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Continuum regulation of the strong-coupling expansion for quantum field theory

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We study two continuum methods of regulating the formal strong-coupling expansion of the Green's functions, obtained by expanding the path integral in powers of the kinetic energy (inverse free propagator). Our continuum regulations amount to introducing either a hard (θ function) or soft (Gaussian) cutoff Λ in momentum space. The cutoff takes the place of the usual spatial cutoff, the lattice spacing, which arises when the path integral is defined as the continuum limit of ordinary integrals on a Euclidean space-time lattice. We find, by investigating free field theory and $g\phi^4$ field theory in one dimension, that the θ-function regulation is more accurate than the Gaussian and, unlike the Gaussian, preserves certain continuum Green's-function identities. The extension to field theories with fermions is trivial and we give the strong-coupling graphical rules for an arbitrary field theory with fermions and bosons in d dimensions.

I. INTRODUCTION

The strong-coupling expansion for quantum field theory is derived from the path-integral representation for the generating functional for the Green's functions by treating the kinetic energy as a perturbation.[1–5] For a general interacting field theory of bosons and fermions, the Lagrangian density in Euclidean spacetime is

$$\mathscr{L}=\bar{\psi}\not{\partial}\psi-\tfrac{1}{2}(\partial_\mu\phi)^2-\mathscr{L}_0(\phi,\bar{\psi},\psi,g,\lambda,m)\ , \tag{1.1}$$

where $\mathscr{L}_0$ contains all the local interactions and mass terms. The idea behind the strong-coupling expansion is to treat $\mathscr{L}_0$ as the unperturbed Lagrangian and

$$\mathscr{L}'=\bar{\psi}\not{\partial}\psi-\tfrac{1}{2}(\partial_\mu\phi)^2 \tag{1.2}$$

as the perturbation. A regulation scheme must be adopted to ensure that (1.2) is bounded, so that an expansion in it makes sense. The choice of regulation is the subject of this paper. In a previous work we discussed difficulties inherent in many continuum regulation schemes.[6] To obtain the formal strong-coupling expansion, sources for the different fields are introduced by adding a term $\bar{\psi}\eta+\bar{\eta}\psi+J\phi$ to $\mathscr{L}_0$ so that the path-integral representation of the generating functional may be written

$$Z[\bar{\eta},\eta,J]=\exp\left[\int d^dx\,d^dy\ \mathscr{L}'\left[\frac{\delta}{\delta\eta},\frac{\delta}{\delta\bar{\eta}},\frac{\delta}{\delta J}\right]\right]\times\int D\phi\ D\psi\ D\bar{\psi}\exp\left\{\int d^dx[\mathscr{L}_0(\phi,\bar{\psi},\psi)+\bar{\psi}\eta+\bar{\eta}\psi+J\phi]\right\}, \tag{1.3}$$

where

$$\mathscr{L}'\left[\frac{\delta}{\delta\eta},\frac{\delta}{\delta\bar{\eta}},\frac{\delta}{\delta J}\right]=-\frac{\delta}{\delta\eta(x)}S^{-1}(x,y)\frac{\delta}{\delta\bar{\eta}(y)}\ +\frac{1}{2}\frac{\delta}{\delta J(x)}G^{-1}(x,y)\frac{\delta}{J(y)}\ , \tag{1.4}$$

$$S^{-1}(x,y)=\partial\!\!\!/\delta(x-y)\ , \tag{1.5}$$

$$G^{-1}(x,y)=\partial^2\delta(x-y)\ , \tag{1.6}$$

and we have suppressed any Lorentz or group-theoretic indices which might be present.

Because $\mathscr{L}_0$ is local one can evaluate the path integrals in (1.3) by considering them to be the limit of a product of ordinary integrals obtained by replacing Euclidean space-time with a hypercubical lattice of spacing a, so that

$$\int d^dx \to a^d \sum_i$$

and

$$\int D\phi \to \prod_i \int_{-\infty}^{\infty} d\phi_i\ ,$$

where i labels the lattice sites. The lattice version of the path integral in (1.3),

$$Z_0[\bar\eta,\eta,J]=\int D\phi\, D\psi\, D\bar\psi\, \exp\left[\int d^dx[\mathscr{L}_0(\phi,\bar\psi,\psi)+\bar\psi\eta+\bar\eta\psi+J\phi]\right]\ , \tag{1.7a}$$

is

$$Z_0[\bar\eta,\eta,J]=\prod_i \int d\phi_i d\psi_i d\bar\psi_i \exp\{a^d[\mathscr{L}_0(\phi_i,\bar\psi_i,\psi_i)+\bar\psi_i\eta_i+\bar\eta_i\psi_i+J_i\phi_i]\}\ . \tag{1.7b}$$

Here Z_0 is the product of ordinary and Grassmann integrals,

$$Z_0[\bar\eta,\eta,J]=\prod_i f(\bar\eta_i,\eta_i,J_i)=\exp\left[\sum_i \ln f(\bar\eta_i,\eta_i,J_i)\right]\ , \tag{1.8}$$

where

$$f(\bar\eta,\eta,J)=\int d\phi\, d\psi\, d\bar\psi\, \exp\{a^d[\mathscr{L}_0(\phi,\bar\psi,\psi)+\bar\psi\eta+\bar\eta\psi+J\phi]\}\ . \tag{1.9}$$

$\ln f(\bar\eta,\eta,J)$ has a Taylor series expansion:

$$\ln f(\bar\eta,\eta,J)=\sum_{k,l=0}^{\infty} v_{kl}\frac{(\bar\eta\eta)^k}{(2k)!}\frac{J^l}{l!}\ . \tag{1.10}$$

The v_{kl}, which are the vertices of the strong-coupling graphs, are the connected Green's functions of the theory with no internal or external kinetic energy, having k fermions, k antifermions, and l bosons. The number of fermions at a point is limited by the Pauli exclusion principle.

Z_0 has a *formal* continuum limit,

$$Z_0=\exp\left[\delta(0)\int d^dx\, \ln f(\bar\eta(x),\eta(x),J(x))\right]\ , \tag{1.11}$$

where we have identified

$$\delta(0)=\frac{1}{a^d}\ . \tag{1.12}$$

Thus, a formal continuum strong-coupling expansion may be obtained from

$$Z[\bar\eta,\eta,J]=\exp\left\{\int d^dx\, d^dy\left[-\frac{\delta}{\delta\eta(x)}\partial\!\!\!/\delta(x-y)\frac{\delta}{\delta\bar\eta(y)}+\frac{1}{2}\frac{\delta}{\delta J(x)}\partial^2\delta(x-y)\frac{\delta}{\delta J(y)}\right]\right\}Z_0[\bar\eta,\eta,J]\ . \tag{1.13}$$

The result is a set of graphical rules organized in terms of the number of free inverse propagators, i.e., in the lines $S^{-1}(x,y)$ and $G^{-1}(x,y)$ defined in (1.5) and (1.6), connecting the vertices v_{kl} defined in (1.10). The explicit rules for $g\phi^4$ field theory in d dimensions are given in Ref. 2.

The lattice can be thought of as a particular way of regulating the Dirac δ functions occurring in the expansion, as well as being the actual *definition* of the formal expressions for the Green's functions obtained from (1.13) by functional differentiation.

II. THE CONTINUUM STRONG-COUPLING DIAGRAM RULES

The formal continuum diagram rules were obtained by making the replacement

$$\sum_i \ln f(\bar{\eta}_i,\eta_i,J_i) \rightarrow \delta(0)\int d^dx \ln f(\bar{\eta}(x),\eta(x),J(x)) \quad (2.1)$$

in Eq. (1.8). However, we will see that this does not result in a completely finite theory, even though a suitable regulation of the Dirac δ function has been chosen. The identification of lattice with continuum sources,

$$\eta_i \rightarrow \eta(x)\ , \quad (2.2)$$

etc., implied by (2.1), introduces divergences in the contribution to Green's functions from the bare vertices. We propose to identify the lattice sources with smeared continuum sources via

$$\eta_i \rightarrow \eta_s(x) = \int d^dz\, \delta(x-z)\eta(z)\ , \quad (2.3)$$

etc., i.e., with the "average over the unit cell."

In order to illustrate this point we write the formal expression for the vertices as

$$\int d^dx \ln f(\bar{\eta}(x),\eta(x),J(x)) = \sum_{k,l=0}^{\infty} \int d^dx_1 \cdots \int d^dx_{2k+l} \frac{V(x_1,\ldots,x_{2k+l})}{(2k)!l!} \bar{\eta}(x_1)\cdots\bar{\eta}(x_k)\eta(x_{k+1})\cdots\eta(x_{2k})J(x_{2k+1})\cdots J(x_{2k+l}) \quad (2.4)$$

where

$$V(x_1,\ldots,x_{2k+l}) = \int d^dx\, v_{kl} \prod_{i=1}^{2k+l} \delta(x-x_i) \quad (2.5)$$

are the local vertices, which are actually the connected Green's functions for the theory without internal or external kinetic energy.

The lattice regulation scheme makes the interpretation $x \rightarrow i$ and

$$\delta(x-y) \rightarrow \delta_a(x-y) \equiv \delta_a(i,j) \equiv \frac{\delta_{ij}}{a^d}\ . \quad (2.6)$$

So the inverse free propagators (1.5) and (1.6) become

$$S^{-1}(x-y) \rightarrow \not{\partial}\delta_a(x-y) \equiv \sum_{\mu=1}^{d} \frac{\gamma_\mu(\delta_{i,j+\hat{\mu}}-\delta_{j,i+\hat{\mu}})}{2a^{d+1}}\ , \quad (2.7)$$

$$G^{-1}(x-y) \rightarrow \partial^2\delta_a(x-y) \equiv \sum_{\mu=1}^{d} \frac{\delta_{i,j+\hat{\mu}}+\delta_{j,i+\hat{\mu}}-2d\delta_{ij}}{a^{d+2}}\ , \quad (2.8)$$

where $\hat{\mu}$ are unit vectors in the μth direction on the lattice. In momentum space, using

$$F(k) = a^d \sum_m F_m e^{ik\cdot ma}\ , \quad (2.9)$$

(2.6)—(2.8) become

$$\delta_a(k) = 1\ , \quad (2.10)$$

$$S_a^{-1}(k) = \sum_{\mu=1}^{d} \gamma_\mu \frac{\sin k_\mu a}{a}\ , \quad (2.11)$$

$$G_a^{-1}(k) = \sum_{\mu=1}^{d} \frac{4}{a^2}\sin^2(k_\mu a/2)\ . \quad (2.12)$$

The lattice cutoff regulates the inverse free propagators as well as the vertices

$$V(x_1,\ldots,x_{2k+l}) = \left[\frac{1}{a^d}\right]^{2k+l-1} v_{kl} \sum_i \prod_{j=1}^{2k+l} \delta_{ij}\ , \quad (2.13)$$

so that (2.4) is just

$$\int d^dx \ln f(\bar{\eta}(x),\eta(x),J(x)) = \sum_{k,l=0}^{\infty} a^d \sum_i \frac{v_{kl}(\bar{\eta}_i\eta_i)^k J_i^{\,l}}{(2k)!l!}$$

and we recover (1.10). The lattice regulated δ function, Eq. (2.6), obeys two properties of the Dirac δ function. First, it obeys the convolution

$$a^d \sum_j \delta_a(i,j)\delta_a(j,k) = \delta_a(i,k)\ . \quad (2.14)$$

Also, the inverse exists,

$$\delta_a^{-1}(i,j) = \delta_a(i,j)\ . \quad (2.15)$$

In attempting to find a continuum regulation of the Dirac δ function, one must at least give up having an inverse exist. The true Dirac δ function obeys

$$\int d^d y\, \delta(x-y)\delta(y-z)=\delta(x-z) \tag{2.16}$$

and

$$\delta(p)=1=\delta^{-1}(p) . \tag{2.17}$$

Even if the Λ-regulated δ function obeys

$$\int d^d y\, \delta_\Lambda(x-y)\delta_\Lambda(y-z)=\delta_\Lambda(x-z) \tag{2.18}$$

so that $\delta_\Lambda(p)$ is a combination of 1 and 0, such as

$$\delta_\Lambda(p)=\theta(\Lambda-|p|) , \tag{2.19}$$

then, except for $\delta(p)=1(\Lambda\to\infty)$, there is no inverse to $\delta_\Lambda(p)$. Thus, if we consider regulating the harmonic-oscillator potential energy

$$\exp\left[\delta(0)\int dx \ln f(J(x))\right]=\exp\left[\delta(0)\int dx \frac{J(x)^2}{2m^2\delta(0)}\right]$$
$$=\exp\left[\int\int dx\, dy\, J(x)\frac{\delta(x-y)}{2m^2}J(y)\right] \tag{2.20}$$

by replacing

$$\delta(x-y)\to\delta_\Lambda(x-y)=\int dz\, \delta_\Lambda(x-z)\delta_\Lambda(z-y) ,$$

it is not true that this comes from smearing the products of fields in the original path integral:

$$Z_0=N\int D\phi \exp\left\{-\int\int dx\, dy[(m^2/2)\phi(x)\delta_\Lambda(x-y)\phi(y)-J(x)\delta_\Lambda(x-y)\phi(y)]\right\} \tag{2.21}$$

because $\delta_\Lambda(x-y)$ has no inverse. Consequently, taking the continuum limit of the lattice-generating functional lnZ and then smearing the sources is not equivalent to smearing the products of fields in the original path integral. Unlike the lattice-regulation scheme, which is the definition of the path integral as the limit of a product of ordinary integrals, any continuum regulation scheme is an *a posteriori* regulation of the formal continuum limit of the lattice strong-coupling expansion.

III. THE GAUSSIAN REGULATION SCHEME

In the Gaussian regulation of the continuum strong-coupling series the Euclidean d-dimensional regulated δ function is defined as

$$\delta_\Lambda(x-y)=\left[\frac{\Lambda}{\sqrt{\pi}}\right]^d e^{-\Lambda^2|x-y|^2} , \tag{3.1}$$

so that

$$\delta_\Lambda(p)=e^{-p^2/4\Lambda^2} . \tag{3.2}$$

The inverse free propagators (1.5) and (1.6) become

$$S_\Lambda{}^{-1}(x,0)=\partial\!\!\!/\delta_\Lambda(x)$$
$$=-2\Lambda^2 x\!\!\!/\delta_\Lambda(x) , \tag{3.3}$$

$$G_\Lambda{}^{-1}(x,0)=\partial^2\delta_\Lambda(x)$$
$$=2\Lambda^2(2\Lambda x^2-d)\delta_\Lambda(x) \tag{3.4}$$

or, in momentum space,

$$S_\Lambda{}^{-1}(p)=ip\!\!\!/e^{-p^2/4\Lambda^2} , \tag{3.5}$$

$$G_\Lambda{}^{-1}(p)=-p^2e^{-p^2/4\Lambda^2} . \tag{3.6}$$

As we have mentioned before, it is necessary to smear the sources via Eq. (2.3) where the δ function is given in (3.1). This has the effect of smearing the vertices $V(x_1,\ldots,x_{2k+l})$ in Eq. (2.5) owing to the replacement

$$\prod_{i=1}^{2k+l}\delta(x-x_i)\to\prod_{i=1}^{2k+l}\delta_\Lambda(x-x_i) . \tag{3.7}$$

In momentum space, this multiplies every line entering or leaving the vertex and carrying momentum p_i by $e^{-p_i^2/4\Lambda^2}$ in addition to the overall momentum-conserving δ function $\delta(\sum_{i=1}^{2k+l}p_i)$.

Because each inverse propagator connects two vertex legs, the effect of smearing the vertices in this way is to modify the inverse propagators represented by internal lines to

$$\tilde{S}_\Lambda^{-1}(p)=i\not{p}e^{-3p^2/4\Lambda^2}\ , \tag{3.8}$$

$$\tilde{G}_\Lambda^{-1}(p)=-p^2e^{-3p^2/4\Lambda^2}\ . \tag{3.9}$$

Hence, the effective momentum cutoff is changed from 2Λ to $2\Lambda/\sqrt{3}$ on internal lines in diagrams.

The Gaussian cutoff is extremely easy to work with in arbitrary space-time dimension d. Even multiloop diagrams only require the Fourier transform of a polynomial times a Gaussian in d dimensions, which is simple. So this regulation scheme would seem ideally suited to carrying out strong-coupling expansions for gauge theories involving fermions where it is desirable to avoid the lattice and necessary to work in arbitrary dimension.[7] Unfortunately, there is a major deficiency of the Gaussian cutoff in that $\delta_\Lambda(x-y)$ of (3.1) does not obey the convolution property. In fact

$$\int d^dy\,\delta_\Lambda(x-y)\delta_\Lambda(y-z)=\delta_{(\Lambda/\sqrt{2})}(x-z)\ , \tag{3.10}$$

$$\delta_\Lambda(p)\delta_\Lambda(p)=\delta_{(\Lambda/\sqrt{2})}(p)=e^{-p^2/2\Lambda^2}\ .$$

Because of this it is possible to get quantities such as $[\delta_\Lambda(p)-\delta_{(\Lambda/\sqrt{2})}(p)]$ appearing in integrals in such a way that the result does not vanish as $\Lambda\to\infty$. This problem will be discussed later in the context of the harmonic oscillator.

IV. THE θ-FUNCTION REGULATION SCHEME

Because of the importance of the convolution property one would like to have at least one continuum regulation scheme that obeys

$$\delta_\Lambda(p)\delta_\Lambda(p)=\delta_\Lambda(p)\ . \tag{4.1}$$

The solution to this equation is either one, the noncutoff δ function, zero, or any combination of one and zero. Thus, a cutoff δ function which statisfies (4.1) is

$$\delta_\Lambda(p)=\theta(\Lambda-|p|)\ . \tag{4.2}$$

In coordinate space, this is

$$\delta_\Lambda(x)=\int\frac{d^dp}{(2\pi)^d}e^{ip\cdot x}\theta(\Lambda-|p|)\ . \tag{4.3}$$

Introducing d-dimensional spherical coordinates,

$$\delta_\Lambda(x)=\int_0^\Lambda\frac{p^{d-1}dp}{(2\pi)^d}\times\int_0^\pi d\theta\,e^{ipr\cos\theta}\sin^{d-2}\theta\frac{2\pi^{(d-1)/2}}{\Gamma\left[\frac{d-1}{2}\right]}\ , \tag{4.4}$$

where $r=|x|$. Then, using

$$\int_0^\pi e^{ipr\cos\theta}\sin^{d-2}\theta\,d\theta=\left[\frac{2}{pr}\right]^{d/2-1}\Gamma\left[\frac{d-1}{2}\right]\Gamma\left[\frac{1}{2}\right]J_{d/2-1}(pr)$$

and

$$\int_0^1 x^{\nu+1}J_\nu(ax)dx=\frac{1}{a}J_{\nu+1}(a)$$

we obtain

$$\delta_\Lambda(x)=\left[\frac{\Lambda}{2\pi|x|}\right]^{d/2}J_{d/2}(\Lambda|x|)\ . \tag{4.5}$$

When $d=1$, using

$$J_{1/2}(x)=(2/\pi x)^{1/2}\sin x$$

we obtain

$$\delta_\Lambda(x)=\frac{\sin\Lambda x}{\pi x}\quad(d=1)\ . \tag{4.6}$$

The Feynman rules in momentum space are very simple for the θ-function regulation. The legs of each vertex carrying momentum p_i have a $\theta(\Lambda-|p_i|)$ attached to them, together with an overall momentum-conserving δ function at the vertex. The internal lines represent

$$S_\Lambda^{-1}(p)=i\not{p}\theta(\Lambda-|p|)\ , \tag{4.7}$$

$$G_\Lambda^{-1}(p)=-p^2\theta(\Lambda-|p|)\ . \tag{4.8}$$

Since

$$[\theta(\Lambda-|p|)]^n=\theta(\Lambda-|p|)\ ,\quad n>1\ , \tag{4.9}$$

the regulation of the vertices does not modify the cutoff on internal lines, but does regulate the theory without kinetic energy. A simple form of θ-function regulation which ignored the need to regulate the vertices is found in Refs. 8 and 9.

In order to understand the difference between the lattice regulation and the continuum Gaussian or θ-function schemes we turn our attention to the harmonic oscillator.

V. THE HARMONIC OSCILLATOR

In order to understand how the lattice regulates the strong-coupling expansion let us first consider the harmonic oscillator. The formal generating functional is

$$Z[J]=\exp\left[\tfrac{1}{2}\int\int dx\,dy\frac{\delta}{\delta J(x)}G^{-1}(x,y)\frac{\delta}{\delta J(y)}\right]\int D\phi\exp\left[-\int dx\left[\frac{m^2\phi^2}{2}+J\phi\right]\right]$$

$$=\exp\left[\tfrac{1}{2}\int\int dx\,dy\frac{\delta}{\delta J(x)}G^{-1}(x,y)\frac{\delta}{\delta J(y)}\right]\exp\left[\int dx\frac{J(x)^2}{2m^2}\right], \tag{5.1}$$

where G^{-1} is given in Eq. (1.6). The two-point function

$$W_2(x,y)=\frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(y)}\ln Z[J]\bigg|_{J=0} \tag{5.2}$$

has a strong-coupling expansion given by the sum of diagrams in Fig. 1, i.e.,

$$W_2(x,y)=\frac{1}{m^2}\delta(x-y)+\frac{1}{m^4}G^{-1}(x,y)+\frac{1}{m^6}\int dz\,G^{-1}(x,z)G^{-1}(z,y)+\cdots . \tag{5.3}$$

Employing a lattice regulation (2.6)–(2.8),

$$\delta(x-y)\to\frac{1}{a}\delta_{ij}, \tag{5.4}$$

$$G^{-1}(x,y)\to(1/a^3)(\delta_{i,j+1}+\delta_{j,i+1}-2\delta_{ij}). \tag{5.5}$$

In momentum space,

$$G^{-1}(p)=-(4/a^2)\sin^2(pa/2) \tag{5.6}$$

and Eq. (5.3) is a geometric series,

$$W_2(p)=(1/m^2)\sum_{n=0}^{\infty}[-(4/m^2a^2)\sin^2(pa/2)]^n=\frac{1}{m^2+(4/a^2)\sin^2(pa/2)}. \tag{5.7}$$

So, in coordinate space,

$$W_2(x,y)=\int_{-\pi/a}^{\pi/a}(dp/2\pi)e^{-ip(x-y)}\frac{1}{m^2+(4/a^2)\sin^2(pa/2)}$$

$$=\int_{-\pi/a}^{\pi/a}(dp/2\pi)e^{-ip(x-y)}(1/m^2)\sum_{n=0}^{\infty}[-(4/m^2a^2)\sin^2(pa/2)]^n\quad(x\equiv ia\,,\;y\equiv ja). \tag{5.8}$$

We note that every term in this strong-coupling series exists for nonzero a. In particular,

$$m^2W_2(x,x)=\frac{1}{a}+\frac{1}{a}\sum_{n=1}^{\infty}\frac{(-1)^n}{(ma)^{2n}}\frac{(2n)!}{(n!)^2}. \tag{5.9}$$

FIG. 1. Strong-coupling diagrams for the two-point function of the harmonic oscillator.

It is obvious from the sum of the geometric series in Eq. (5.8) that as $a\to 0$

$$W_2(x,y)\underset{a\to 0}{\to}\int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{-ip(x-y)}\frac{1}{m^2+p^2} \tag{5.10}$$

and

$$W_2(y,y)\underset{a\to 0}{\to}\frac{1}{2m}. \tag{5.11}$$

We can extrapolate the series in (5.9) to obtain the correct continuum ($a \to 0$) result (5.11) as follows. Let

$$x \equiv \frac{1}{m^2 a^2} , \tag{5.12}$$

then

$$m^2 W_2(y,y) = m\sqrt{x} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)! x^n}{(n!)^2} \tag{5.13}$$

and the lowest-order approximant to the right-hand side,

$$m\sqrt{x}(1-2x) \cong m \left[\frac{x}{1+4x} \right]^{1/2} \underset{x\to\infty}{\longrightarrow} \frac{m}{2} , \tag{5.14}$$

is exact.

We notice that the lattice does two things. First it regulates the inverse free propagator $G^{-1}(x,y)$ so that the momentum-space loop integrals such as

$$G^{-1}(x,x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} G^{-1}(p) \to -\int_{-\pi/a}^{\pi/a} \frac{dp}{2\pi} \frac{4}{a^2} \sin^2 \frac{pa}{2} = \frac{-2}{a^3} \tag{5.15}$$

exist. But it also regulates the bare two-point vertex

$$\frac{1}{m^2} \delta(x-y) \to \frac{1}{m^2} \frac{\delta_{ij}}{a} . \tag{5.16}$$

This second property is necessary if one wants to evaluate $W_2(x,x)$ from a regulated strong-coupling series. We pursue this point further. We have seen [Eq. (5.11)] that $W_2(x,x)$ is related to the ground-state energy of the harmonic oscillator. So, in general, how do we regulate the formal strong-coupling series for

$$W_2(x,y) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{e^{-ip(x-y)}}{m^2} \sum_{n=0}^{\infty} \left[\frac{G^{-1}(p)}{m^2} \right]^n \tag{5.17}$$

such that each term exists for finite values of the cutoff? First, note that

$$W_2(x,x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{1}{m^2 - G^{-1}(p)} = \int_{-\infty}^{\infty} \frac{dp}{2\pi} W_2(p) . \tag{5.18}$$

If all we did was to regulate the kinetic energy $G^{-1}(p)$, for example, by means of a soft Gaussian cutoff Λ in momentum space (3.6),

$$G_\Lambda{}^{-1}(p) = -p^2 e^{-p^2/4\Lambda^2} ; \tag{5.19}$$

then the integral for $W_2(x,x)$ in (5.18) would diverge. Thus, it is insufficient to regulate the kinetic energy alone. In a related attempt at continuum regulation, Benzi, Martinelli, and Parisi[10] obtained

$$W_{2\Lambda}(p) = \frac{1}{m_0{}^2 + (\beta/2^{d/2}) e^{-p^2 R^2/4}} \tag{5.20}$$

which also suffers from having the wrong ultraviolet behavior as $p^2 \to \infty$ for fixed cutoff R.

In quantum mechanics one can calculate the ground-state energy from the Green's functions using certain formal identities proven in Ref. 11. We will show that only the θ-function regulation preserves these identities and that for some forms of these identities the Gaussian regulation leads to an asymptotic, rather than convergent, sequence of approximants.

For quantum mechanics ($d=1$ field theory) with potential $g\phi^{2N}$, one can prove[11] that the ground-state energy obeys

$$2Ng \frac{dE}{dg} = \int_{-\infty}^{\infty} \frac{dp}{2\pi} G^{-1}(p)[G(p) + W_2(p)] \tag{5.21}$$

in the continuum, where

$$G^{-1}(p) = -p^2 \tag{5.22}$$

and W_2 is the exact two-point function. The harmonic oscillator has $N=1$, $g=m^2/2$, and

$$W_2(p) = \frac{1}{p^2+m^2} . \tag{5.23}$$

So

$$2m^2 \frac{dE}{dm^2} = \int_{-\infty}^{\infty} \frac{dp}{2\pi} p^2 \left[\frac{1}{p^2} - \frac{1}{p^2+m^2} \right] = \frac{m^2}{2\pi} \int_{-\infty}^{\infty} \frac{dp}{p^2+m^2} = \frac{m}{2} \tag{5.24}$$

and

$$m^2 \frac{dE}{dm^2} = \frac{m^2}{2} W_2(x,x) . \tag{5.25}$$

Thus, in the continuum we have the identity

$$2m^2 \frac{dE}{dm^2} = \delta(0) + \int_{-\infty}^{\infty} \frac{dp}{2\pi} G^{-1}(p) W_2(p) = m^2 W_2(x,x) . \tag{5.26}$$

We would like to see under what conditions the theory with a continuum cutoff also obeys relations such as Eq. (5.26).

In general,

$$G_\Lambda{}^{-1}(p) = -p^2\delta_\Lambda(p) \tag{5.27}$$

so that the strong-coupling expansion (5.17) is

$$W_2(p) = \frac{\delta_\Lambda(p)^2}{m^2}\sum_{n=0}^{\infty}\left[\frac{-p^2\delta_\Lambda(p)^3}{m^2}\right]^n = \frac{\delta_\Lambda(p)^2}{m^2+p^2\delta_\Lambda(p)^3}\,, \tag{5.28}$$

where we have used the smeared two-point vertex $(1/m^2)\delta_\Lambda(p)^2$. Thus using

$$\delta_\Lambda(0) = \int_{-\infty}^{\infty}\frac{dp}{2\pi}\delta_\Lambda(p)\,, \tag{5.29}$$

we have

$$\delta_\Lambda(0) + \int_{-\infty}^{\infty}\frac{dp}{2\pi}G_\Lambda{}^{-1}(p)W_{2\Lambda}(p) = \int_{-\infty}^{\infty}\frac{dp}{2\pi}\left[\delta_\Lambda(p) - \frac{p^2\delta_\Lambda(p)^3}{m^2+p^2\delta_\Lambda(p)^3}\right]$$

$$= m^2\int_{-\infty}^{\infty}\frac{dp}{2\pi}\frac{\delta_\Lambda(p)}{m^2+p^2\delta_\Lambda(p)^3} + \int_{-\infty}^{\infty}\frac{dp}{2\pi}\frac{p^2[\delta_\Lambda(p)^4-\delta_\Lambda(p)^3]}{m^2+p^2\delta_\Lambda(p)^3}\,, \tag{5.30}$$

whereas from (5.28)

$$m^2W_2(x,x) = m^2\int_{-\infty}^{\infty}\frac{dp}{2\pi}\frac{\delta_\Lambda(p)^2}{m^2+p^2\delta_\Lambda(p)^3}\,.$$

Hence, for the formal continuum identity to hold in the cutoff theory we need

$$\delta_\Lambda(p)^2 = \delta_\Lambda(p) \tag{5.31a}$$

or, equivalently,

$$\int dz\,\delta_\Lambda(x-z)\delta_\Lambda(z-y) = \delta_\Lambda(x-y)\,. \tag{5.31b}$$

As we have already remarked, this can only happen if $\delta_\Lambda(p)$ is a combination of zero and one, such as $\delta_\Lambda(p) = \theta(\Lambda - |p|)$. We conclude that only the Dirac δ function itself and the step function preserve the continuum Green's-function identity (5.26).

Of course, it would be sufficient for the extra term in Eq. (5.30),

$$\int_{-\infty}^{\infty}\frac{dp}{2\pi}\frac{p^2[\delta_\Lambda(p)^4-\delta_\Lambda(p)^3]}{m^2+p^2\delta_\Lambda(p)^3}\,,$$

to vanish as $\Lambda\to\infty$. This may not be the case. In fact, for the Gaussian regulation scheme $\delta_\Lambda(p) = e^{-p^2/4\Lambda^2}$ this integral actually diverges as $\Lambda\to\infty$.[6] Nevertheless, we will see that the sequence of approximants obtained by expanding these integrands in powers of p^2 and then performing the integration is, at least, asymptotic even for the Gaussian regulation.

First, let us calculate the ground-state energy of the harmonic oscillator using the Gaussian cutoff. If we use one part of the continuum identity (5.26) and Eq. (5.28),

$$2m^2\frac{dE}{dm^2} = m^2W_2(x,x) = m^2\int_{-\infty}^{\infty}\frac{dp}{2\pi}\left[\frac{e^{-p^2/2\Lambda^2}}{m^2+p^2e^{-3p^2/4\Lambda^2}}\right]\,, \tag{5.32}$$

then it can be shown that

$$\int_{-\infty}^{\infty}\frac{dp}{2\pi}\left[\frac{e^{-p^2/2\Lambda^2}}{m^2+p^2e^{-3p^2/4\Lambda^2}}\right]\underset{\Lambda\to\infty}{\longrightarrow}\frac{1}{2m}\,. \tag{5.33}$$

Thus, we want to see how the extrapolants of

$$\Sigma(\Lambda) = \frac{2}{m}\int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{-p^2/2\Lambda^2}\sum_{n=0}^{\infty}\left[\frac{-p^2}{m^2}e^{-3p^2/4\Lambda^2}\right]^n \tag{5.34}$$

behave. Substituting

$$y \equiv \frac{4\Lambda^2}{m^2} \tag{5.35}$$

we have

$$\Sigma(\Lambda)=\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})}\frac{\sqrt{2}y^{n+1/2}}{(2+3n)^{n+1/2}}$$

$$=\frac{y^{1/2}}{\sqrt{2\pi}}\left[1-\left[\frac{2}{5}\right]^{1/2}\frac{y}{10}+\frac{3y^2}{512}-\frac{15}{10\,648}\left[\frac{2}{11}\right]^{1/2}y^3+\frac{15}{87\,808}\frac{y^4}{\sqrt{7}}-\frac{945}{45\,435\,424}\left[\frac{2}{17}\right]^{1/2}y^5\right.$$
$$+\frac{2079}{1\,638\,400\,000}\left[\frac{2}{5}\right]^{1/2}y^6-\frac{135135}{435\,817\,657\,216}\left[\frac{2}{23}\right]^{1/2}y^7$$
$$+\frac{155925}{4\,112\,286\,810\,112}\left[\frac{2}{26}\right]^{1/2}y^8-\frac{34459425}{7\,427\,658\,739\,644\,928}\left[\frac{2}{29}\right]^{1/2}y^9+\frac{654729075}{4\,611\,686\,018\,427\,387\,904}y^{10}$$
$$\left.-\frac{11223927}{161\,414\,428\,000\,000\,000}\left[\frac{2}{35}\right]^{1/2}y^{11}+\frac{16643902275}{1\,954\,382\,235\,431\,342\,178\,304}\left[\frac{2}{38}\right]^{1/2}y^{12}+\cdots\right]. \tag{5.36}$$

Using our usual procedure[2] to extrapolate this series to $y=\infty$, we obtain

1.1217, 1.1121, 1.1057, 1.1001, 1.0973, 1.0944, 1.0920,

1.0900, 1.0881, 1.0866, 1.0852, 1.0840, . . . ,

a sequence which is converging very slowly to one, just as the integral (5.33) converges very slowly as $\Lambda\to\infty$. If, instead, we use the other part of the continuum identity (5.26),

$$2m^2\frac{dE}{dm^2}=\delta_\Lambda(0)+\int_{-\infty}^{\infty}\frac{dp}{2\pi}G_\Lambda{}^{-1}(p)W_{2\Lambda}(p) \tag{5.37}$$

which actually diverges as $\Lambda\to\infty$, the corresponding series is

$$\Sigma(\Lambda)=\frac{2}{m}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\left[\frac{m^2e^{-p^2/4\Lambda^2}+p^2(e^{-p^2/\Lambda^2}-e^{-3p^2/4\Lambda^2})}{m^2+p^2e^{-3p^2/4\Lambda^2}}\right]$$
$$=\frac{y^{1/2}}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\left\{\frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})(3n+1)^{n+1/2}}y^n+\frac{\Gamma(n+3/2)}{\Gamma(\frac{1}{2})}\left[\frac{y^{n+1}}{(3n+4)^{n+3/2}}-\frac{y^{n+1}}{(3n+3)^{n+3/2}}\right]\right\}$$
$$=\frac{y^{1/2}}{\sqrt{\pi}}\left[1-\frac{\sqrt{3}y}{18}+\frac{\sqrt{6}y^2}{288}-\frac{5y^3}{5832}+\frac{(35\sqrt{3})y^4}{663\,552}-\frac{(7\sqrt{15})y^5}{2\,700\,000}+\frac{(385\sqrt{2})y^6}{483\,729\,408}-\frac{(715\sqrt{21})y^7}{25\,615\,481\,472}\right.$$
$$\left.+\frac{(25\,025\sqrt{6})y^8}{4\,174\,708\,211\,712}-\frac{(425\,425\sqrt{3})y^9}{433\,811\,768\,034\,816}+\cdots\right]. \tag{5.38}$$

We would like to compare this with the *correct* continuum result $\Sigma(\Lambda=\infty)=1$. The series (5.38) does not converge as $y\to\infty$. However, the first five extrapolants are

1.2861, 1.1468, 1.0770, 1.0357, 1.0008

and appear to be converging to one. But the next four, 0.9888,0.9740,0.9625,0.9532, . . .

overshoot. So the series of extrapolants is asymptotic, at best. Our explanation for this behavior is the following. Because of the failure of the convolution property (5.31) for a Gaussian regulation scheme the second integral in (5.30) diverges instead of vanishing as $\Lambda \to \infty$. This divergence is coming from the ultraviolet region $p^2 >> \Lambda^2$ and so the low orders in the p^2 expansion do not know about this ultimate divergence and give quite reasonable results.

Next, let us turn to the θ-function cutoff. Now,

$$W_{2\Lambda}(p)=\frac{\theta(\Lambda-|p|)}{m^2+p^2\theta(\Lambda-|p|)}, \tag{5.39}$$

$$G_\Lambda{}^{-1}(p)=-p^2\theta(\Lambda-|p|), \tag{5.40}$$

and

$$\delta_\Lambda(0)=\int_{-\infty}^{\infty}\frac{dp}{2\pi}\theta(\Lambda-|p|). \tag{5.41}$$

Because this cutoff preserves the convolution property,

$$\delta_\Lambda(0)+\int_{-\infty}^{\infty}\frac{dp}{2\pi}G_\Lambda{}^{-1}(p)W_{2\Lambda}(p)$$
$$=m^2\int_{-\infty}^{\infty}\frac{dp}{2\pi}\frac{\theta(\Lambda-|p|)}{m^2+p^2\theta(\Lambda-|p|)}$$
$$=m^2W_{2\Lambda}(x,x). \tag{5.42}$$

To obtain the strong-coupling expansion we use

$$\frac{m^2}{2\pi}\int_{-\Lambda}^{\Lambda}\frac{dp}{p^2+m^2}=\frac{1}{2\pi}\int_{-\Lambda}^{\Lambda}dp\sum_{n=0}^{\infty}\left[\frac{-p^2}{m^2}\right]^n$$
$$=\frac{m}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n y^{2n+1}}{2n+1} \tag{5.43}$$

with

$$y=\frac{\Lambda}{m}. \tag{5.44}$$

Thus, we wish to compare the extrapolants of

$$\frac{2}{\pi}y\sum_{n=0}^{\infty}\frac{(-1)^n y^{2n}}{2n+1}$$

with unity. We obtain the following sequence as $y\to\infty$,

0.7797,0.8524,0.8931,0.9331,

0.9439,0.9517,0.9576,0.9622,

0.9659,0.9690,0.9715, . . .

which appears to be converging to the right answer.

Thus, in this example we see the advantage of the θ function over the Gaussian regulation scheme. Because it obeys the convolution property, the correct ultraviolet behavior of the Green's functions is ensured and one expects that continuum Ward identities will be preserved in the cutoff theory. With the Gaussian scheme there is always a danger of having terms which are not quite zero for $p^2 >> \Lambda^2$ and one expects eventual asymptotic rather than convergent sequences of approximants.

VI. THE ANHARMONIC OSCILLATOR

The calculation of the ground-state and the first-excited-state energies of the anharmonic oscillator with Hamiltonian

$$H=\frac{p^2}{2m}+\frac{g}{4}x^4 \tag{6.1}$$

using a lattice-regulated strong-coupling expansion was discussed in detail in Ref. 2. The exact numerical result for the first excited state is

$$M=1.0808\ldots g^{1/3} \tag{6.2}$$

and for the ground state

$$4g\frac{dE}{dg}=0.569\,473g^{1/3}. \tag{6.3}$$

In this section we would like to compare results from continuum regulation schemes with the same-order results from the lattice calculation.

The formal generating functional for the strong-coupling expansion of $g\phi^4$ field theory[2] is

$$Z[J]=N\exp\left[\frac{1}{2}\int d^dx\,d^dy\frac{\delta}{\delta J(x)}\times G^{-1}(x,y)\frac{\delta}{\delta J(y)}\right]Z_0[J], \tag{6.4}$$

where

$$Z_0[J]=\exp a^{-d}\int d^dx\ln\frac{F(J(x))}{F(0)}, \tag{6.5}$$

$$F(J)=\int_{-\infty}^{\infty}dx\exp\left[-a^d\left[\frac{g}{4}x^4+\frac{m^2}{2}x^2-Jx\right]\right]$$
$$=\sum_{n=0}^{\infty}A_{2n}\frac{J^{2n}}{(2n)!}, \tag{6.6}$$

and

$$A_{2n}=2(a^d)^{2n}\left[\frac{4}{ga^d}\right]^{n/2+1/4}$$
$$\times\sum_{l=0}^{\infty}\left[\frac{-m^2a^2}{(ga^{4-d})^{1/2}}\right]^l\frac{\Gamma(n/2+l/2+1/4)}{l!}\ . \tag{6.7}$$

Here a is the lattice spacing. The formal vertices of the theory are defined by

$$\ln Z_0[J]=a^{-d}\int d^dx\,\ln\sum_{n=0}^{\infty}\frac{A_{2n}}{A_0}\frac{[J(x)]^{2n}}{(2n)!}$$
$$=a^{-d}\int d^dx\sum_{n=1}^{\infty}\frac{L_{2n}}{(2n)!}[J(x)]^{2n}\ . \tag{6.8}$$

The first few vertices L_{2n} are

$$\begin{aligned}L_2&=g^{-1/2}a^{d/2}2R\ ,\\ L_4&=g^{-1}a^{2d}(1-12R^2)\ ,\\ L_6&=g^{-3/2}a^{7d/2}(240R^3-24R)\ ,\\ L_8&=g^{-2}a^{5d}(-10\,080R^4+1344R^2-30)\ ,\end{aligned} \tag{6.9}$$

where

$$R=\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}=0.337\,989\,120\ . \tag{6.10}$$

Our prescription for performing a continuum regulation of (6.8) is to replace

$$\frac{1}{a^d}\to\delta_\Lambda(0)\ , \tag{6.11a}$$

$$G^{-1}(x,y)=\partial^2\delta(x-y)\to\partial^2\delta_\Lambda(x-y) \tag{6.11b}$$

and to smear the vertices via [cf. Eq. (2.3)]

$$J(x)\to J_s(x)=\int d^dz\,\delta_\Lambda(x-z)J(z)\ . \tag{6.11c}$$

To calculate the two-point function it is sufficient to realize that $W_2(p)$ obeys a Dyson equation in the continuum

$$W_2(p)=\Lambda_2(p)+\Lambda_2(p)G^{-1}(p)W_2(p)\ , \tag{6.12}$$

where $\Lambda_2(p)$ is the sum of diagrams one-particle irreducible (1PI) in G^{-1}. Because of our smearing process the two external legs of $W_2(p)$ each pick up a factor of $\delta_\Lambda(p)$ while each internal line picks up a factor of $\delta_\Lambda(p)^2$. So $G_\Lambda{}^{-1}(p)$ is effectively replaced by

$$\widetilde{G}_\Lambda{}^{-1}(p)=\delta_\Lambda(p)^2G_\Lambda{}^{-1}(p) \tag{6.13}$$

and the continuum-regulated Dyson equation is

$$W_{2\Lambda}(p)=\delta_\Lambda(p)^2[\Lambda_{2\Lambda}(p)+\Lambda_{2\Lambda}(p)\widetilde{G}_\Lambda{}^{-1}(p)W_{2\Lambda}(p)]$$
$$=\frac{\delta_\Lambda(p)^2}{\Lambda_{2\Lambda}{}^{-1}(p)-\widetilde{G}_\Lambda{}^{-1}(p)}\ . \tag{6.14}$$

Here $\Lambda_{2\Lambda}(p)$ is the sum of all 1PI diagrams with the momentum dependence of the two external legs removed as shown in Fig. 2. That is, as in Ref. 2, and setting $d=1$,

$$\Lambda_{2\Lambda}=L_2+\frac{L_2}{2}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\widetilde{G}_\Lambda{}^{-1}(p)$$
$$+\frac{L_6}{8}\left[\int_{-\infty}^{\infty}\frac{dp}{2\pi}\widetilde{G}_\Lambda{}^{-1}(p)\right]^2$$
$$+\frac{L_2L_4}{2}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\left[\widetilde{G}_\Lambda{}^{-1}(p)\right]^2+\cdots\ . \tag{6.15}$$

The only difference here is that $\widetilde{G}_\Lambda{}^{-1}(p)$ replaces $G_\Lambda{}^{-1}(p)$ for the case of a Gaussian regulation scheme. The first-excited state is the lowest zero in $[\Lambda_{2\Lambda}{}^{-1}(p)-\widetilde{G}_\Lambda{}^{-1}(p)]_{p^2=M^2}$.

In the Gaussian regulation scheme

$$\widetilde{G}_\Lambda{}^{-1}(p)=-p^2e^{-3p^2/4\Lambda^2}\ , \tag{6.16a}$$

$$\delta_\Lambda(p)=e^{-p^2/4\Lambda^2}\ ,$$
$$\delta_\Lambda(x)=\frac{\Lambda e^{-\Lambda^2x^2}}{\sqrt{\pi}}\ , \tag{6.16b}$$

and

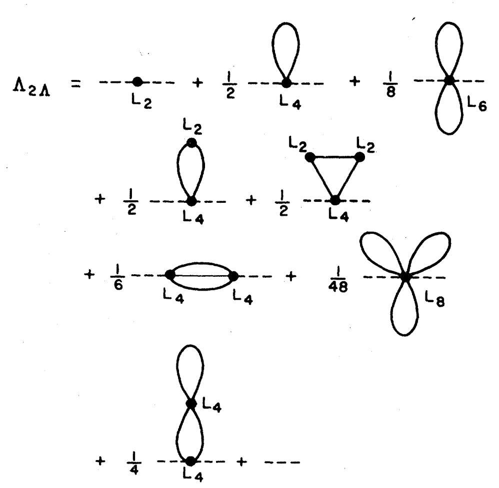

FIG. 2. One-particle irreducible strong-coupling diagrams for the self-energy in $g\phi^4$ field theory.

$$\frac{g^{2/3}}{2RM^2}x^{1/3}=1+\frac{\sqrt{3}\,\pi(12R^2-1)x}{18R}+\frac{\pi^2}{36}[8-\sqrt{6}+(80-12\sqrt{6})R^2]x^2$$
$$+\frac{\pi^3}{52\,488R}[476\sqrt{3}-243\sqrt{2}+R^2(17496\sqrt{2}-22\,656\sqrt{3}-2880)$$
$$+R^4(172\,224\sqrt{3}-151\,632\sqrt{2}+34\,560)]x^3+\cdots\ , \qquad (6.17)$$

where

$$\delta_\Lambda(0)=\frac{\Lambda}{\sqrt{\pi}}=g^{1/3}x^{2/3}\ . \qquad (6.18)$$

Using the same extrapolation procedure described in Ref. 2 and the value of R given in Eq. (6.10), we obtain the following three $x\rightarrow\infty$ estimates for M:

$$M_1=1.21\,729g^{1/3}\ \ (12.6\%\text{ error})\ ,$$
$$M_2=1.14\,662g^{1/3}\ \ (6.1\%\text{ error})\ , \qquad (6.19)$$
$$M_3=1.10\,961g^{1/3}\ \ (2.7\%\text{ error})\ ,$$

as compared with the lattice cutoff calculation[2]

$$\alpha_1=1.1194\ \ (3.6\%\text{ error})\ ,$$
$$\alpha_2=1.1021\ \ (2.0\%\text{ error})\ , \qquad (6.20)$$
$$\alpha_3=1.0973\ \ (1.5\%\text{ error})\ .$$

The exact answer is $1.0808g^{1/3}$. So we see that the Gaussian regulation scheme is quite reliable in determining the position of the pole, but not as good as the lattice. If, instead, we use a θ-function regulation, where

$$\delta_\Lambda=\theta(\Lambda-|p|)\ ,$$
$$\qquad (6.21)$$
$$\delta_\Lambda(x)=\frac{\sin\Lambda x}{\pi x}$$

so that

$$\delta_\Lambda(0)=\frac{\Lambda}{\pi} \qquad (6.22)$$

and

$$G_\Lambda{}^{-1}(p)=-p^2\theta(\Lambda-|p|)\ , \qquad (6.23)$$

the series we obtain is

$$\frac{g^{2/3}}{2RM^2}x^{1/3}=1+\frac{\pi^2(12R^2-1)x}{12R}+\frac{\pi^4}{15}(7R^2-1)x^2+\cdots \qquad (6.24)$$

which has the following two extrapolants,

$$M_1=1.03026\ \ (4.7\%\text{ error})\ ,$$
$$\qquad (6.25)$$
$$M_2=1.07483\ \ (0.6\%\text{ error})\ .$$

Thus we see that the θ-function regulation scheme is quite accurate at second order.

A more stringent test of our procedure is to calculate the ground-state energy of the anharmonic oscillator as $g\rightarrow\infty$, since this is not just obtained from the position of the pole in $W_2(p)$. From Ref. 2 we have the continuum formal identity

$$4g\frac{dE}{dg}=\delta(0)+\int_{-\infty}^{\infty}\frac{dp}{2\pi}W_2(p)G^{-1}(p) \qquad (6.26)$$

which we regulate as

$$\delta_\Lambda(0)+\int_{-\infty}^{\infty}\frac{dp}{2\pi}W_{2\Lambda}(p)G_\Lambda{}^{-1}(p)\ , \qquad (6.27)$$

remembering that in the case of the harmonic oscillator with Gaussian cutoff this prescription only leads to an asymptotic sequence of approximants. $W_{2\Lambda}(p)$ is obtained as a power series in x using Eq. (6.14). The diagrams which contribute are shown in Fig. 3. For the case of Gaussian regulation we obtain

$$4g\frac{dE}{dg}=g^{1/3}x^{2/3}\left[1-\frac{4\pi R\sqrt{3}}{9}x+\frac{2\pi^2}{27}[1+(6\sqrt{6}-24)R^2]x^2+\cdots\right]\ . \qquad (6.28)$$

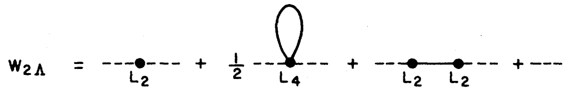

FIG. 3. Strong-coupling diagrams for the full two-point function in $g\phi^4$ field theory.

The first two extrapolants are

$$\begin{aligned} &0.8729g^{1/3}\ \ (53\%\ \text{high})\ , \\ &0.6948g^{1/3}\ \ (22\%\ \text{high}) \end{aligned} \tag{6.29}$$

compared with the exact answer $0.5695g^{1/3}$. Using the θ-function regulation, which should lead to a convergent sequence of approximants if the harmonic oscillator is a good guide, we obtain

$$4g\frac{dE}{dg}=g^{1/3}x^{2/3}\left[1-\frac{2\pi^2R}{3}x+\frac{\pi^4}{90}(5+12R^2)x^2+\cdots\right]. \tag{6.30}$$

The first two extrapolants are

$$\begin{aligned} &0.4479g^{1/3}\ \ (21.3\%\ \text{low})\ , \\ &0.4810g^{1/3}\ \ (15.5\%\ \text{low})\ . \end{aligned} \tag{6.31}$$

The sequence obtained from the lattice calculation[2] was

$$\begin{aligned} &\alpha_1=0.6242g^{1/3}\ \ (9.6\%\ \text{high})\ , \\ &\alpha_2=0.5861g^{1/3}\ \ (2.9\%\ \text{high}) \end{aligned} \tag{6.32}$$

compared with the exact result $0.5695g^{1/3}$. We see that the lattice calculation is more accurate than either continuum scheme for the ground state. Remember that for the harmonic oscillator the lattice cutoff has exact extrapolants.[11]

VII. CONCLUSIONS

We have shown how to carry out continuum strong-coupling expansions for quantum field theory. In particular, we showed that it is necessary to regulate both the kinetic energy and the interaction vertices in order to have a finite theory. The Gaussian and θ-function schemes were investigated in detail for one-dimensional field theories. We found that the θ function is generally more accurate numerically and has the advantage of preserving certain formal continuum Green's-function identities. Neither is as accurate as the lattice cutoff at low orders. However, we take the existence of a well-behaved θ-function regulation scheme as an encouraging sign for the development of a continuum strong-coupling expansion for fermionic field theories which avoids species doubling.

*On leave from Los Alamos National Laboratory.

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