Higher-order terms in the strong-coupling expansion for the renormalized effective potential

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This is the third of three papers on the strong-coupling expansion of the renormalized effective potential in $g\phi^4$ quantum field theory in d-dimensional Euclidean space-time. We first assume that the renormalized strong-coupling expansion for the coefficients of the effective potential in the continuum field theory is a series in powers of M^{4-d}/g, where M is the renormalized mass and g is the bare coupling constant. The first term in this series was obtained in the previous paper. Here we estimate the next terms in the series. The results indicate that our assumption is unlikely to be true. Some alternatives are briefly discussed.

I. INTRODUCTION

In the previous two papers of this series we assumed that in the continuum $g\phi^4$ field theory the strong-coupling expansion for γ_{2n}, the mass-renormalized dimensionless scattering amplitudes at vanishing momentum on the external legs, is a series in powers of M^{4-d}/g. We then computed the first term in this expansion for γ_4, γ_6, and γ_8. In this paper we concentrate on the problem of computing the next three terms in the series for γ_4. Specifically, in Sec. II we write down the mass-renormalized series for γ_4 on the lattice and independently extrapolate each of the coefficients of the power series in M^{4-d}/g. We give extensive numerical results. We also calculate the critical exponent ν for each of these series. It does not appear that the dimensionless coupling constant is a power series in M^{4-d}/g for $d=1$ and 2, while our series are too noisy to conclude anything for $d=3$ and 4. In Sec. III, we discuss the possibility that the corrections to the $g\to\infty$ limit of γ_{2n} do not take the form of a power series in M^{4-d}/g. We show how the correction to the $g\to\infty$ limit could have the form $(M^{4-d}/g)^{\alpha}$, with $\alpha<1$, for example. In Sec. IV we assume that γ_4 is a power series in M^{4-d}/g. We discuss the β function and show that it can be written as a power series in $(\tilde{G}|_{g=\infty}-\tilde{G})$, where $\tilde{G}|_{g=\infty}$ is the dimensionless renormalized coupling constant at the fixed point $g=\infty$. This leads to the wrong slope for β at its zero.

II. HIGHER TERMS IN THE STRONG-COUPLING EXPANSION

In the first paper of this series[1] we showed how to develop the lattice strong-coupling expansion for the coefficients γ_{2n} of the effective potential. We expressed γ_{2n} as a double series in powers of the two variables $\alpha=-m^2a^{d-2}/g$ and $\delta=gm^{-4}a^{-d}$ [m is the bare mass, a the lattice spacing, g the bare coupling, and d the dimension of space-time; see (4.42) of Ref. 1]:

$$M^2a^2=\alpha^{-1}\sum_{k=0}^{N}\sum_{l=0}^{L}b_{kl}(d)\alpha^k\delta^l\,, \tag{2.1}$$

$$\gamma_{2n}=\alpha^{d/2-dn/2}\sum_{k=0}^{N}\sum_{l=0}^{L}b_{kl}^{(n)}(d)\alpha^k\delta^l\,. \tag{2.2}$$

In the above series N is the maximum number of lines in the diagrams contributing to γ_{2n} and L is the number of corrections to the Ising field theory.

In the second paper of this series[2] we showed for the purpose of mass renormalization how to solve for α and δ in terms of the renormalized mass M using (2.1). The general form of the answer is

$$\delta=x\sum_{k=0}^{N}\sum_{l=0}^{L}h_{kl}(d)y^kx^l \tag{2.3}$$

and

$$\alpha=y\sum_{k=0}^{N}\sum_{l=0}^{L}j_{kl}(d)y^kx^l\,, \tag{2.4}$$

where $x=M^{4-d}y^{-d/2}/g$ and $y=M^{-2}a^{-2}$. Below we list the first few terms in the series:

$$\delta = [1 + 4dy + (12d^2 - 12d + 4)y^2/3 - (180d^2 - 300d + 136)y^4/15 + (720d^3 - 1200d^2 + 544d)y^5/15]x$$
$$+ [-2 - 20dy + (-72d^2 + 20d - 8)y^2 + (-112d^3 + 104d^2 - 40d)y^3$$
$$- (2880d^4 - 5760d^3 + 1800d^2 + 3060d - 2056)y^4/45 + (2160d^3 - 4680d^2 + 3128d - 720)y^5/45]x^2$$
$$+ [-1 + (216d^2 + 16)y^2/3 + (1248d^3 - 432d^2 + 296d)y^3/3 + (15\,120d^4 - 14\,400d^3 + 8520d^2 - 960d - 92)y^4/15$$
$$+ (51\,840d^5 - 95\,040d^4 + 60\,960d^3 + 5040d^2 - 18\,728d + 3600)y^5/45]x^3 + \cdots \quad (2.5)$$

and

$$\alpha = y - 2dy^2 + (12d^2 + 6d - 2)y^3/3 + (-8d^3 - 12d^2 + 4d)y^4 + (80d^4 + 240d^3 - 20d^2 - 70d + 26)y^5/5$$
$$+ [y + 4dy^2 + (-4d + 2)y^3 + (8d^2 - 4d)y^4 - (720d^3 + 360d^2 - 1260d + 626)y^5/45]x$$
$$+ [2y + 12dy^2 + (72d^2 - 36d + 10)y^3/3 + (48d^3 - 72d^2 + 20d)y^4/3 + (-24d^2 + 48d - 22)y^5]x^2$$
$$+ [14y + 140dy^2 + (560d^2 - 140d + 46)y^3 + (1120d^3 - 840d^2 + 276d)y^4$$
$$+ (3360d^4 - 5040d^3 + 1656d^2 + 1256d - 802)y^5/3]x^3 + \cdots . \quad (2.6)$$

We then eliminated α and δ in (2.2) in favor of x and y. This gives a series for $\tilde{G}$, the coupling constant at zero external momentum, of the form

$$\tilde{G} = 24\gamma_4 = y^{-d/2} \sum_{k=0}^{N} \sum_{l=0}^{L} P_{kl}^{(4)}(d) y^k x^l . \quad (2.7)$$

To order y^5 and x^3 this series is

$$\tilde{G} = y^{-d/2}\{2 + 8dy + (8d^2 - 20d)y^2 + 32dy^3 + (-160d^2 + 60d)y^4 + (512d^3 + 208d^2 - 384d)y^5$$
$$+ [-2 - 16dy - (144d^2 - 120d + 8)y^2/3 - (192d^3 - 480d^2 + 256d)y^3/3$$
$$- (480d^4 - 2400d^3 + 1800d^2 + 640d - 272)y^4/15 - (36\,480d^2 - 37\,568d)y^5/45]x$$
$$+ [-2 - 24dy + (-120d^2 + 36d)y^2 - (960d^3 - 864d^2 + 128d)y^3/3$$
$$- (21\,600d^4 - 38\,880d^3 + 12\,600d^2 + 4860d - 304)y^4/45$$
$$- (5760d^5 - 17\,280d^4 + 9120d^3 + 6480d^2 - 3904d - 480)y^5/15]x^2$$
$$+ [-14 - 224dy - (4704d^2 - 1104d + 128)y^2/3 - (18\,816d^3 - 13\,248d^2 + 2944d)y^3/3$$
$$- (235\,200d^4 - 331\,200d^3 + 133\,280d^2 + 440d - 3736)y^4/15$$
$$- (376\,320d^5 - 883\,200d^4 + 579\,840d^3 + 15\,680d^2 - 85\,376d - 480)y^5/15]x^3 + \cdots\} . \quad (2.8)$$

To obtain the continuum limit of (2.7) or (2.8) we replace x by $y^{-d/2}M^{4-d}/g$ and assume that we can extrapolate the series in y multiplying each power of M^{4-d}/g to $y = \infty$. With this assumption the final continuum result is a series of the form

$$\tilde{G} = \tilde{G}(\infty) + \sum_{l=1}^{\infty} \tilde{G}_l (M^{4-d}/g)^l . \quad (2.9)$$

Note that we have already performed the extrapolation leading to $\tilde{G}(\infty)$ in Sec. III of Ref. 2.

The assumption that the renormalized strong-coupling series takes the rather natural form in (2.9) is motivated in part by the lowest-order continuum mean-field-theory result for $\tilde{G}$ in $g\phi^4$ field theory:

$$\tilde{G} = 2(g/M^{4-d})\left[1 + g \int \frac{d^dk}{(2\pi)^d}(k^2 + M^2)^{-2}\right]^{-1} .$$

When $d < 4$, the integral is proportional to M^{d-4}. Thus, $\tilde{G}$ to leading order in the mean-field approximation has a series expansion of the form (2.9). We begin by assuming that in going from the leading-order mean-field-theory result to the exact result the coefficients in (2.9) change but that the analytic structure remains the same.

This assumption is apparently incorrect in one dimension because of the effect of instantons. For example, in one dimension mean-field theory leads to an algebraic relation between the bare and renormalized masses; namely,

$$M^2 = m_0^2 + g \int \frac{dK}{(2\pi)(K^2 + M^2)} = m_0^2 + \frac{g}{2M} .$$

However, a semiclassical calculation which can be done using the notion of instantons gives an *exponentially* small mass gap as $g\to\infty$.[3] Specifically, if we take $g/|m|^3 \ll 1$,

$$\frac{M^2}{g^{2/3}}=\left(\frac{|m|^3}{g}\right)^{5/3}\left(\frac{64e}{\sqrt{2}\pi^2}\right)\exp\left(-\frac{4\sqrt{2}}{3}\frac{|m|^3}{g}\right).$$

The process of mass renormalization consists of eliminating m in favor of M using the above relation. Thus, we are not surprised to find logarithmic departures from the form of the series in (2.9). However, it may be that when $d\neq 1$ there are no such logarithmic terms and that the series in (2.9) is correct.

To better understand whether the extrapolation procedure gives a finite expression for $\tilde{G}_l$ in (2.9), we compute the critical exponent for the series in (2.8) multiplying $(M^{4-d}/g)^l$ before computing $\tilde{G}_l$. There are two methods for computing the critical exponent ν: (i) directly extrapolating to $y=\infty$ or (ii) conformally mapping the point at $y=\infty$ to 1 and then computing Padé approximants for the logarithmic derivative of the resulting series. Both of these methods are extensively discussed in Secs. III and IV of Ref. 2. We determine the critical indices ν_l by fitting the behavior of $\tilde{G}_l$ at small lattice spacing a (large y) by an algebraic form

$$\tilde{G}_l \sim (M^2a^2)^{\nu_l} \quad (a\to 0)\,. \qquad (2.10)$$

Since the series for $\tilde{G}_l$ already contains a term $y^{-(l+1)d/2}$, it is convenient to determine the index ρ_l of the power series in y without this term. That is, we assume that

$$\sum_{k=0}^{\infty} P_{kl}^{(4)}(d)y^k \sim y^{\rho_l} \quad (y\to\infty)\,, \qquad (2.11)$$

where the connection between the critical indices ρ_l and ν_l is

$$\nu_l=(l+1)d/2-\rho_l\,. \qquad (2.12)$$

Recall that the lattice expression for $\tilde{G}(\infty)$ had an index ρ_0 which extrapolated between the two straight lines $\rho=d/2$ and $\rho=2$. That is, the extrapolants to ρ_0 approach the line $\rho=d/2$ for $d<2.5$, which gives $\nu\equiv 0$, and for $d\geq 4$ they approach the line $\rho=2$, which gives $\nu=d/2-2$. Thus, for $d\leq 2.5$ $G(g)$ approaches a constant as $g\to\infty$, and for $d>4$, $\tilde{G}\to$ zero as $g\to\infty$. We will look for a similar behavior for the correction terms $\tilde{G}_l$.

Calculation of G_1 and its critical index ρ_1

The series for ρ_1 is obtained by taking the logarithmic derivatives of (2.11):

$$\rho_1=y\frac{d}{dy}\ln\sum_{k=0}^{\infty} P_{k1}^{(4)}(d)y^k\,. \qquad (2.13)$$

The first five terms in this series are

$$\begin{aligned}\rho_1=&8dy-(48d^2+120d-8)y^2/3+(32d^3+240d^2+96d)y^3\\&-(2880d^4+43\,200d^3+77\,040d^2-8640d+1792)y^4/45\\&+(1152d^5+28\,800d^4+117\,600d^3+34\,560d^2-17\,440d)y^5/9+\cdots\,.\end{aligned} \qquad (2.14)$$

As in Sec. IV of Ref. 2, we find that retaining the highest power of d for each power of y in (2.14) gives a simple geometric series. The sum of this series is

$$\rho_1(d\to\infty)=\lim_{d\to\infty}\frac{8dy}{1+2dy}=4\,. \qquad (2.15)$$

Thus, from (2.12)

$$\nu_1\sim d-4 \quad (d\to\infty)\,. \qquad (2.16)$$

If we apply the same reasoning as in Sec. IV of Ref. 2, we expect that $\tilde{G}_1$ vanishes for $d\geq 4$. [We conclude in Ref. 2 that $\tilde{G}(\infty)$ also vanishes for $d\geq 4$.]

Here are the first five extrapolants to ρ_1 evaluated at $y=\infty$ as functions of d:

$$\begin{aligned}(\rho_1)_1(\infty)&=\frac{24d^2}{6d^2+15d-1}\,,\\(\rho_1)_2(\infty)&=8d^2\left(\frac{3}{12d^4+141d^2-30d+1}\right)^{1/2},\\(\rho_1)_3(\infty)&=12d^2\left(\frac{20}{540d^6+2025d^4+25\,515d^3-12\,177d^2+1125d-25}\right)^{1/3},\\(\rho_1)_4(\infty)&=24d^2\left(\frac{1}{1296d^8+74\,520d^5+217\,323d^4-203\,220d^3+37\,242d^2-2100d+35}\right)^{1/4},\\(\rho_1)_5(\infty)&=4d^2\left(\frac{3024}{3024d^{10}-83\,160d^7+1\,275\,435d^6+1\,686\,573d^5-2\,766\,878d^4+874\,650d^3-91\,532d^2+3635d-49}\right)^{1/5}.\end{aligned} \qquad (2.17)$$

For all values of d, $0 \leqslant d \leqslant 4$, these five approximants lie above the line $\rho = d$ for $0 \leqslant d \lesssim 2.5$ and slightly below the line $\rho = d$ for $2.5 \leqslant d \leqslant 4$. They become level, $\rho = 4$, for $d > 4$, just as in Fig. 3 of Ref. 2 (see Fig. 1).

Next, we conformally map the series for ρ_1 and compute the $[2,2]$ (fifth-order) Padé extrapolant at 1. The result is

$$(\rho_1)_5[2,2] = \frac{(810\,000d^7 - 3\,866\,400d^6 + 10\,790\,100d^5 - 12\,085\,020d^4 + 4\,064\,100d^3 + 380\,940d^2 - 404d + 2840)}{(202\,500d^2 - 966\,600d^6 + 2\,697\,525d^5 - 996\,255d^4 - 2\,718\,750d^3 + 1\,984\,560d^2 - 143\,666d + 15\,904)} . \tag{2.18}$$

In Table I we compare the two extrapolation methods by evaluating $(\rho_1)_5(\infty)$ in (2.17) and $(\rho_1)_5[2,2]$ in (2.18) for various values of d. The close numerical agreement gives us confidence in the consistency of our approach.

In Table I we see that $\nu_1 = d - \rho_1$ is slightly negative for $d < 2$, which suggests that the extrapolants might diverge there. Such a divergence would indicate that the first correction to $\tilde{G}(\infty)$ goes to zero slower than M^{4-d}/g. This possibility is discussed in detail in Sec. III.

The analytic form for the first five extrapolants to $\tilde{G}_1$ at $y = \infty$ and also the $[3,2]$ Padé extrapolant for the conformally transformed series for $\tilde{G}_1$ evaluated at 1 are

$$\begin{aligned}
(\tilde{G}_1)_1(\infty) &= -2(8)^d ,\\
(\tilde{G}_1)_2(\infty) &= -2\left(\frac{-48d^2 + 264d + 8}{3d}\right)^{d/2} ,\\
(\tilde{G}_1)_3(\infty) &= -2\left(\frac{32d^3 - 336d^2 + 960d + 96}{d}\right)^{d/3} ,\\
(\tilde{G}_1)_4 &= -2\left(\frac{-2880d^5 + 41\,280d^4 - 238\,320d^3 + 452\,160d^2 + 101\,888d + 640}{45d^2}\right)^{d/4} ,\\
(\tilde{G}_1)_5 &= -2\left(\frac{1152d^6 - 19\,200d^5 + 165\,600d^4 - 670\,240d^3 + 884\,960d^2 + 391\,680d + 8000}{9d^2}\right)^{d/5} ,\\
(\tilde{G}_1)_5[3,2] &= -2[(1920d^{10} + 172\,800d^9 + 1\,563\,840d^8 + 8\,164\,800d^7 - 54\,093\,000d^6\\
&\quad + 75\,828\,000d^5 - 35\,921\,120d^4 + 3\,967\,680d^3 + 361\,568d^2 - 48\,000d\\
&\quad + 800)/(2160d^{10} + 131\,040d^9 - 118\,440d^8 + 505\,560d^7 + 4\,666\,305d^6\\
&\quad + 8\,281\,620d^5 - 5\,367\,420d^4 + 1\,455\,900d^3 - 232\,744d^2 + 8240d - 100)]^d .
\end{aligned} \tag{2.19}$$

The numerical values of the extrapolants to $\tilde{G}_1$ in (2.19) for various values of d are given in Table II. Observe that the extrapolants $(\tilde{G}_1)_n(\infty)$ appear to converge for increasing n when $d \lesssim 3.0$ and do not converge when $d \gtrsim 3.0$. The region of d where

TABLE I. Comparison of the two different fifth-order extrapolants to ρ_1 for various values of d. The numerical values of $(\rho_1)_5(\infty)$ and $(\rho_1)_5[2,2]$ obtained by two totally different extrapolation procedures are quite consistent.

Dimension d	$(\rho_1)_5(\infty)$	$(\rho_1)_5[2,2]$
$d=1$	1.2784	1.2798
$d=1.5$	1.7217	1.6442
$d=2$	2.0728	2.0369
$d=2.5$	2.4231	2.4154
$d=3$	2.7587	2.7632
$d=3.5$	3.0499	3.0664
$d=4$	3.2825	3.3154
$d=\infty$	4	4

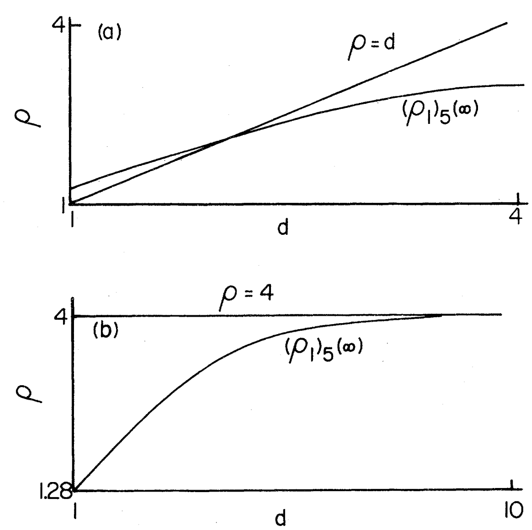

FIG. 1. Plot of $(\rho_1)_5(\infty)$ vs d. In (a) we compare $(\rho_1)_5(\infty)$ with $\rho = d$ for $1 \leqslant d \leqslant 4$. In (b) we compare $(\rho_1)_5(\infty)$ with $\rho = 4$ for $1 \leqslant d \leqslant 10$.

TABLE II. The extrapolants in (2.19) to $\tilde{G}_1$ evaluated for d ranging from 0 to 4. The [3,2] Padé extrapolant has a zero between $d=0$ and $d=1$ so it is not reliable until $d=1.5$. The approximants appear to converge well until $d=3$. For values of d larger than 3 the approximants do not converge with increasing order; this is consistent with the value of the critical exponent ν_1 being positive. A comparison with the second coefficient in (2.12) of Ref. 2 shows that the result for $d=0$ is exact.

Dimension d	$(\tilde{G}_1)_1(\infty)$	$(\tilde{G}_1)_2(\infty)$	$(\tilde{G}_1)_3(\infty)$	$(\tilde{G}_1)_4(\infty)$	$(\tilde{G}_1)_5(\infty)$	$(\tilde{G}_1)_5[3,2]$
0	−2	−2	−2	−2	−2	−2
0.5	−5.657	−6.078	−6.316	−6.848	−6.612	complex
1.0	−16.00	−17.28	−18.19	−18.85	−19.35	−2.912
1.5	−45.25	−46.15	−48.66	−50.43	−51.64	−59.20
2.0	−128.0	−114.7	−115.9	−122.5	−123.5	−131.6
2.5	−362.0	−259.7	−269.0	−262.5	−255.9	−259.6
3.0	−1024	−522.9	−544.0	−468.6	−435.8	−492.2
3.5	−2896	−897.3	−986.7	−581.2	−584.2	−869.2
4.0	−8192	−1217	−1622	−24.53	−879.8	−1443

the extrapolants converge (do not converge) is the same as the region where ν is near zero (positive). Note that $(\tilde{G}_1)_4(\infty)$ rises until it reaches a maximum at $d=3.4$ and then decreases to zero at $d=4.01$. This behavior is similar to that of several of the extrapolants to $\tilde{G}(\infty)$ (see Ref. 2, Fig. 1) and, as we will see, it also resembles the behavior of some of the extrapolants to $\tilde{G}_2$ and $\tilde{G}_3$.

Also observe that the convergence of the extrapolants for $\tilde{G}_1$ is not as good as the extrapolants for $\tilde{G}(\infty)$ in Table II of Ref. 2. We will see that the convergence of the extrapolants for $\tilde{G}_2$ and $\tilde{G}_3$ is even worse. To get the same accuracy for the higher-order corrections to G we would have had to compute more terms in the lattice strong-coupling expansion for γ_4.

For completeness, we have listed the series for ρ_2 and ρ_3 together with a brief analysis in Appendix A.

To understand the convergence of higher-order extrapolants, we restricted our attention to integral dimensions ($d=1,2,3,4$). Here we were able to use the tables of Kincaid, Baker, and Fullerton[4] to extract results for up to ten orders. The technique is described in Appendix B and the results are presented in Table III. These results are also a useful consistency check on our results for arbitrary dimension.

We see that the series for $\tilde{G}_1$ are much too noisy for $d=3$ or $d=4$ to conclude anything about convergence on the basis of only ten lines. The estimates for $\tilde{G}_1$ for $d=1,2$ do seem quite stable, although the nonzero value of ν_1 for $d=1$ makes the result for $d=1$ suspect.

Similarly for $\tilde{G}_2$, the series for $d=1,2$ do seem to be converging, while those for $d=3$ and 4 are very noisy. For $\tilde{G}_2$, however, ν_2 appears far from 0 both for $d=1$ and 2.

The negative values of ν_1 and ν_2 for $d=1$ and of ν_2 for $d=2$ suggest that the presumed expansion (2.9) is incorrect. For $d=1$, we have already argued why this is so. If however, we take the estimates in Table III seriously, we see that the estimates for $\tilde{G}_l$ grow rapidly with l, so that (2.9) is a series with zero radius of convergence, just like the renormalized weak-coupling[5] expansions, but unlike the unrenormalized strong-coupling expansion.[5] We have verified[6] that this is the case in zero dimension, where

$$G_l \sim -\frac{1}{\pi\sqrt{2e}}\,4^l\,\Gamma(l)\,.$$

Thus the series (2.9) for $d=0$ is not even Borel summable, because the coefficients do not alternate in sign.

III. POSSIBLE DEVIATION FROM POWER SERIES IN M^{4-d}/g BEHAVIOR

In Sec. II we saw that our numerical predictions for the critical exponents ν_n are negative for $d<2$ and we remarked that this might be a signal of the lack of power-series behavior. In this section we discuss the possibility that the lattice series for $\tilde{G}-\tilde{G}(\infty)$, which has the form

$$\tilde{G}-\tilde{G}(\infty)=\sum_{l=1}^{\infty} G_l(y)(M^{4-d}/g)^l\,, \tag{3.1}$$

might sum to a quantity which, as $a\to 0$, has the leading behavior $(M^{4-d}/g)^\alpha$.

Suppose (3.1) does sum as $y\to\infty (a\to 0)$ to $(M^{4-d}/g)^\alpha$. Then in the vicinity of the lattice limit (large y, small a) one might find, for example, that

$$\tilde{G}-\tilde{G}(\infty)\sim[M^{4-d}/g+(M^2a^2)^r]^\alpha \quad (a\to 0)\,. \tag{3.2}$$

However, prior to taking the lattice limit we treat a as a large parameter. Thus, on the lattice, the right side of (3.2) has the form

$$(M^2a^2)^{r\alpha}+\sum_{l=1}^{\infty}C_l(M^2a^2)^{r(\alpha-l)}(M^{4-d}/g)^l\,. \qquad (3.3)$$

This is a lattice series which for large y yields *negative* indices ν_l of the form

$$\nu_l = r\alpha - rl \quad (l>0)\,. \qquad (3.4)$$

At $d=1$ we find that the critical exponent ν_0 for the series for $\tilde{G}(\infty)$ is -0.008, which suggests that $\nu_0=0$. However the ν_l are negative for $l\geq 1$ and the extrapolants do not appear to be approaching zero. In particular, we find that

$$\nu_1=-0.289\,,\quad \nu_2=-1.30\,,\quad \nu_3=-2.09\,,\ldots\,. \qquad (3.5)$$

Those indices ν_l might well have a behavior similar to that in (3.4), namely a negative linear growth with l:

$$\nu_l=-bl+c\,,\quad l\geq 1\,. \qquad (3.6)$$

The behavior of ν_l in (3.6) is consistent with an *algebraic* dependence of $\tilde{G}-\tilde{G}(\infty)$ upon M^{4-d}/g. Assuming (3.6) (and taking $d=1$) we have for small Ma

$$\begin{aligned}\tilde{G}-\tilde{G}(\infty) &= \sum \tilde{G}_l(M^2a^2)(M^3/g)^l \\ &\sim \sum C_l(M^2a^2)^{\nu_l}(M^3/g)^l \\ &\sim \sum (M^2a^2)^c f[(M^2a^2)^{-b}M^3/g]\,. \end{aligned}\qquad (3.7)$$

We know that for $d<4$, $\tilde{G}$ is bounded as $g\rightarrow\infty$. Thus the only way for (3.7) to give a finite and nonzero result is that for small Ma

$$f[(M^2a^2)^{-b}M^3/g]\sim[(M^2a^2)^{-b}M^3/g]^{c/b}\,.$$

This shows that the first correction to $\tilde{G}-\tilde{G}(\infty)$ has the form

$$\tilde{G}-\tilde{G}(\infty)\sim(M^3/g)^{c/b}\,. \qquad (3.8)$$

It is clear that as long as our extrapolants give negative exponents, which do not converge to zero, then we must conclude that $\tilde{G}-\tilde{G}(\infty)$ is not a power series in M^{4-d}/g, but instead has an algebraic behavior like that in (3.5), and perhaps even a logarithmic behavior[3] like $1/\ln(g/M^3)$. This question can be settled only by calculating more terms in the series for the critical indices ν_l to see whether or not $\nu_l\rightarrow 0$, just as the first critical index ν_0 was very close to zero for $d\leq 2.5$.

IV. CALCULATION OF THE β FUNCTION

The Callan-Symanzik β function is defined as

$$\beta\equiv M\frac{\partial}{\partial M}\tilde{G}\bigg|_{g,a\text{ fixed}}\,. \qquad (4.1)$$

In our lattice expansion (2.8) $\tilde{G}$ is a function of y and $M^{4-d}/g=u$. Using the chain rule we find

$$\beta=M\frac{\partial}{\partial M}\tilde{G}\bigg|_{y\text{ fixed}}-2y\frac{\partial}{\partial y}\tilde{G}\bigg|_{u\text{ fixed}}\,. \qquad (4.2)$$

But, as $y\rightarrow\infty$, $\tilde{G}$ is finite. Thus, the second term in (4.2) must vanish as $a\rightarrow 0$. This implies that the order in which we perform the two operations, $a\rightarrow 0$ and $M\partial/\partial M$, is irrelevant. It is convenient to take the continuum limit first and apply $M\partial/\partial M$ to $\tilde{G}$ in (2.9) [here we assume the validity of (2.9)]. Doing this, we obtain

$$\beta=(4-d)\sum_{l=1}^{\infty}l\tilde{G}_l(M^{4-d}/g)^l\,. \qquad (4.3)$$

It remains to express β as a function of $\tilde{G}$. To do this we solve (2.9) to obtain M^{4-d}/g as a series in powers of $\tilde{G}(\infty)-\tilde{G}$:

$$M^{4-d}/g=-\frac{\tilde{G}(\infty)-\tilde{G}}{\tilde{G}_1}-\frac{\tilde{G}_2}{\tilde{G}_1^{\,3}}[\tilde{G}(\infty)-\tilde{G}]^2+\frac{\tilde{G}_3\tilde{G}_1-2\tilde{G}_2^{\,2}}{\tilde{G}_1^{\,5}}[\tilde{G}(\infty)-\tilde{G}]^3+\cdots\,. \qquad (4.4)$$

Substituting the right side of (3.4) into (3.3) gives β in terms of the renormalized coupling constant $\tilde{G}$:

$$\beta(\tilde{G})=(4-d)\left(-[\tilde{G}(\infty)-\tilde{G}]+\frac{\tilde{G}_2}{\tilde{G}_1^{\,2}}[\tilde{G}(\infty)-\tilde{G}]^2-[\tilde{G}(\infty)-\tilde{G}]^3\,\frac{2\tilde{G}_3\tilde{G}_1-2\tilde{G}_2^{\,2}}{\tilde{G}_1^{\,4}}+\cdots\right)\,. \qquad (4.5)$$

This implies that

$$\beta'(G(\infty))=4-d\,.$$

But weak-coupling renormalization-group calculations give[7]

$$\begin{aligned}\beta'[G(\infty)]&=1.3\pm 0.2\quad(d=2)\,,\\ &=0.79\pm 0.03\quad(d=3)\,.\end{aligned}$$

This is further argument against the validity of (2.9).

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APPENDIX A

In analogy with (2.14), the first five terms in the series for the critical index ρ_2 are

$$\rho_2 = 12dy - (24d^2+36d)y^2 + (48d^3+216d^2+64d)y^3 - (4320d^4+38\,880d^3+50\,040d^2-9720d+608)y^4/45 + (192d^5+2880d^4+8560d^3-240d^2-448d-80)y^5+\cdots . \quad \text{(A1)}$$

This series is summable for large d; the analogs of (2.15) and (2.16) are

$$\rho_2(d\to\infty) = \lim_{d\to\infty} \frac{12dy}{1+2dy} = 6 \quad \text{(A2)}$$

and

$$\rho_2 \sim 3d/2 - 6 \quad (d\to\infty). \quad \text{(A3)}$$

Here, as is the case with ν in (4.9) of Ref. 2 and ν_1 in (2.16), we see indications that ν_2 is positive for $d \geq 4$. This suggests that $\tilde{G}_2$ like $\tilde{G}(\infty)$ and $\tilde{G}_1$ vanishes for $d\geq 4$.

We list below the first five extrapolants to ρ_2 computed at $y=\infty$:

$$\begin{aligned}
(\rho_2)_1(\infty) &= \frac{12d}{2d+3}, \\
(\rho_2)_2(\infty) &= 12\left(\frac{3d}{12d^2+49}\right)^{1/2}, \\
(\rho_2)_3(\infty) &= 6\left(\frac{180d^4}{180d^4+945d^2+540d+76}\right)^{1/3}, \\
(\rho_2)_4(\infty) &= 12\left(\frac{27d^5}{432d^5+22\,680d^2-17\,447d+2544}\right)^{1/4}, \\
(\rho_2)_5(\infty) &= 6\left(\frac{3024d^6}{3024d^6+37\,800d^3+535\,395d^2-579\,264d+126\,488}\right)^{1/5}.
\end{aligned} \quad \text{(A4)}$$

Observe that these extrapolants are consistent with (A2) in that they approach 6 as $d\to\infty$.

The sequence of extrapolants in (A4) has the property that for $d \lesssim 3.0$ they all lie above and are apparently decreasing towards, the line $\rho = 3d/2$ (see Table III). However, to this order in perturbation theory ρ is not yet very close to $3d/2$, so ν_2 in (2.12) is not yet near zero. This suggests that the extrapolants to $\tilde{G}_2$ will converge slowly, or that the series is trying to have a different form than that suggested by mean-field theory.

In analogy with (2.14) and (A1), the first five terms in the series for the critical index ρ_3 are

$$\rho_3 = 16dy - (672d^2+1104d-128)y^2/21 + (448d^3+2209d^2+448d)y^3/7 - (282\,240d^4+2\,782\,080d^3+3\,470\,400d^2-725\,040d+197\,872)y^4/2205 + (112\,896d^5+1\,854\,720d^4+5\,596\,800d^3+620\,640d^2-412\,160d-5040)y^5/441+\cdots . \quad \text{(A5)}$$

Once again, the series for the critical exponent is summable for large d, the analog of (2.15) and (A2) being

$$\rho_3(d\to\infty) = \lim_{d\to\infty}\frac{16dy}{1+2dy} = 8 . \quad \text{(A6)}$$

Evidently, the formula for all n is

$$\lim_{d\to\infty} \rho_n = 2n+2 . \quad \text{(A7)}$$

Also, the general formula for ν_n is

$$\nu_n \sim (d-4)\frac{n+1}{2} \quad (d\to\infty). \quad \text{(A8)}$$

The first four extrapolants to ρ_3 are

TABLE III. Values of $\tilde{G}_1$, ν_1, $\tilde{G}_2$, and ν_2 up to ten internal lines for $d=1$, 2, 3, and 4. These numbers are calculated using method 1. The first number in each column is the extrapolation at $y=\infty$. The second number is the improved extrapolant (Ref. 8) evaluated at $y=n^2$, where n is the number of internal lines.

For $d=1$

Number of lines	$\tilde{G}_1$		$\nu_1=1-\rho_1$		$\tilde{G}_2$		$\nu_2=3/2-\rho_2$	
2	−16.00	−18.00			−45.25	−54.00		
3	−17.28	−17.75	−0.2000	−0.0435	−57.46	−59.44	−0.9000	−0.5000
4	−18.19	−18.37	−0.2443	−0.1962	−65.81	−66.58	−1.116	−1.008
5	−18.85	−18.94	−0.2667	−0.2432	−71.78	−72.16	−1.316	−1.228
6	−19.34	−19.40	−0.2761	−0.2623	−76.27	−76.49	−1.374	−1.318
7	−19.74	−19.77	−0.2799	−0.2708	−79.80	−79.93	−1.357	−1.322
8	−20.06	−20.08	−0.2823	−0.2759	−82.64	−82.73	−1.319	−1.297
9	−20.32	−20.34	−0.2853	−0.2805	−85.00	−85.06	−1.297	−1.281
10	−20.54	−20.56	−0.2893	−0.2855	−86.98	−87.03	−1.295	−1.283

For $d=2$

Number of lines	$\tilde{G}_1$		$\nu_1=2-\rho_1$		$\tilde{G}_2$		$\nu_2=3-\rho_2$	
2	−128.0	−162.0			−1024	−1458		
3	−114.7	−122.8	0.1887	−0.3729	−1222	−1327	−0.4286	0
4	−119.9	−123.4	−0.0994	−0.0228	−1493	−1539	−1.2207	−1.0058
5	−122.5	−124.4	−0.1435	−0.1053	−1696	−1721	−1.3015	−1.1931
6	−123.5	−124.7	−0.0604	−0.0425	−1854	−1870	−0.9690	−0.9271
7	−124.7	−125.5	−0.0370	−0.0263	−1984	−1994	−0.9299	−0.9047
8	−125.8	−126.4	−0.0702	−0.0619	−2093	−2100	−1.1089	−1.0850
9	−126.7	−127.1	−0.0914	−0.0848	−2186	−2191	−1.1480	−1.1289
10	−127.5	−127.8	−0.0773	−0.0725	−2266	−2270	−1.0039	−0.9932

For $d=3$

Number of lines	$\tilde{G}_1$		$\nu_1=3-\rho_1$		$\tilde{G}_2$		$\nu_2=9/2-\rho_2$	
2	−1024	−1458			−23 170	−39 370		
3	−522.9	−602.8	0.7959	0.9813	−17 160	−20 090	0.5000	−0.9000
4	−544.0	−579.6	0.3117	0.4001	−20 680	−22 020	−0.4764	−0.2727
5	−468.6	−489.4	0.1940	0.2423	−21 780	−22 530	−0.5276	−0.4287
6	−435.8	−449.9	0.2677	0.2920	−22 960	−23 440	−0.2860	−0.2416
7	−443.8	−453.5	0.2368	0.2543	−24 080	−24 410	−0.3433	−0.3129
8	−392.8	−401.2	0.1636	0.1775	−24 900	−25 130	−0.4330	−0.4090
9	−526.5	−530.6	−0.0385	−0.0208	−29 360	−29 520	imaginary	imaginary
10	−396.5	−402.0	−0.0595	−0.5341	−26 370	−26 510	−0.3512	−0.3449

For $d=4$

Number of lines	$\tilde{G}_1$		$\nu_1=4-\rho_1$		$\tilde{G}_2$		$\nu_2=6-\rho_2$	
2	−8192	−13 120			−524 300	−1 063 000		
3	−1217	−1 651	1.5226	1.7006	−128 000	−171 100	1.6364	2.0000
4	−1622	−1 840	0.9282	1.0165	−142 200	−161 700	0.6446	0.8209
5	−24.53	−144.3	0.7628	0.8130	−80 330	−89 210	0.6064	0.6904
6	−879.8	−954.1	0.7887	0.8165	−80 390	−86 170	0.7206	0.7636
7	−575.0	−635.3	0.6848	0.7064	−59 390	−63 420	0.6154	0.6463
8	imaginary	imaginary	0.5814	0.6002	−44 800	−47 980	0.5673	0.5903
9	−2608	−2614	0.6266	0.6392	−56 000	−58 360	0.5850	0.6015
10	imaginary	imaginary	0.6024	0.6127	−84 440	−85 980	0.3484	0.3669

$$(\rho_3)_1(\infty)=\frac{336d^2}{42d^2+69d-8},$$

$$(\rho_3)_2(\infty)=112d^2\frac{3}{(588d^4+2913d^2-1104d+64)^{1/2}},$$

$$(\rho_3)_3(\infty)=336d^2\left(\frac{5}{370\,440d^6+1\,950\,480d^4+3\,485\,565d^3-3\,684\,159d^2+662\,400d-25\,600}\right)^{1/3}, \tag{A9}$$

$$(\rho_2)_4(\infty)=48d^2\left(\frac{343}{444\,528d^8+15\,973\,200d^5-956\,007d^4-14\,840\,424d^3+6\,556\,128d^2-706\,560d+20\,480}\right)^{1/4}.$$

Notice that as $d\to\infty$ each of these extrapolants approaches the limiting value 8 which agrees with the exact result for ρ_3 in (A6).

APPENDIX B

In this appendix, we indicate how to use the tables of Kincaid, Baker, and Fullerton[4] to derive the series in the texts for integer dimension. These tables give the high-temperature series expansion coefficients for the susceptibility χ, second moment μ_2, and second derivative of the susceptibility with respect to an external field $\chi^{(2)}$, for any continuous-spin Ising model with an even spin-density function $F(s)$. The results are given in terms of the moments I_{2n} of the spin-density function

$$I_{2n}=\frac{\int_{-\infty}^{\infty}s^{2n}F(s)ds}{\int_{-\infty}^{\infty}F(s)ds}. \tag{B1}$$

The translation to our language for simple cubic lattices of dimension d is as follows. In Euclidean space, the propagator in momentum space is given by

$$G(p^2)\equiv\sum_{\vec{x}}e^{i\vec{p}\cdot\vec{x}}\langle s(0)s(\vec{x})\rangle$$

$$=\frac{Z_3}{(pa)^2+(Ma)^2}+O(p^4), \tag{B2}$$

where a is the lattice spacing. Clearly

$$Z_3^{-1}=\frac{1}{a^2}\frac{d^2}{dp^2}G^{-1}(p^2)\bigg|_{p=0}, \tag{B3}$$

$$M^2a^2=Z_3G^{-1}(0). \tag{B4}$$

The susceptibility χ is given by

$$\chi=\sum_{\vec{x}}\langle s(0)s(\vec{x})\rangle=G(0), \tag{B5}$$

while μ_2 is defined by

$$\mu_2=\sum_{\vec{x}}\left(\frac{\vec{x}}{a}\right)^2\langle s(0)s(\vec{x})\rangle=-\frac{1}{a^2}\nabla_p^2G(p^2)\bigg|_{p=0}. \tag{B6}$$

Observing that on a function of p^2

$$\nabla_p^2|_{p=0}=2d\frac{d}{dp^2}, \tag{B7}$$

we find that

$$Z_3^{-1}=\frac{\mu_2}{2d\chi^2} \tag{B8}$$

and

$$M^2a^2=\frac{2d\chi}{\mu_2}. \tag{B9}$$

The coupling constant g_R is defined by

$$g_Ra^{4-d}=Z_3^2[G^{-1}(0)]^4G_4(0), \tag{B10}$$

where

$$G_4(0)=\sum_{\vec{x}_1,\vec{x}_2,\vec{x}_3}\langle s(0)s(\vec{x}_1)s(\vec{x}_2)s(\vec{x}_3)\rangle_c \tag{B11}$$

$$=-\chi^{(2)} \tag{B12}$$

so that

$$g_Ra^{4-d}=-\frac{4d^2\chi^{(2)}}{\mu_2^2}. \tag{B13}$$

The dimensionless quantity $\tilde{G}$ is given by

$$\tilde{G}=\frac{g_R}{M^{4-d}}=\frac{-4d^2\chi^{(2)}}{\mu_2^2(Ma)^{4-d}}. \tag{B14}$$

To compute the moments of the spin-density function, recall that the interaction term in our Lagrangian is

$$\sum_i a^d\left(\frac{g\phi_i^4}{4}-\frac{g\mu^{d-2}\phi_i^2}{2}+\frac{d\phi_i^2}{a^2}\right), \tag{B15}$$

where ϕ_i is the field ϕ at lattice site i. The last term of (B15) comes from absorbing the "diagonal part" of the kinetic energy term into the local interaction.

We are interested in the large-g limit of

$$I_{2n}=(a^{d-2})^n\,\frac{\int_0^\infty \phi^{2n}\exp[-a^d(g\phi^4/4-g\mu^{d-2}\phi^2/2+d\phi^2/a^2)]d\phi}{\int_0^\infty \exp[-a^d(g\phi^4/4-g\mu^{d-2}\phi^2/2+d\phi^2/a^2)]d\phi}\,. \tag{B16}$$

For large g, each integral is dominated by its saddle point at

$$\phi^2=\phi_0{}^2\equiv\mu^{d-2}\,. \tag{B17}$$

Define a new integration variable u by

$$\phi=\phi_0(1+u\sqrt{\delta})\,, \tag{B18}$$

$$\delta=\frac{1}{ga^d\phi_0{}^4}\,. \tag{B19}$$

This gives

$$I_{2n}=\alpha^n\,\frac{\int du(1+u\sqrt{\delta})^{2n}\exp\left\{-\left[\frac{u^4\delta}{4}+u^3\sqrt{\delta}+u^2+d\alpha(2u\sqrt{\delta}+u^2\delta)\right]\right\}}{\int du\exp\left\{-\left[\frac{u^4\delta}{4}+u^3\sqrt{\delta}+u^2+d\alpha(2u\sqrt{\delta}+u^2\delta)\right]\right\}}\,, \tag{B20}$$

where

$$\alpha=(\mu a)^{d-2}\,. \tag{B21}$$

With these identifications, we can use the tables of Ref. 4, to generate the series (2.1), (2.2) and proceed as in the text.

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