Fermionic contribution to $\Lambda_{\rm lattice}$

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The ratio $\Lambda_{\rm MO}/\Lambda_{\rm latt}$ is computed for quantum chromodynamics (QCD) with massless fermions present. The result obtained, using the Wilson form of the lattice action, is $\Lambda_{\rm MO}/\Lambda_{\rm latt}=117.5$. We show how our methods can readily be extended to other forms of the action. Appendices give details of the technology of QCD perturbation theory on the lattice. Included are derivations of Feynman rules and Ward identities as well as general power-counting arguments and a discussion of the renormalization group.

I. INTRODUCTION

Monte Carlo lattice calculations performed within the framework of an SU(3) gauge theory without fermions[1,2] have numerically related the string tension K to a parameter $\Lambda_{\rm latt}$ defined in close analogy to the Λ parameter of the ordinary continuum theory by

$$g_0{}^2(a)=1/[\beta_0\ln(1/a^2\Lambda_{\rm latt}{}^2)+\cdots]\,.$$

Here, a is the lattice spacing, g_0 is the gauge coupling constant, and β_0 is the beta function. Hasenfratz and Hasenfratz[3] and Dashen and Gross[4] have shown that it is possible to express the ratio of $\Lambda_{\rm latt}$ to the usual Λ parameter of the continuum theory, defined in some renormalization scheme [minimal subtraction ($\overline{\rm MS}$), momentum-space subtraction (MOM), etc.][5,6] as a perturbative expansion in g. For an SU(3) gauge theory without fermions, these authors obtain $\Lambda_{\rm MO}/\Lambda_{\rm latt}=83.5$. This calculation is done in the Feynman gauge and the attendant subtractions are performed at the symmetric point.[6] By combining this result with $\Lambda_{\rm latt}/\sqrt{K}$, Creutz[1] obtained $\Lambda_{\rm MO}\simeq170\pm50$ MeV in reasonable agreement with experiment.

Recently, Weingarten and Petcher, Fucito *et al.*, and Scalapino and Sugar[7] have demonstrated the feasibility of doing Monte Carlo studies with quarks present. Therefore, it is natural to extend the result of Hasenfratz and Hasenfratz to include fermions and this is what we will do in what follows. Our approach emphasizes two points. The first point is related to the well-known fact that there are many different forms of the fermionic part of a lattice gauge theory which all yield the same continuum limit although not the same $\Lambda_{\rm latt}$. We *explicitly* separate those parts of the calculation which do and do not depend on the particular form of the lattice action, and it is found that very few modifications have to be made if the fermionic part of the action is changed. Second, we emphasize the numerical accuracy of our method. It happens that the analytically computable parts of our calculation dominate the numerical parts. This significantly reduces the effect of numerical inaccuracy in the four-dimensional Gaussian integrations which must be performed.

The plan of our paper is as follows. In Sec. II we introduce and discuss the Wilson action[8] upon which our calculation is based. In Sec. III we describe our calculational procedure in five steps: (1) relating $\Lambda_{\rm MO}/\Lambda_{\rm latt}$ (the fermion part) to the propgator $\Pi_{\mu\nu}$, (2) showing that only $\mathcal{Q}=\partial_\rho\partial^\rho(p^4\Pi)^\mu{}_\mu|_{p=0}$ needs to be calculated by lattice regularization, (3) obtaining the value of $[p^4\Pi^\mu{}_\mu-(p^2/8)\mathcal{Q}]$ by dimensional regularization, (4) separating from $\mathcal{Q}$ the $m\to0$ contribution, and (5) numerically computing $\mathcal{Q}$ minus the $m\to0$ contribution. This final step consists of evaluating, by quadrature, a straightforward finite, four-dimensional integral which is independent of all scales (lattice spacing, mass, and external momentum). Only steps (4) and (5) would be modified if the action is changed. Details of the calculation are relegated to four appendices. Appendix A explains the connection between Green's functions and Λ ratios. In the literature, this connection is often stated[3,4] but is rarely discussed in detail. Appendix B describes the derivation of Feynman rules for the lattice theory. In Appendix C, these rules are used to obtain various lattice Ward identities which are employed extensively in our calculations. Finally, Appendix D explains power counting and lattice regularization. These concepts enable us to identify lattice-regularized quantum chromodynamics (QCD) with, for instance, dimensionally regularized QCD. We also discuss the problem of fermion doubling which occurs when "convergent" integrals do not have the same continuum limits as in the dimensionally regularized theory. Our result, finally, is

$\Lambda_{\text{MO}}/\Lambda_{\text{latt}} = \exp\{(1/\beta_0)[\Pi_F(-0.01416)+(0.30816)]\}$. For four flavors, this expression has the value 117.5 ± 0.1.

During the course of our calculation, two other derivations of this quantity were published (Weisz[9] and Kawai *et al.*[10]). The techniques and emphases in these works differ from one another and from ours but, within numerical accuracy, the values obtained for $\Lambda_{\text{MO}}/\Lambda_{\text{latt}}$ all agree. (The number that we present is probably the most accurate of the three but that is of no practical importance.)

II. THE ACTION

As previously stated, the form of the action for a lattice gauge theory is not uniquely specified by specifying its continuum limit. In this paper we choose to work with the Wilson action S in Euclidean space-time. Explicitly,

$$S = -\sum_n \bar\psi_n\psi_n + K\sum_n\sum_\mu \bar\psi_n(1-\gamma_\mu)U_{n\mu}\psi_{n+\hat\mu} + K\sum_n\sum_\mu \bar\psi_{n+\hat\mu}(1+\gamma_\mu)U^\dagger_{n\mu}\psi_n + \frac{1}{2g_0^2}\sum_n\sum_\mu\sum_{\nu\neq\mu}\mathrm{Tr}(U_{n\mu}U_{n+\hat\mu,\nu}U^\dagger_{n+\hat\nu,\mu}U^\dagger_{n\nu}).$$

Here $U_{n\mu}$ is an element of SU(N) and the A fields are defined by

$$U_{n\mu} = \exp\left(iag_0\sum_b A^b_{n\mu}T_b\right).$$

K is defined in terms of the quark mass m by $K=1/(8+2ma)$, and the matrices T which generate SU(N) are normalized so that $\mathrm{Tr}(T^aT^b)=T(R)\delta^{ab}$.

The classical continuum limit of S is obtained by making the identification

$$\psi_n \simeq (a^3/2K)^{1/2}\psi(na),$$
$$A^b_{n\mu}\simeq A^b_\mu(na),$$
$$\sum a^4 = \int d^4x.$$

In that limit,

$$S_{\text{cont}} = -\int d^4x\left\{\bar\psi(x)\left[\not\partial + m + ig_0\sum_b \not A_b(x)T_b\right]\psi(x) + \tfrac14\sum_b F^b_{\mu\nu}(x)F^{\mu\nu b}(x)\right\}$$
$$+\text{an irrelevant constant},$$

where

$$F^b_{\mu\nu}(x) = \partial_\mu A^b_\nu(x) - \partial_\nu A^b_\mu(x) - g_0 f^{bcd}A^c_\mu(x)A^d_\nu(x).$$

It should be noted that the quantum mechanics of the continuum limit implied by S requires the introduction of gauge fixing and Faddeev-Popov ghost terms for its complete specification. A detailed discussion of these terms will not be necessary for the purposes of this paper, however, for these terms are not relevant in treating the fermionic effects upon which we focus. A more serious complication of the quantum theory is that we cannot blithely take the $a\to 0$ limit. In a g expansion of the Lagrangian there are terms which are proportional to a^m (those terms are known as "irrelevant operators") which formally go to 0 as $a\to 0$. However, these terms contain nonrenormalizable factors which can induce divergences of $O(1/a^m)$. After multiplying by a^m we are left with a finite nonzero quantity. Sharatchandra[11] has shown, for QED, that these "anomalous-vertex" contributions can be absorbed into renormalization counterterms[12] which, in turn, affect the Λ ratios. Thus, although there are numerous ways[13-15] of writing lattice theories which have the usual classical continuum limit, these theories may differ in their anomalous vertices.

However, it is worthwhile to point out, in concluding this section, that some actions may be better suited than others for doing Monte Carlo studies.[16] At present, fermions are being examined[7] using only the standard Wilson action but we must be prepared to easily modify our Λ-ratio results in case other actions are used in Monte Carlo analyses. For that reason, our calculation is explicitly separated into a part which depends on the lattice regularization and a part which does not. The latter part will not change when irrelevant operators are added to the action.

III. PROCEDURE

The calculation of the fermionic contribution to $\Lambda_{\text{MO}}/\Lambda_{\text{latt}}$ proceeds in five steps which follow. The first three steps are independent of the regularization procedure and hence will not change if anomalous vertices are added to the action.

Step 1. We will demonstrate that it suffices to compute the gauge boson propagator $\Pi^{\mu\nu}_{ab}(p)$ on the lattice. First define the propagator and vertex functions as (remember, all momenta are Euclidean, i.e., $p^\mu p_\mu = \sum_{i=0}^3 p_i^2 > 0$)

$$\Pi^{\mu\nu}_{ab}(p) \equiv \delta_{ab}\underline{\Pi}^{\mu\nu}(p) = i\delta_{ab}\left[\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)\frac{1}{p^2}h(p^2) + \frac{p^\mu p^\nu}{p^4} + O(a)\right], \tag{1a}$$

where

$$h(p^2) = 1 + g_0^2(a)[b_1\ln(a\mu)+b_2] - (Z_3-1). \tag{1b}$$

Also, for $p^2=q^2=r^2=\mu^2$ [see Eq. (A16a)],

$$\Gamma^{abc}_{\mu\nu\omega}(p,q,r)=g_0 f_{abc}\{[g_{\mu\nu}(p-q)_\omega+\text{cyclic}][G_0(\mu^2)+Z_1] - (q-r)_\mu(r-p)_\nu(p-q)_\omega G_1(\mu^2) - (r_\mu p_\nu q_\omega - r_\nu p_\omega q_\mu)G_2(\mu^2)\}+O(a), \tag{2a}$$

where[17]

$$G_0(\mu^2)=g_0{}^2(a)[c_1\ln(a\mu)+c_2]. \tag{2b}$$

Define $a_2=c_2+\frac{3}{2}b_2$. Then as explained in Appendix A [see (A14)],

$$\Lambda_{\rm MO}/\Lambda_{\rm latt}=\exp(a_2/\beta_0). \tag{3}$$

Here, $a_2=a_2^{\rm NF}+a_2^{\rm F}$, with NF and F denoting the nonfermionic and fermionic parts, respectively. β_0 is the β function defined as usual by

$$\beta_0=(11-\tfrac{2}{3}n_F)/16\pi^2. \tag{4}$$

This is for n_F quark flavors. $a_2^{\rm NF}$ has been computed[3,4,9,10] for SU(3) as

$$a_2^{\rm NF}=0.308\,162\,2\pm0.000\,000\,1. \tag{5}$$

Our task is to find $a_2^{\rm F}$.

That task is considerably simplified by noticing that $b_2^{\rm F}$ and $c_2^{\rm F}$ can be related by using the Ward identity discussed in Appendix C [(C10)]:

$$p_\mu\Gamma^{\mu\nu\omega}_{abc}(p,q,r)=-ig_0(a)f_{abc}\times[q^4\underline{\Pi}^{\nu\omega}(q)-r^4\underline{\Pi}^{\nu\omega}(r)]+O(a^2). \tag{6}$$

Applying this to (2a) and (2b) and using both conservation of momentum and the fact that $p^2=q^2=r^2=-2p\cdot q$, etc., we find that

$$g_0{}^2(a)[-(b_1^{\rm F}+c_1^{\rm F})\ln(a\mu)+(b_2^{\rm F}+c_2^{\rm F})]=\frac{\mu^2}{2}G_2^{\rm F}(\mu^2). \tag{7}$$

But G_2 is finite in the limit that $a\to0$ (because it is the coefficient of a tensor which is trilinear in external momenta) hence it can be computed by dimensional regularization. The result is quoted in Ref. 6 [Eq. (14c)] as

$$\frac{\mu^2}{2}G_2^{\rm F}(\mu^2)=\frac{n_F T(R)}{16\pi^2}(-\tfrac{2}{9}-\tfrac{8}{9}I), \tag{8}$$

where $I=2.343\,907\,2\cdots$. Then

$$\begin{aligned}a_2^{\rm F}&=(c_2^{\rm F}+\tfrac{3}{2}b_2^{\rm F})\\&=\tfrac{1}{2}b_2^{\rm F}+(c_2^{\rm F}+b_2^{\rm F})\\&=\tfrac{1}{2}b_2^{\rm F}+\frac{n_F T(R)}{16\pi^2}(-2.305\,695\,3).\end{aligned} \tag{9}$$

In summary, it has been shown that the Λ ratio can be computed from the fermionic part of the gluon propagator. Equation (9) gives $a_2^{\rm F}$ in terms of $b_2^{\rm F}$ [Eq. (1b)] and then $\Lambda_{\rm MO}/\Lambda_{\rm latt}$ is found by applying (3)–(5). From now on, the superscript F will be dropped and only fermionic contributions will be considered.

Step 2. Next, the divergent part of $\Pi^{\mu\nu}$ is isolated and it is shown that only $\partial_\rho\partial^\rho(p^4\Pi)^\mu{}_\mu|_{p=0}$ need be computed by lattice regularization. Because of the transversality of the propagator (see Appendices B and C) it suffices to compute $\Pi(p^2)$ defined by

$$\Pi(p^2)=\tfrac{1}{3}(p^4\Pi)^\mu{}_\mu(p). \tag{10}$$

(Notice that Π is defined so it has dimension 2.) In Appendix C [(C5)] it is shown that

$$\Pi(0)=0. \tag{11}$$

Furthermore, by inspection of the propagator (B10a) and (B10b), Π is seen to be even in p. Hence these facts imply that a Taylor expansion (which is possible only when $m\neq0$) gives

$$\Pi(p^2)=0+\frac{p^\mu p^\nu}{2}[\partial_\mu\partial_\nu\Pi(p^2)]\Big|_{p=0}+O(p^3). \tag{12}$$

In fact, since Π is a function of p^2, we have

$$\Pi(p^2)=\frac{p^2}{8}\partial^\rho\partial_\rho\Pi(p^2)\Big|_{p=0}+O(p^3). \tag{13}$$

The term of order p^3 should be finite by power counting (Appendix D). Actually, one must be careful with this argument;

$$\Pi_{(3)}=\left[\Pi(p)-\frac{p^2}{8}\partial_\rho\partial^\rho\Pi\Big|_{p=0}\right]$$

is a Taylor series starting with the third derivative of Π. Following Appendix D, write

$$\Pi=a^N\int_{-\delta/a}^{\delta/a}dk\,\ell(k,p)[1+O(2m)+O(a(p-k))+\cdots]+\int_{-\pi/a}^{-\delta/a}\cdots+\int_{\delta/a}^{\pi/a}\cdots, \tag{14}$$

where ℓ is the $a\to0$ limit of the integrand and N is the number of factors of a that must be removed so the integrand is finite as $a\to0$. Symbolically write $\partial\partial\partial$ to denote a third (partial) derivative. Then

$$\partial\partial\partial\Pi=a^N\int_{-\delta/a}^{\delta/a}dk\{[\partial\partial\partial\ell][1+O(am)]+[\partial\partial\ell][\partial O(am)]+\cdots\}. \tag{15}$$

It is possible to convince oneself [by staring at the explicit form (B10), of the integrals, and using the methods of Appendix D] that the first term of the right-hand side dominates as $a\to0$. The argument generalizes to higher derivatives. Thus, since $\int_{-\infty}^{\infty}\partial\partial\partial\ell$ (and higher derivatives of ℓ) is known to be finite, it can be computed dimensionally.

Recapitulating, it has been shown that

$$\tfrac{1}{3}p^4\Pi^{\mu}{}_{\mu}(p)=\Pi(p^2)=\frac{p^2}{8}\,\partial^{\rho}\partial_{\rho}\Pi(p^2)\Big|_{p=0}+\tilde{\Pi}(p^2)\,, \tag{16a}$$

where

$$\tilde{\Pi}(p^2)=\Pi(p^2)-\frac{p^2}{8}\partial^{\rho}\partial_{\rho}\Pi(p^2)\Big|_{p=0} \tag{16b}$$

can be computed by dimensional regularization, since it is finite as $a\to 0$. Only $\partial^{\rho}\partial_{\rho}\Pi(p^2)|_{p=0}$ needs to be computed on the lattice.

Step 3. The value of $\tilde{\Pi}(p^2)$, as defined in (16b), is found from the dimensionally regularized Π. See, for instance, Refs. 18 and 19. In $N=4+\epsilon$ dimensions

$$\Pi^{\rm DR}_{ab}(p^2)=A(N)\,p^2\left\{\frac{1}{6}+\frac{\epsilon}{12}\ln m^2 -\frac{\epsilon}{12}\left[-\left(1-\frac{2m^2}{p^2}\right)(1+4m^2/p^2)^{1/2}\ln\{[1+(1+4m^2/p^2)^{1/2}]/[-1+(1+4m^2/p^2)^{1/2}]\}+\frac{5}{3}-4\frac{m^2}{p^2}\right]\right\}, \tag{17a}$$

where

$$A(N)=\frac{-i8\delta_{ab}T(R)}{3(2\pi)^N}\,g_0^{\,2}n_F\,\pi^{N/2}\Gamma(2-N/2)(N-1)\,. \tag{17b}$$

Then

$$\tilde{\Pi}^{\rm DR}(p^2)=\lim_{N\to 4}\left[\Pi^{\rm DR}(p^2)-\frac{p^2}{8}\,\partial^{\rho}\partial_{\rho}\Pi^{\rm DR}(p^2)\Big|_{p=0}\right]$$
$$=\lim_{N\to 4}A(N)p^2\left\{-\frac{\epsilon}{12}\left[-\left(1-\frac{2m^2}{p^2}\right)(1+4m^2/p^2)^{1/2}\ln\{[1+(1+4m^2/p^2)^{1/2}]/[-1+(1+4m^2/p^2)^{1/2}]\}+\frac{5}{3}-4\frac{m^2}{p^2}\right]\right\}. \tag{18}$$

We are interested in the $m\to 0$ limit:

$$\tilde{\Pi}_{ab}(p^2)=\frac{i4p^2\delta_{ab}T(R)g_0^{\,2}n_F}{3\times(16\pi^2)}\,[\ln(p^2/m^2)-\tfrac{5}{3}]+O(m^2)\,. \tag{19}$$

Step 4. We next isolate the $m\to 0$ contribution of $\partial_{\rho}\partial^{\rho}\Pi(p^2)|_{p=0}$. There is a logarithmic divergence in "am" which will be seen to combine with the logarithm from (19) to give a term proportional to $\ln(a^2p^2)$ [see Eq. (1b)]. Parenthetically, one should note that this is the first occasion where we must use the explicit Wilson action. Barring the occurrence of an extraordinary theoretical pathology, steps (1)–(3) will yield the same results for *any* action whose continuum limit is as usual. Now we must invoke the specific form of the propagator as written in Appendix B [(B10a)]:

$$p^4\Pi^{\mu\nu}_{ab}(p)=i\delta_{ab}\,g_0^{\,2}n_F T(R)\frac{4K}{a^6}\int d^4k\,{\rm Tr}\left[v^{\mu(1)}(k)f\left(k-\frac{p}{2}\right)v^{\nu(1)}(k)f\left(k+\frac{p}{2}\right)\right]+\text{term}(\propto\Pi_B)\text{ independent of }p, \tag{20}$$

where $v^{\mu(1)}$ and f are the vertex and propagator, respectively (see Appendix B).

We will examine $\partial_{\rho}\partial^{\rho}(p^4\Pi^{\mu}_{\mu})|_{p=0}$. This involves a variety of terms. In order to illustrate the procedure of isolating the m dependence, we focus on just one term in the integrand (the variable of integration has been rescaled to go from 0 to π):

$$T=32K^2[4\alpha^2-4K\alpha C_1+(4K^2-\alpha^2)C_2+4K\alpha C_3-4K^2C_4]\times[64K^2-4\alpha^2+8K\alpha C_1+(-24K^2+2\alpha^2)C_2-8K\alpha C_3+8K^2C_4]/D^4\cdots\,, \tag{21}$$

where $\alpha=-1+2KC_1$, $C_N=\sum_{i=0}^{3}\cos^N(k_i)$, and $D=\alpha^2+4K^2\sum_{i=0}^{3}\sin^2(k_i)$. Since we are interested in the $m\to 0$ limit, let us do an expansion of T around $m=0$ and $k=0$ (remember, the divergence in m comes from the infrared limit of integration). These expansions are easily performed with the help of MACSYMA (Ref. 20) and it is found that

$$T\sim\hat{T}=\frac{64\{k^2[16(ma)^2+32(ma)^3]-k^4(-8-16ma)\}}{[k^2+(ma)^2]^4}+\cdots\,. \tag{22}$$

Now consider $\int_{|k|<\pi} \hat{T}\, d^4k$. We will do this term by term in (22):

(a) $$\int d^4k\, k^2(am)^3 \Big/ [k^2+(ma)^2]^4 = \int_0^\delta d\Omega\, dk\, k^5(am)^3 \Big/ [k^2+(ma)^2]^4 + (am)^3 \int_{|k|>\delta,\, k_i<\pi} (\cdots)\,. \quad (23)$$

The second integral on the right is clearly $O((am)^3)$. As will be seen by following the discussion below, $(am)^3 \int_0^\delta dk\, k^5/[k^2+(ma)^2]^4 = O(am)$. Hence this term goes to zero as $am \to 0$.

(b) Similarly, $\int d^4k\, k^4(am)/D^4 = O(am\ln(am))$ and goes to 0 as $am \to 0$.

(c) $$\int d^4k\, k^2(am)^2 \Big/ [k^2+(ma)^2]^4 = 2\pi^2 \int_0^\delta dk\, kk^4 \Big/ [k^2+(ma)^2]^4 + O(am)$$

$$= 2\pi^2(am)^2 \int_0^\delta dk\, k \frac{\{[k^2+(ma)^2]^2 - 2(ma)^2[(ma)^2+k^2]+(ma)^4\}}{[k^2+(ma)^2]^4} + O(am)$$

$$= \frac{2\pi^2(am)^2}{2}\left\{ -\frac{1}{k^2+(ma)^2}\Bigg|_0^\delta + \frac{2(ma)^2}{2[k^2+(ma)^2]^2}\Bigg|_0^\delta - \frac{(ma)^4}{3}\frac{1}{[k^2+(ma)^2]^3}\Bigg|_0^\delta \right\} + O(am)$$

$$= \pi^2/3 + O(am)\,. \quad (24)$$

(d) $$\int_{|k|<\pi} d^4k\, k^4 \Big/ [k^2+(ma)^2]^4$$

$$= 2\pi^2 \int_{|k|<\pi} dk\, k \frac{\{[k^2+(ma)^2]^3 - 3(ma)^2[k^2+(ma)^2]^2 + 3(ma)^4[k^2+(ma)^2] - (ma)^6\}}{[k^2+(ma)^2]^4} + O(am)$$

$$= 2\pi^2 \int_{|k|<\pi} dk\, k[k^2+(ma)^2]^3 \Big/ [k^2+(ma)^2]^4 + 2\pi^2 \int_{|k|<\delta} \text{other terms} + O(am)$$

$$= \pi^2 \ln[\pi^2/(ma)^2] - 11\pi^2/6 + O(am)\,. \quad (25)$$

By putting together the information obtained in (*a*) through (*d*) above, we finally have

$$\int_{|k|<\pi} d^4k\, \hat{T} = 256(2\pi^2)(-\tfrac{7}{12} + \tfrac{1}{2}\ln\pi^2) - 256\pi^2\ln(ma)^2 + O(am)\,. \quad (26)$$

Now write

$$\int d^4k\, T = \int d^4k[T - \hat{T}\theta(\pi - |k|)] + \int_{|k|\le\pi} d^4k\, \hat{T}\,. \quad (27)$$

The left-hand integral has the property (which can be shown by the above power-counting arguments) that

$$\int d^4k \lim_{(am\to 0)} [T - \hat{T}\theta(\pi - |k|)] = \lim_{am\to 0} \int d^4k[T - \hat{T}\theta(\pi - |k|)]\,. \quad (28)$$

Hence our final evaluation of T in the $am \to 0$ limit becomes

$$\int d^4k\, T = \int d^4k\, \tilde{T} + \int_{|k|<\pi} d^4k\, \hat{T}\,, \quad (29a)$$

where

$$\tilde{T} = [T - \hat{T}\theta(\pi - |k|)]\,|_{m=0} \quad (29b)$$

and $\int_{|k|<\pi} d^4k\, \hat{T}$ has been evaluated analytically [Eq. (26)]. The form (used in $\tilde{T}$) for $\hat{T}|_{m=0}$ is particularly simple and is in fact

$$\hat{T}|_{m=0} = 512/k^8\,. \quad (30)$$

The integral of $\tilde{T}$ then remains to be calculated. It is independent of m, p, and a; its range of integration is finite ($0 \le k \le \pi$) and it has no divergences. Thus it is easy to do numerically.

Recall that $\partial_\rho \partial^\rho \Pi(p^2)|_{p=0}$ is a sum of contributions similar to T. Just as above, the mass dependence can be isolated so that

$$\partial_\rho \partial^\rho \Pi(p^2)|_{p=0} = \mathcal{T}_1 + \mathcal{T}_2(m)\,, \quad (31)$$

where $\mathcal{T}_1$ is independent of a, m, and, of course, p. The decomposition (following the method described above for T) is well defined provided that the step function used in defining $\mathcal{T}_2$ is $\theta(\pi - |k|)$ [see (29)]. Then $\mathcal{T}_2(m)$ is found to be

$$\mathcal{T}_2(m) = \frac{i}{16\pi^2}\delta_{ab} T(R) n_F g_0^{\,2}(a)[\tfrac{40}{9} - \tfrac{32}{3}\ln\pi^2 + \tfrac{32}{3}\ln(am)^2]\,. \quad (32)$$

Step 5. The final step is to compute $\mathcal{T}_1$ in (31). This term is of the same form as $\tilde{T}$ defined in (29b) and we evaluated it numerically by Gaussian-Legendre quadrature. The symmetry of the integrals allows them to be evaluated in the quadrant $0 \le k_i \le \pi$. Further accuracy could have been obtained[21] by restricting to the region $0 \le k_1 \le k_2 \le k_3 \le k_4$ and then symmetrizing. However, that was not tried. In order to eliminate spurious numerical infrared divergences the integrals were written in polar coordinates where the integrands are explicitly smooth and finite over the entire range of integration. Also, special care was taken at the boundary [see (29b)], $|k| = \pi$, of the θ function (namely, quadrature was done separately in the regions $0 \le |k| \le \pi$ and $\{|k| \ge \pi;\ k_i \le \pi\}$). The result obtained is

$$\mathcal{T}_1 = \frac{i\delta_{ab} T(R) g_0^2(a) n_F}{16\pi^2} \times (3.11 \pm 0.04)\,. \qquad (33)$$

The error (±0.04) is estimated by inspecting the sequence of answers coming from 6-, 12-, 24-, and 48-point quadrature.

The five steps of the calculation can now be combined to give the answer. We have [(16a)]

$$\begin{aligned}\Pi(p^2) &= \frac{p^2}{8}\partial^\rho\partial_\rho\Pi(p^2)\Big|_{p=0} + \tilde{\Pi}(p^2)\\ &= \tilde{\Pi}(p^2) + \frac{p^2}{8}[\mathcal{T}_2(m) + \mathcal{T}_1] \quad [\text{by (31)}]\\ &= K'[\tfrac{4}{3}\ln(p^2/m^2) - \tfrac{20}{9} + \tfrac{5}{9} - \tfrac{4}{3}\ln\pi^2 + \tfrac{4}{3}\ln(a^2m^2) + (3.11 \pm 0.04)/8] \quad \text{by (19), (32), and (33)}\\ &= K'[\tfrac{4}{3}\ln(a^2p^2) - 4.719\,279\,7 + (0.389 \pm 0.005)]\\ &= K'[\tfrac{4}{3}\ln(a^2p^2) - 4.330 \pm 0.005]\,,\end{aligned} \qquad (34)$$

where

$$K' = \frac{iT(R)p^2 g_0^2 n_F}{16\pi^2}\,\delta_{ab}\,.$$

It is important to notice that the term which can be computed analytically far outweighs the term which had to be done numerically. Of course, this need not have been the case, but it has the fortunate effect of greatly reducing the numerical error.

Going back to (1b) we see that

$$b_1^{\text{F}} = \frac{n_F T(R)}{16\pi^2}\left(\frac{8}{3}\right) \qquad (35)$$

and

$$b_2^{\text{F}} = \frac{n_F T(R)}{16\pi^2} \times (-4.330 \pm 0.005)\,. \qquad (36)$$

b_1^{F} is, of course, the usual fermionic contribution to the β function. From (9),

$$\begin{aligned}a_2^{\text{F}} &= \frac{n_F T(R)}{16\pi^2}[\tfrac{1}{2}(-4.330 \pm 0.005) - 2.305\,695\,3]\\ &= \frac{n_F T(R)}{16\pi^2}(-4.471 \pm 0.003)\,.\end{aligned} \qquad (37)$$

Again, the analytically computable terms conspire to minimize the relative error:

$$\begin{aligned}a_2 &= a_2^{\text{F}} + a_2^{\text{NF}}\\ &= n_F(-0.014\,16 \pm 0.000\,01)\\ &\quad + (0.308\,162\,2) \quad [\text{see (5)}]\,.\end{aligned}$$

Then $\Lambda_{\text{MO}}/\Lambda_{\text{latt}} = \exp(a_2/\beta_0)$ where β_0 is given by (4). For four flavors $\Lambda_{\text{MO}}/\Lambda_{\text{latt}} = 117.5 \pm 0.1$. The ratios for $0 \le n_F \le 8$ are given in Table I. This result (for four flavors) may also be written[6] as $\Lambda_{\text{MOM}}/\Lambda_{\text{latt}} = (\Lambda_{\text{MOM}}/\Lambda_{\text{MO}}) \times (\Lambda_{\text{MO}}/\Lambda_{\text{latt}}) = 141.4 \pm 0.1$, where Λ_{MOM} is the Landau-gauge momentum-subtracted Λ. "MOM" rather than "MO" is the renormalization scheme used in phenomenological analyses[18] but they are so similar that it is unimportant which one we use. The conversion between the two schemes for an arbitrary number of flavors is given in Ref. 6.

IV. CONCLUSIONS

We have described in detail a method for computing the fermionic contribution to $\Lambda_{\text{MO}}/\Lambda_{\text{latt}}$.

TABLE I. Values of $\Lambda_{\text{MO}}/\Lambda_{\text{latt}}$ for $0 \le n_F \le 8$. The numerical accuracy decreases as n_F increases since the error in a_2^{F} is proportional to n_F. For $n_F = 4$, the error is ±0.1, but for $n_F = 8$, the error is ±0.5.

n_F	$\Lambda_{\text{MO}}/\Lambda_{\text{latt}}$
0	83.4
1	89.4
2	96.7
3	105.8
4	117.5
5	132.8
6	153.7
7	183.5
8	228.3

The importance of our technique is that it is easily applied to any action which has the usual continuum limit. In fact, the Wilson action may turn out not to be most suited for Monte Carlo calculations and if that is the case, then only steps 4 and 5 need to be modified in the procedure described above. Another advantage of our method is that (fortuitously) the analytically computable part of the result dominates the numerical part. This leads to a reduction in the relative numerical errors. Perhaps if another action were to be used we might not be so lucky, although we might have some control of this effect through our choice of the θ function in the definition of $\mathcal{T}_2$ [see Eqs. (29) and (31)].

Our result for SU(3) (using the nonfermionic contribution from Refs. 3 and 4) is $\Lambda_{\rm MO}/\Lambda_{\rm latt}|_{4\ \rm flavors} = 117.5 \pm 0.1$ or $\Lambda_{\rm MOM}/\Lambda_{\rm latt}|_{4\ \rm flavors} = 141.4 \pm 0.1$. For $0 \le n_F \le 8$, see Table I.

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APPENDIX A: THE CONNECTION BETWEEN GREEN'S FUNCTIONS AND Λ RATIOS

The relation [cf. Eq. (14)], which exists between the Green's functions of the theory and the Λ ratio which has been our primary interest in this paper, follows as a result of the twin requirements of renormalizability and the validity of perturbation theory for sufficiently small nonvanishing values of the coupling constants. To see this, first note that renormalizability implies

$$\lim_{a\to 0} g(\mu) = Z(a\mu, g(\mu)) g_0(a)\,, \tag{A1}$$

where a is the lattice spacing and μ is the renormalization mass. In what follows, $1/a$ and μ will both be taken to be large (i.e., g_0 and g will be small).

In perturbation theory,

$$Z(a\mu, g(\mu)) = 1 + A(a\mu) g^2(\mu) + B(a\mu) g^4(\mu) + \cdots\,. \tag{A2}$$

Define

$$\beta(g) = \lim_{a\to 0} \mu \left.\frac{dg(\mu)}{d\mu}\right|_{g_0(a)}. \tag{A3}$$

For notational convenience we will temporarily suppress the arguments of Z, A, etc.,

$$\mu \left.\frac{dg}{d\mu}\right|_{g_0(a)} = \mu \frac{d}{d\mu}[Z g_0(a)] = \mu Z^{-1} g \left(Z_{,(1)} a + Z_{,(2)} \left.\frac{dg}{d\mu}\right|_{g_0(a)}\right), \tag{A4}$$

where

$$Z_{,(1)} = A' g^2 + B' g^4 + \cdots\,, \tag{A5a}$$

$$Z_{,(2)} = 2gA + 4g^3 B + \cdots\,. \tag{A5b}$$

Rewrite (A4) and make a Taylor expansion to obtain

$$\mu \left.\frac{dg}{d\mu}\right|_{g_0(a)} = a\mu [g^3 A' + g^5 (AA' + B') + \cdots]\,. \tag{A6}$$

Next consider

$$\left.\left[a \frac{dg_0(a)}{da}\right]\right|_{g(\mu)} = a \frac{d}{da}(Z^{-1} g)\Big|_{g(\mu)} = -a\mu Z_{,(1)} Z^{-1} g_0\,. \tag{A7}$$

Expanding, we have

$$\left.\left[-a \frac{dg_0}{da}\right]\right|_{g(\mu)} = a\mu [g_0{}^3 A' + g_0{}^5 (AA' + B') + \cdots]\,. \tag{A8}$$

Notice that the right-hand sides of Eqs. (A6) and (A8) are equal (through this order of perturbation theory). Hence g_0 as a function of $1/a$ and g as a function of μ satisfy the same differential equation [for this we rely on the fact that the right-hand sides of Eqs. (A6) and (A8) depend on $a\mu$ only implicitly through g and g_0].

Of course, Eq. (A6) specifies the β function and by definition

$$\beta_0 = -a\mu A'\,, \tag{A9a}$$

$$\beta_1 = -a\mu (AA' + B')\,. \tag{A9b}$$

A well-known solution to Eqs. (6) and (8) is[3,4,6]

$$g^2(\mu) = \frac{1}{\beta_0 \ln(\mu^2/\Lambda^2)} - \frac{\beta_1 \ln\ln(\mu^2/\Lambda^2)}{\beta_0{}^3 \ln^2(\mu^2/\Lambda^2)} + O(1/\ln^3)\,, \tag{A10a}$$

$$g_0{}^2(a) = \frac{1}{\beta_0 \ln(1/a^2\tilde{\Lambda}^2)} - \frac{\beta_1 \ln\ln(1/a^2\tilde{\Lambda}^2)}{\beta_0{}^3 \ln^2(1/a^2\tilde{\Lambda}^2)} + O(1/\ln^3)\,. \tag{A10b}$$

Λ and $\tilde{\Lambda}$ are constants of integration uniquely specified by the definitions of g and g_0 *and* by the fact that no other terms of $O(1/\ln^2)$ are present in Eqs. (10).

By inverting Eqs. (10) we have

$$\Lambda/\tilde{\Lambda}=a\mu\exp\left\{-\frac{1}{2\beta_0}\left[\frac{1}{g^2(\mu)}-\frac{1}{g_0^{\,2}(a)}\right]\right\}\times\left[\frac{g^2(\mu)}{g_0^{\,2}(a)}\right]^{-\beta_1/2\beta_0^{\,2}}[1+O(g^2)]\,. \qquad (A11)$$

Later, it will be explicitly shown that

$$A=-a_1\ln(a\mu)+a_2\,,$$

but for now we simply use this expression, in conjunction with Eq. (9), to obtain

$$A=-\beta_0\ln(a\mu)+a_2\,. \qquad (A12)$$

Recalling Eqs. (A1) and (A2), this implies

$$g^2(\mu)=g_0^{\,2}(a)\{1+2[-\beta_0\ln(a\mu)+a_2]g_0^{\,2}(a)+\cdots\}\,. \qquad (A13)$$

Substituting this into Eq. (A11) the final result is

$$\Lambda/\tilde{\Lambda}=\lim_{a\to 0}\exp(a_2/\beta_0)[1+O(g_0^{\,2})f(a\mu)] =\exp(a_2/\beta_0)\,. \qquad (A14)$$

We have exploited[6] the fact that $\lim_{a\to 0}g_0(a)=0$ [and that μ can be made large so that $a\mu=O(1)$].

Thus, in order to compute Λ ratios it is necessary only to compute a_2. That involves computing $A=Z_1^{-1}Z_3^{3/2}$. To do this, we will define the Z_i by momentum-space subtraction and calculate the propagator and vertex functions[6] on the lattice, in the Feynman gauge. For $p^2=\mu^2$ (recall that the momenta are Euclidean)

$$\Pi^{\mu\nu}_{ab}(p)=+i\delta_{ab}\left\{\left[\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)\frac{1}{p^2}\right]h(p^2)+\frac{p^\mu p^\nu}{p^4}+O(a)\right\}, \qquad (A15a)$$

where

$$h(p^2)=1+g_0^{\,2}(a)[b_1\ln(a\mu)+b_2]-(Z_3-1)\,. \qquad (A15b)$$

Also, for $p^2=q^2=r^2=\mu^2$,

$$\Gamma^{abc}_{\mu\nu\omega}(p,q,r)=g_0f_{abc}\{[g_{\mu\nu}(p-q)_\omega+g_{\nu\omega}(q-r)_\mu+g_{\omega\mu}(r-p)_\nu][G_0(\mu^2)+Z_1]+\text{other tensors}+O(a)\}\,, \qquad (A16a)$$

where

$$G_0(\mu^2)=g_0^{\,2}(a)[c_1\ln(a\mu)+c_2]\,. \qquad (A16b)$$

$Z_1^{\rm MO}$ and $Z_3^{\rm MO}$ are then defined by

$$Z_3^{\rm MO}=1+g_0^{\,2}(a)[b_1\ln(a\mu)+b_2] \qquad (A17a)$$

and

$$Z_1^{\rm MO}=1-g_0^{\,2}(a)[c_1\ln(a\mu)+c_2]\,. \qquad (A17b)$$

Using $A=Z_1^{-1}Z_3^{3/2}$ we find

$$A^{\rm MO}=(c_1+\tfrac{3}{2}b_1)\ln(a\mu)+(c_2+\tfrac{3}{2}b_2) \qquad (A18)$$

and

$$a_2=c_2+\tfrac{3}{2}b_2\,, \qquad (A19a)$$

$$\beta_0=c_1+\tfrac{3}{2}b_1\,. \qquad (A19b)$$

Note: The superscript MO denotes "momentum subtraction in the Feynman gauge" as distinct from MOM which is "momentum subtraction in the Landau gauge." The results of Ref. 6 allow us to directly compute $\Lambda_{\rm MOM}/\Lambda_{\rm MO}$ (=1.20 for four flavors).

APPENDIX B: FEYNMAN RULES

The Feynman rules necessary to the analysis of the fermionic effects we have considered in this paper arise from an $O(g_0^{\,2})$ expansion, $S_f^{(2)}$, of the fermionic part of the Wilson action:

$$S_f^{(2)}=\bar\psi Q\psi+K\sum_{n\mu}\bar\psi_n(1-\gamma_\mu)\left[iag_0A^b_{n\mu}T_b+\frac{(iag_0)^2(A_{n\mu}\cdot T)(A_{n\mu}\cdot T)}{2}\right]\psi_{n+\hat\mu}+K\sum_{n\mu}\bar\psi_{n+\hat\mu}(1+\gamma_\mu)\left[(-iag_0)A_{n\mu}\cdot T+(iag_0)^2\frac{(A_{n\mu}\cdot T)(A_{n\mu}\cdot T)}{2}\right]\psi_n\,. \qquad (B1)$$

Here, Q is a quadratic form independent of the A's. These rules are obtained by introducing Grassmann variable sources η, $\bar\eta$ via

$$S_f^{(2)}\to S_f^{(2)}+\bar\eta\psi+\bar\psi\eta \qquad (B2)$$

and taking appropriate functional derivatives of the usual generating functional. The propagator $-Q^{-1}$ is then given by

$$Q^{-1\,j'j}_{\;\bar n'\bar n}=\int dk\,e^{iak(\bar n'-\bar n)}f^{j'j}(k) \qquad (B3a)$$

with

$$f(k)=a^4\left\{\left[-1+2K\sum_\mu \cos(ak\cdot\hat{\mu})\right] - 2iK\sum_\mu \gamma_\mu \sin(ak\cdot\hat{\mu})\right\}^{-1} \tag{B3b}$$

and

$$\int dk \equiv \frac{1}{(2\pi)^4}\int_{-\pi/a}^{\pi/a} dk_0 \cdots \int_{-\pi/a}^{\pi/a} dk_3 . \tag{B4}$$

To derive the other Feynman rules, one may follow the procedure described by Coleman in Ref. 22. Start by noting that

$$\int \prod_n d\psi_n\, d\bar{\psi}_n \exp(\bar{\psi}Q\psi+\bar{\eta}\psi+\bar{\psi}\eta) = -\exp(-\bar{\eta}\, Q^{-1}\eta)\det Q . \tag{B5}$$

One can also prove that (for Grassmann variables)

$$e^{f(-\partial/\partial\eta,\,\partial/\partial\bar{\eta})} e^{-\bar{\eta}_i M_{ij}\eta_j} = e^{(-\partial/\partial b_i M_{ij})\partial/\partial \bar{b}_j} \times e^{-i(\bar{b}\cdot\eta+\bar{\eta}\cdot b)} e^{f(-\bar{b},b)}\Big|_{b=0} . \tag{B6}$$

Equations (B5) and (B6), as well as corresponding equations for the Bose variables,[22] directly give the Feynman rules. Care is needed in getting the correct combinatoric factors and signs but these do follow from the rules. One useful fact when going to momentum space is

$$\sum_n e^{-iak\cdot n} = \frac{\delta(k)}{a^4} . \tag{B7}$$

As usual on a lattice all functions are periodic, and energy-momentum is conserved modulo $2\pi/a$. However, all integrals are integrated over one period only, so the periodicity of energy-momentum conservation (umklapp processes) does not create any technical difficulties (for divergent quantities, the "interpretation" of energy-momentum conservation is somewhat different than it would be in the continuum case[11] but such an interpretation is meaningless anyway). Also, because of this periodicity, shifts of variables can be performed at will, despite the finite limits of integration.

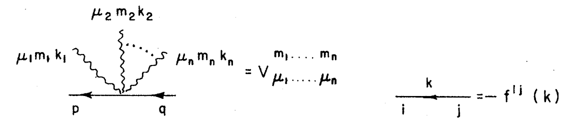

FIG. 1. The vertices and propagator of the fermionic part of the action.

The vertices which involve fermions are[23] (see Fig. 1)

$$V^{m_1\cdots m_n}_{\mu_1\cdots\mu_n}(p,q) = 2K(ag_0)^n\left[\frac{1}{n!}\sum_{\text{perms},\pi} T_{\pi(1)}\cdots T_{\pi(n)}\right]\sum_\mu \delta_{\mu\mu_1}\delta_{\mu\mu_2}\cdots\delta_{\mu\mu_n} v_\mu^{(n)}(\tfrac{1}{2}(p+q)) , \tag{B8}$$

where

$$v_\mu^{(1)}(r) = i\sin(ar_\mu) - \gamma^\mu\cos(ar_\mu) , \tag{B8a}$$

$$v_\mu^{(2)}(r) = -i\sin(ar_\mu)\gamma_\mu + \cos(ar_\mu) , \tag{B8b}$$

$$v_\mu^{(n+2)}(r) = v_\mu^{(n)}(r) . \tag{B8c}$$

Fermion loops have an overall minus sign as usual. When using these Feynman rules, the fields must be renormalized so that (for Euclidean p^2)

$$\Pi^{(0)\mu\nu}_{ab(p)} = \frac{i\delta_{ab}}{p^2} g^{\mu\nu} \tag{B9}$$

in the Feynman gauge ($\Pi^{(0)}$ is the free-field propagator). With that normalization

$$\Pi^{(2)}(p) = [\Pi_A(p)+\Pi_B(p)]\frac{i}{p^2}\delta_{ab} g_0^{\,2} n_F T(R) , \tag{B10}$$

where

$$\Pi^{\mu\nu}_{A_{ab}}(p) = \frac{4K^2}{a^6 p^2}\int dk'\, \mathrm{Tr}\left[v^{(1)\mu}(k')f\left(k'-\frac{p}{2}\right) \times v^{(1)\nu}(k')f\left(k'+\frac{p}{2}\right)\right] \tag{B10a}$$

and

$$\Pi^{\mu\nu}_{B_{ab}}(p) = \frac{-2K}{a^2 p^2}\int dk'\, \mathrm{Tr}[v^{(2)\mu}(k')f(k')]\delta^{\mu\nu} . \tag{B10b}$$

$v^{\mu(1)}$ and $v^{\mu(2)}$ are given in Eqs. (B8), and in (B10a) we have used the fact that shifts of variables are permitted. $\Pi^{\mu\nu}_A$ and $\Pi^{\mu\nu}_B$ are the Feynman diagrams of Fig. 2. We see that Π_B is due to the "anomalous vertex" ($q\bar{q}gg$-coupling) and that it is p independent. Thus it has the same structure as a mass-insertion diagram (and, in fact, it is needed so that the gluon does *not* obtain a mass).

In Appendix C there is a discussion of Ward identities. From these we deduce that [through $O(a^0)$]

$$\Pi^{\mu\nu}(p) = (g^{\mu\nu} - p^\mu p^\nu/p^2)\Pi(p^2) .$$

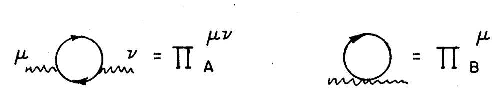

FIG. 2. Fermionic contributions to the propagator.

Hence, it suffices to compute $\Pi^{\mu}_{\mu}(p)=3\Pi(p^2)$. This is what is actually computed.

Although we do not evaluate $\Gamma^{\mu\nu\omega}_{abc}(p,q,r)$, we will (in Appendix C) demonstrate a Ward identity which relates Γ to Π. The (fermion contributions to) diagrams contributing to $\Gamma^{\mu\nu\omega}$ are given in Fig. 3. Figures 3(c) and 3(d) involve anomalous vertices but can be shown to be zero. That must be the case because each of those graphs involve the structure constants "d_{abc}." To see this, note in (B8) that the vertex $V^{ab}_{\mu\nu}$ involves $(T^aT^b+T^bT^a)$ and that the vertex $V^{abc}_{\mu\nu\omega}$ involves $(T^aT^bT^c+$permutations). Figure 3(c) is proportional to $\mathrm{Tr}[(T^aT^b+T^bT^a)T^c]$ and Fig. 3(d) is proportional to

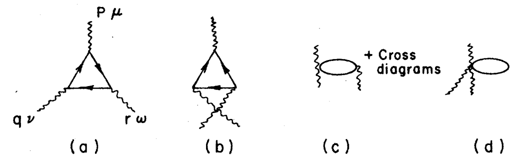

FIG. 3. (a) and (b) Fermionic contributions to the vertex. (c) and (d) are shown to be 0.

$\mathrm{Tr}[(T^aT^bT^c+\text{perms})]$. Each of these traces is proportional to d^{abc}. Also, it is possible to show directly [invoking evenness and oddness of $\cos(k)$ and $\sin(k)$] that Figs. 3(c) and 3(d) are zero. Hence, we are left to evaluate

$$\text{Fig. 3(a)}+\text{Fig. 3(b)}=\Gamma^{\mu\nu\omega(2)}_{abc}(p,q,r)$$
$$=\frac{-8iK^3}{a^9}f_{abc}T(R)n_Fg_0{}^3\int dk\,\mathrm{Tr}\left[f(k)v^{(1)}_{\mu}\left(k+\frac{p}{2}\right)f(k+p)v^{(1)}_{\nu}\left(k+\frac{p}{2}-\frac{r}{2}\right)\right.$$
$$\left.\times f(k-r)v^{(1)}_{\omega}\left(k-\frac{r}{2}\right)\right]. \tag{B11}$$

This is normalized so that

$$\Gamma^{\mu\nu\omega(0)}_{abc}(p,q,r)=g_0f_{abc}[g_{\mu\nu}(p-q)_\omega+g_{\nu\omega}(q-r)_\mu+g_{\omega\mu}(r-p)_\nu]. \tag{B12}$$

APPENDIX C: SOME WARD IDENTITIES

Various Ward-Slavnov identities are expected to hold for lattice QCD since this theory is constructed to be gauge invariant. Becchi-Rouet-Stora (BRS) transformations[24] can be used to derive these identities, but it is important to check the legitimacy of formal manipulations which disregard divergences. Hence we have directly checked those Ward identities which we need for the calculation of the Λ ratio. Those identities follow. Much of this discussion has been given by Karsten and Smit.[23]

To begin, note that

$$\frac{\partial}{\partial k_\mu}f^{-1}(k)=2Ka^{-3}iv^{(1)}_{\mu}(k) \tag{C1}$$

and

$$\frac{\partial}{\partial k_\mu}\frac{\partial}{\partial k_\nu}f^{-1}(k)=-2Ka^{-2}\delta_{\mu\nu}v^{(2)}_{\mu}(k), \tag{C2}$$

where $v^{(1)}_{\mu}$ and f are given in Appendix B. Then [see (B10)]

$$[p^2\Pi^{\mu\nu}_{Aab}](0)=\frac{4K^2}{a^6(2Ka^{-3}i)^2}\int dk'\,\mathrm{Tr}\left\{\left[\frac{\partial}{\partial k'_\mu}f^{-1}(k')\right]f(k')\left[\frac{\partial}{\partial k'_\nu}f^{-1}(k')\right]f(k')\right\}$$
$$=\int dk'\,\mathrm{Tr}\left[\frac{\partial}{\partial k'_\mu}f(k')\right]\frac{\partial}{\partial k'_\nu}f^{-1}(k'). \tag{C3}$$

We can integrate by parts. Notice that by periodicity

$$f(k')\left[\frac{\partial}{\partial k'_\mu}f(k')\right]\Bigg|_{-\pi/a}^{\pi/a}=0,$$

so we finally have

$$[p^2\Pi^{\mu\nu}_{Aab}](0)=-\int dk'\,\mathrm{Tr}\left\{f(k')\left[\frac{\partial}{\partial k'_\mu}\frac{\partial}{\partial k'_\nu}f(k')\right]\right\}$$
$$=\frac{2K}{a^2}\int dk'\,\mathrm{Tr}[f(k')v^{(2)\mu}(k')]\,\delta^{\mu\nu}\quad[\text{by (C2)}]$$
$$=-[p^2\Pi^{\mu\nu}_{Bab}](0). \tag{C4}$$

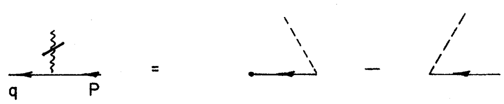

FIG. 4. The crossed gluon line denotes $\sin[\frac{1}{2}a(q-p)_\mu]\,W_\mu(p+q)$ where W_μ is defined in Appendix C. The dashed lines play no role except to carry momentum.

Hence

$$[p^4\Pi](0)=0\,. \tag{C5}$$

This is the condition that the gluon is massless. The role of the anomalous vertex is simply to enforce this condition (which is due to gauge invariance).

We can also derive a Ward identity which guarantees transversality of the one-loop gluon propagator. That is a bit more difficult because the propagator is not Lorentz covariant (except in the $a\to 0$ limit). The identity to be derived is

$$\sum_\mu \sin(\tfrac{1}{2}ap_\mu)I^{\mu\nu}(a,ap)=0\,, \tag{C6}$$

where $I^{\mu\nu}(a,ap)\equiv p^4\Pi^{\mu\nu}(p)$. In the limit $ap\to 0$

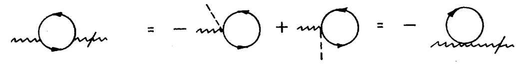

FIG. 5. Schematic derivation of the Ward identity (C6).

this identity becomes

$$\sum_\mu (ap_\mu)I^{\mu\nu}(a,ap)=0\,. \tag{C7}$$

The derivation depends on the observation

$$\sum_\mu f(p)\sin\left(\frac{a}{2}[q-p]_\mu\right)W_\mu(p+q)f(q)=\frac{ia^4}{4K}[f(q)-f(p)]\,, \tag{C8}$$

where $W_\mu(r)=v_\mu^{(1)}(r/2)$. That equation is easily proven by invoking various trigonometric identities such as $\sin a\cos b=\frac{1}{2}[\sin(a+b)+\sin(a-b)]$. Equation (C8) is the analog of the familiar continuum identity[25]

$$\frac{1}{\not p-m}[(q-p)_\mu\gamma^\mu]\frac{1}{\not q-m}=\frac{1}{\not p-m}-\frac{1}{\not q-m}\,.$$

In Fig. 4 we schematically describe Eq. (C8) and in Fig. 5 we show how this is used to derive (C6). Explicitly, from (C8)

$$\begin{aligned}\sum_\mu \sin(\tfrac{1}{2}ap_\nu)[p^2\Pi_{A(p)}^{\mu\nu}]&=\frac{ia^4}{4K}\times\frac{4K^2}{a^6}\int dk'\,\mathrm{Tr}\left[v^{(1)\mu}(k')f\left(k'+\frac{p}{2}\right)-v^{(1)\mu}(k')f\left(k'-\frac{p}{2}\right)\right]\\&=\frac{iK}{a^2}\int dk'\,\mathrm{Tr}\left\{f(k')\left[v^{(1)\mu}\left(k'-\frac{p}{2}\right)-v^{(1)\mu}\left(k'+\frac{p}{2}\right)\right]\right\}\\&=-\sum_\mu \sin(\tfrac{1}{2}ap_\nu)[p^2\Pi_B^{\mu\nu}(p)]\,,\end{aligned} \tag{C9}$$

where the last equation follows directly from applying trigonometric identities to $v^{(2)}$. Having proven this, it is not yet immediate that transversality follows. By inspecting the integrals Π_A and Π_B we can convince ourselves that the most general tensors are a linear combination of $(p_\mu{}^2+p_\nu{}^2)$, $p_\mu{}^2\delta_{\mu\nu}$, $p_\mu p_\nu$, $p^2\delta_{\mu\nu}$, and p^2. But the most general combination of these which satisfies (C6) [or (C7)] is, in fact, $(-p_\mu p_\nu+p^2\delta_{\mu\nu})$. Thus $\Pi^{\mu\nu}$ *is* transverse.

There is one more Ward identity of interest to us:

$$\sum_\mu \sin\left(\frac{a}{2}p_\mu\right)\Gamma_{abc}^{\mu\nu\omega}(p,q,r)=g_0{}^3T(R)f_{abc}n_F[q^2\Pi_A^{\nu\omega}(q)-r^2\Pi_A^{\nu\omega}(r)]\frac{a}{2}+O(a^3)\,. \tag{C10}$$

This will be proven using techniques similar to those used in deriving (C9) (see Fig. 5). Let

$$\begin{aligned}I^{\mu\nu\omega}&=\mathrm{Tr}\left[\int dk\,f(k)W_\mu(2k+p)f(k+p)W_\nu(2k+p-r)f(k-r)W_\omega(2k-r)\right]\,,\\ \sum_\mu \sin\left(\frac{a}{2}p_\mu\right)I^{\mu\nu\omega}&=\hat I_1^{\nu\omega}-\hat I_2^{\nu\omega}\,,\end{aligned} \tag{C11a}$$

where

$$\hat I_1^{\nu\omega}=\frac{ia^4}{4K}\left[\mathrm{Tr}\int dk\,f(k+p)W_\nu(2k+p-r)f(k-r)W_\omega(2k-r)\right] \tag{C11b}$$

and

$$\hat I_2^{\nu\omega}=\frac{ia^4}{4K}\left[\mathrm{Tr}\int dk\,f(k)W_\nu(2k+p-r)f(k-r)W_\omega(2k-r)\right]\,. \tag{C11c}$$

This follows from (C8):

$$\hat{I}_1^{\nu\omega} = \mathrm{Tr}\left\{\int dk\, f(k-q)W_\nu(2k-q)f(k)[W_\omega(2k-q)+W_\omega(2k-q-p)-W_\omega(2k-q)]\right\}$$
$$= \hat{\Pi}^{\nu\omega}(q) + R_1^{\nu\omega}(q,p)\,, \tag{C12a}$$

where

$$\hat{\Pi}^{\nu\omega}(q) = \frac{a^6q^2}{4K^2}\,\Pi_A^{\nu\omega}(q) \tag{C12b}$$

and

$$R_1^{\nu\omega}(q,p) = \mathrm{Tr}\int dk\, f(k-q)W_\nu(2k-q)f(k)[W_\omega(2k-q-p)-W_\omega(2k-q)]\,. \tag{C12c}$$

Similarly,

$$\hat{I}_2^{\nu\omega} = \hat{\Pi}^{\nu\omega}(r) + R_1^{\omega\nu}(r,-p)\,. \tag{C13}$$

The term of interest is $R_1^{\nu\omega}$. It is easy to see that this is analytic around 0 in the external variables $(r,\ p,\ q)$. It will shortly be shown that it suffices to demonstrate

$$R_1^{\nu\omega}(q,p) - R_1^{\omega\nu}(r,-p) = O(a^8)\,,$$
$$W_\omega(2k-q-p) - W_\omega(2k-q) = \sin\left(\frac{ap_\omega}{4}\right)\left\{-i\cos\left[\frac{a}{2}\left(2k-q-\frac{p}{2}\right)_\omega\right] - \gamma_\omega \sin\left[\frac{a}{2}\left(2k-q-\frac{p}{2}\right)_\omega\right]\right\}. \tag{C14}$$

Thus

$$R_1^{\nu\omega}(q,p) = \sin\left(\frac{ap_\omega}{4}\right)\hat{R}_1^{\nu\omega}(q,p)\,, \tag{C15a}$$

where

$$\hat{R}_1^{\nu\omega}(q,p) = \mathrm{Tr}\int dk\, f(k-q)W_\nu(2k-q)f(k)\left\{-i\cos\left[\frac{a}{2}\left(2k-q-\frac{p}{2}\right)_\omega\right] - \gamma_\omega\sin\left[\frac{a}{2}\left(2k-q-\frac{p}{2}\right)_\omega\right]\right\}. \tag{C15b}$$

At $p=0$, the integral is, after shifting variables,

$$\mathrm{Tr}\int dk\, f\left(k-\frac{q}{2}\right)W_\nu(2k)f\left(k+\frac{q}{2}\right)$$
$$\times[-i\cos(ak_\omega) - \gamma_\omega\cos(ak_\omega)]\,. \tag{C16}$$

This can be shown to be 0 by expanding the propagators and vertex and then noting that terms (or, sometimes, pairs of terms) with an even number of γ matrices are odd under $k \to -k$. Thus R_1 is at least linear in ap. Furthermore, by shifting variables it is also possible to see that R_1 is odd in the external momenta. Hence, if it can be shown that $\hat{R}_1^{\nu\omega}(0,p) + \hat{R}_1^{\omega\nu}(0,-p) = 0$, then $\hat{R}$ is at least of order 3 in the external momentum.

Indeed, if $\nu \neq \omega$, then $\hat{R}_1^{\nu\omega}(0,p) = 0$ by oddness [$\int \sin(ap_\nu)\sin(ap_\omega) = 0$ if $\nu \neq \omega$, etc.]. If $\nu = \omega$, then $\hat{R}_1^{\nu\omega}(0,p) = -\hat{R}_1^{\omega\nu}(0,-p)$ by oddness. Hence

$$R_1^{\nu\omega}(0,p) - R_1^{\omega\nu}(0,-p) = 0\,.$$

Finally, note that by rescaling k and remembering that $f \propto a^4$, $\hat{R}$ is seen to have dimensions a^4. By the above arguments, R has been shown to be at least quartic in the external momenta ($\sin[ap_\mu/4] \sim ap_\mu/4$ and $\hat{R}$ is trilinear in the momenta) so

$$R_1^{\nu\omega}(q,p) - R_1^{\omega\nu}(r,-p) = O(a^4\mu^4)\times O(a^4)$$
$$= O(a^8)\,, \tag{C17}$$

where μ denotes an external momentum.

The proof of the Ward identity (C10) is completed by putting together Eq. (B7) and Eqs. (C11a)–(C17). It will be seen in the text that

$$[q^2\Pi_A^{\nu\omega}(q) - r^2\Pi_A^{\nu\omega}(r)]\frac{a}{2} = O(a)\,, \tag{C18}$$

hence terms of $O(a^3)$ can be dropped in the continuum limit. It is interesting to note that this Ward identity, unlike (C9), appears to involve a residual term of $O(\boldsymbol{a})$.

APPENDIX D: POWER COUNTING

Not all of the statements here will be proven rigorously, but there is presumably no difficulty in doing so. The main point to be established is the following: when a Feynman integral, or certain combinations of Feynman integrals, appear to be convergent in the "unregulated" theory, then the lattice-regulated integrals converge to the "unregulated" integrals (when $a \to 0$). Technically this amounts to saying that convergent integrals have the same values in all regularization schemes. If that is true, then Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) renormalization can be carried out for the lattice theory.

These arguments turn out to be slightly nontrivial. If the most naive fermion action had been used, the resultant *convergent* Feynman integrals would have displayed the famous "doubling"[13] problem, i.e., each fermion loop would be multiplied by a factor of 2^4 over the corresponding dimensionally regularized result. This fact will be explained shortly. In what follows, only one-loop fermionic integrals will be considered.

Let

$$I=\int d^4k'\,N(k',p_1,p_2,\ldots,p_n)f^{m_1}(k')f^{m_2}(k'-p_1)\cdots \times f^{m_n}(k'-p_n)\,, \qquad \text{(D1)}$$

where N is some numerator involving vertices such as those of Appendix B and p_i are combinations of external momenta. Recall that I is "power-counting finite" if and only if $\sum_i m_i > 4$. What happens in the lattice theory? Divide the integration region into region 1 with $|k'|<\delta/a$ and region 2 with $|k'|\geq \delta/a$. In region 1, expand N and f's around 0. Note that $0\leq ak'\leq\delta$, so if a is small (i.e., ap_i are small) (we always assume am is small, since we will eventually take $m\to 0$) and δ is small, this expansion is formally justified provided that there are no divergences.

From Eq. (B3), using the identification $K=1/(8+2ma)$

$$f(k)=\frac{4a^3(-m+i\not{k})+O(k^2,m^2)}{(m^2+k^2)}\,. \qquad \text{(D2)}$$

Suppose for a moment that $N=1$. Then the integral in region 1 is

$$I_1\propto a^{3\Sigma m_i}\int_0^{\delta/a} d^4k'\,S^{m_1}(k',m)\cdots S^{m_n}(k'-p_n,m)\,, \qquad \text{(D3)}$$

where S are the usual (unregulated) Dirac propagators. *If* $\sum m_i>4$, then it is known that the above integral converges and, in particular, the dependence of I upon δ/a *vanishes* as $a\to 0$. Hence

$$I\,|_{\text{region }1}\propto a^{3\Sigma m_i}\,. \qquad \text{(D4)}$$

Continuing with $N=1$ in region 2, rescale the variables so that $k=ak'$. Then

$$I_2=\frac{1}{a^4}a^{4\Sigma m_i}\int_\delta^\pi d^4k\bar{f}^{m_1}(k)\cdots\bar{f}^{m_n}(k-p_n)\,, \qquad \text{(D5)}$$

where $\bar{f}(k,p)$ is defined as $(1/a^4)f(k,p)$ except that k is not multiplied by a. Now suppose that $f(k)$ has a pole *only* when $k=0$. Then $\lim_{a\to 0}\bar{f}(k,p_i)$ is bounded away from ∞ in the region $\delta\leq k\leq\pi$ (of course, ma and $ap_i\to 0$ in this limit). Thus, to do the power count we can replace a by 0 everywhere in the integrand and find that

$$I\,|_{\text{region }2}\propto a^{4(\Sigma m_i-1)}\,. \qquad \text{(D6)}$$

If $\sum m_i>4$, then by (D4) and (D6), I_2 has more powers of a then I_1, so I_1 dominates the integral I. Since δ and a can be made as small as possible, the unregulated propagators can be used, as in (D3). Of critical importance in this derivation was the fact that f had no pole except at 0. The "naive" propagator[13,23] $[1/\sum_\mu \sin(k_\mu\cdot a)\gamma^\mu]$, on the other hand, has poles at $k_\mu=\pm\pi/a$. The region structure would have to be modified and the result would be "fermion doubling." This is the reason for the form of the Wilson action as used in this paper.

Up to this point we have considered only the case when the numerator N is equal to 1. Suppose instead that $N|_{\text{region }1}=C+O(ak)$. Although powers of k increase the degree of divergence, these are accompanied by powers of a. Hence, if the denominator has a degree of convergence d, then

$$\int_0^{\delta/a}\frac{d^4k\,a^mk^m}{\text{denominator}}\sim a^m\left(\frac{a}{\delta}\right)^{d-m}\propto a^d\,. \qquad \text{(D7)}$$

[By convention $(a/\delta)^0\sim\ln(a/\delta)$.] Thus the endpoint contribution can be ignored as before, and the overall contribution (in region 1) of such a term is down by powers of a^m. In region 2, as before such a term can be dropped. The same arguments can be used to justify dropping higher-order terms in the expansion of $f(k)$, etc.

Finally there is the situation where $N|_{\text{region }1}=O(ak)$. This can only occur if all the vertices are anomalous (i.e., $v_m^{(c)}$ where $c>1$ in Appendix B). But anomalous vertices have no usual continuum limit and must be considered separately.

The above analysis can be extended to more complicated situations where there are formally convergent combinations of integrals (such as when doing BPHZ subtractions). Care must be exercised in taking these combinations. The following illustrates the procedure. Let

$$I=\frac{1}{a^8}\int dk\,\mathrm{Tr}\left[v_\mu^{(1)}\left(k-\frac{p}{2}\right)f(k-p)v^{(1)\mu}\left(k-\frac{p}{2}\right)f(k)\right]\,. \qquad \text{(D8)}$$

The factor of a^8 is explicitly written in order to cancel the $(a^4)^2$ associated with the definition of f. Then, as discussed in the text, the quantity $\hat{I}$ is formally convergent (I is even in p), where

$$\hat{I}\equiv I-I(0)-(p_\rho p_\omega/2)\frac{\partial^2}{\partial p_\rho\,\partial p_\omega}I(p)\bigg|_{p=0}\,. \qquad \text{(D9)}$$

In region 1 the argument is the same as in the continuum case and, as in (D2)–(D4), we see that

$$\hat{I}_1 \propto 1/a^2 \,, \tag{D10}$$

where $\hat{I}_1$ denotes $\hat{I}|_{\text{region 1}}$. However, it is necessary to demonstrate that

$$\hat{I}_2 = O(a^N)\frac{1}{a^2}\,, \quad \text{where } N>0\,. \tag{D11}$$

That is not immediately obvious. However, in region 2 we rescale the variables so that all of the vertices and propagators depend on a and p_i *only* in the combination ap_i. (It is legitimate, for now, to set K to a constant.) Then

$$a\frac{d}{da}F(ap_i) = \sum_\mu p^\mu \frac{\partial}{\partial p^\mu}F(ap_i)\,, \tag{D12}$$

where F is a vertex or propagator. Thus (in region 2)

$$\tilde{I}_2(a) - \tilde{I}_2(0) - \frac{a^2}{2}\frac{d^2}{da^2}\tilde{I}_2\bigg|_0 = a^4\hat{I}_2\,, \tag{D13}$$

where $\tilde{I}_2 = a^4 I_2$ ($I_2 \propto 1/a^4$ due to rescaling). But the left-hand side is proportional to a^3 (because of the Taylor expansion) so $\hat{I}_2 \propto 1/a$. Thus condition (D11) is satisfied, and $\hat{I}_1$ dominates $\hat{I}_2$ by powers of a.

The property that I_1 and I_2 are even depends on a particular choice of variables [see (B10)], i.e., this property can be lost under a shift of variables. However, we will need this fact only when we compute $\hat{I}$ by dimensional regularization. In that case the answer is independent of shifting variables, so we have finessed the problem. Had we wanted to, we could have computed $\hat{I}_1$ on the lattice and taken $a \to 0$ but then the shift would have had to be done carefully.

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