

[11]I am indebted to Dr. Ted Barnes for reminding me of this fact.

[12]J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962).

[13]I. T. Drummond, Nucl. Phys. B72, 41 (1974).

[14]In particular, a ground-state solution with $\phi=0$ ($\psi=0$) must have $\psi=0$ ($\phi=0$).