Coherent-state representation of a non-Abelian charged quantum field

K.-E. Eriksson, N. Mukunda,* and B.-S. Skagerstam†
Institute of Theoretical Physics, S-412 96 Göteborg, Sweden
(Received 18 November 1980)

Quasicoherent states, previously defined for bosons with SU(2) gauge charge (and no other degree of freedom), are now defined for the general (field-theoretical) case. The coherent states thus constructed form a complete basis in the Fock space. They transform according to irreducible representations of the SU(2) group and are at the same time eigenstates of isosinglet pair and isosinglet three-particle annihilation operators. Some physical applications are indicated in the contexts of multiple particle production and gluon bremsstrahlung by a quark line in e^+e^- annihilation.

I. INTRODUCTION

In a previous paper two of the present authors discussed an extension of the concept of coherent states to a situation where a non-Abelian charge is involved.[1] For one degree of freedom a construction was given of *quasi-coherent states* transforming irreducibly under isotopic spin rotations. In the present paper we will extend the results in Ref. 1 to the field-theoretical case. Also some applications are discussed at the end of the paper. We use the term "quasicoherent states" rather than "generalized coherent states" since the latter term has already been used for a similar construction in the literature.[2]

As in Ref. 1 we will only consider the rotation group (the isospin group). A more general treatment of gauge groups such as SU(3) will be discussed in a forthcoming paper. For the reader's convenience we here briefly recall the conventional treatment of coherent states[3] for an isovector boson field with N kinematical states available. Reference 1 describes simply the case $N=1$. In the present paper we keep N finite only for convenience; clearly $N\to\infty$ (field theory) can be treated in the same way.

The Fock space for the isovector boson can be generated from a vacuum state $|0\rangle$ by creation operators $a^\dagger_{i\alpha}$, where $i=1,\ldots,N$. The corresponding annihilation operators are $a_{i\alpha}$. An orthonormal basis for the Fock space is

$$|\vec{n}_1,\ldots,\vec{n}_N\rangle=\prod_{i=1}^{N}[(n_{i1}!\,n_{i2}!\,n_{i3}!)^{-1/2}\times(a^\dagger_{i1})^{n_{i1}}(a^\dagger_{i2})^{n_{i2}}(a^\dagger_{i3})^{n_{i3}}]|0\rangle\ . \tag{1}$$

We introduce in the usual way the unitary operator

$$U(\underline{x})=e^{\underline{a}^\dagger\cdot\underline{x}-\underline{x}^*\cdot\underline{a}}\ , \tag{2}$$

where the $3N$-vector $\underline{x}$ describes a one-particle state and where $\underline{x}^*\cdot\underline{a}=x^*_{i\alpha}a_{i\alpha}$. Here i runs over 1 to N and α over 1 to 3. The conventional coherent state $|x\rangle$ is then given by[3]

$$|\underline{x}\rangle=U(\underline{x})|0\rangle=e^{-\underline{x}^*\cdot\underline{x}/2}e^{\underline{a}^\dagger\cdot\underline{x}}|0\rangle,\quad a_{i\alpha}|\underline{x}\rangle=x_{i\alpha}|\underline{x}\rangle\ . \tag{3}$$

The scalar product then is

$$\langle\underline{y}|\underline{x}\rangle=e^{-(\underline{x}^*-\underline{y}^*)\cdot(\underline{x}-\underline{y})/2}\,e^{(\underline{y}^*\cdot\underline{x}-\underline{x}^*\cdot\underline{y})/2}\ . \tag{4}$$

In particular

$$\langle\underline{x}|\underline{x}\rangle=1\ . \tag{5}$$

The coherent states form an overcomplete basis with the completeness relation

$$\int d^{6N}\underline{x}|\underline{x}\rangle\langle\underline{x}|=1\ ,\qquad d^{6N}\underline{x}=\prod_{i=1}^{N}\prod_{\alpha=1}^{3}\left(\frac{1}{\pi}\,d\,\mathrm{Re}\{x_{i\alpha}\}\,d\,\mathrm{Im}\{x_{i\alpha}\}\right)\ . \tag{6}$$

Given a $3N$-vector $\underline{x}$ we may define a matrix $J(\underline{x})$ serving as a "tensor of inertia" in the isospin space

$$J_{\alpha\beta}(\underline{x})=x^*_{i\alpha}x_{i\beta}\quad(\alpha,\beta=1,2,3)\ . \tag{7}$$

Then

$$J(\underline{x})^\dagger=J(\underline{x})\ ,\qquad \mathrm{Tr}[J(\underline{x})]=\underline{x}^*\cdot\underline{x}\ . \tag{8}$$

The tensor of inertia (7) will turn out to be useful in the normalization of a certain basis of quasicoherent states.

In the Fock space we can define the isospin rotation operators ($\hbar=1$)

$$I_\alpha=\frac{1}{i}\epsilon_{\alpha\beta\gamma}a^\dagger_{k\beta}a_{k\gamma} \tag{9}$$

which obey

$$[I_\alpha,I_\beta]=i\epsilon_{\alpha\beta\gamma}I_\gamma\ . \tag{10}$$

Below we will give a decomposition of the state $|\underline{x}\rangle$ in terms of a complete set of states (the quasicoherent states) which are diagonal in $\vec{I}^2$, I_3, and the isosinglet pair and three-particle annihilation operators

$$\omega_{ij} = \vec{a}_i \cdot \vec{a}_j \tag{11}$$

and

$$\Omega_{ijk} = \epsilon_{\alpha\beta\gamma} a_{i\alpha} a_{j\beta} a_{k\gamma} \ . \tag{12}$$

To obtain such a decomposition we make use of the fact that the Wigner $D^l_{\mu\nu}$ functions form a complete basis for the continuous functions on the rotation group.[4] For $R \in SO(3)$ we normalize the invariant group measure d^3R so that

$$\int_{SO(3)} d^3R = 1 \ . \tag{13}$$

The $D^l_{\mu\nu}$ functions satisfy (summation over repeated matrix indices is understood)

$$D^l_{\mu\nu}(R_1) D^l_{\nu\lambda}(R_2) = D^l_{\mu\lambda}(R_1R_2) \ , \tag{14}$$

$$D^l_{\mu\nu}(R)^* = D^l_{\nu\mu}(R^\sim) \ , \tag{15}$$

$$(2l+1)\int_{SO(3)} d^3R D^{l'}_{\mu'\nu'}(R)^* D^l_{\mu\nu}(R) = \delta_{l',l}\delta_{\mu',\mu}\delta_{\nu',\nu} \ , \tag{16}$$

$$\sum_{l=0}^{\infty} (2l+1) D^l_{\mu\nu}(R)^* D^l_{\mu\nu}(R') = \delta^3(R;R') \ , \tag{17}$$

where the group δ function satisfies

$$\int_{SO(3)} d^3R' f(R')\delta^3(R;R') = f(R) \ . \tag{18}$$

Now let M be an arbitrary 3×3 matrix and let R be a 3×3 SO(3) matrix ($R = R^*$; $R_{\alpha\gamma}R_{\beta\gamma} = \delta_{\alpha\beta}$). We shall need the SO(3) decomposition of exp $[\mathrm{Tr}(RM^\sim)]$.

Using completeness (17) and orthonormality (16) we find

$$e^{\mathrm{Tr}(RM^\sim)} = \sum_{l=0}^{\infty} (2l+1)\phi^l_{\mu\nu}(M) D^l_{\mu\nu}(R) \ , \tag{19}$$

where

$$\phi^l_{\mu\nu}(M) = \int_{SO(3)} d^3R \, D^l_{\mu\nu}(R)^* e^{M_{\alpha\beta}R_{\alpha\beta}} = D^l_{\mu\nu}\left(\frac{\partial}{\partial M^*}\right)^* I(M) \tag{20}$$

with the transformation property

$$\phi^l_{\mu\nu}(RMR') = D^l_{\mu\mu'}(R)^* \phi^l_{\mu'\nu'}(M) D^l_{\nu'\nu}(R')^* \ . \tag{21}$$

In (20) $I(M)$ is the integral

$$I(M) = \int_{SO(3)} d^3R \, e^{M_{\alpha\beta}R_{\alpha\beta}} \tag{22}$$

and the functions $D^l_{\mu\nu}(M)$ are defined as homogeneous polynomials of degree l in the matrix elements $M_{\alpha\beta}$ (see Appendix A),

$$D^l_{\mu\nu}(M) = \frac{2^l}{(2l+1)!} (t^l_\mu)^*_{\alpha_1\cdots\alpha_l} (t^l_\nu)_{\beta_1\cdots\beta_l} M_{\alpha_1\beta_1} \cdots M_{\alpha_l\beta_l} \ , \tag{23}$$

where $(t^l_\mu)_{\alpha_1\cdots\alpha_l}$ are the expansion coefficients of the spherical harmonics, defined in Ref. 1 as homogeneous polyomials of an arbitrary three-vector,

$$r^l Y^l_\mu(\theta,\varphi) = Y^l_\mu(\vec{r}) = \frac{1}{l!}(t^l_\mu)_{\alpha_1\cdots\alpha_l} r_{\alpha_1}\cdots r_{\alpha_l} \ , \tag{24}$$

where $\vec{r} = r(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$. From the definition (22) it immediately follows that $I(M)$ is invariant under left-handed as well as right-handed multiplication of M by SO(3) matrices

$$I(M) = I(R_1MR_2),\quad R_1, R_2 \in SO(3) \ . \tag{25}$$

Therefore $I(M)$ is a function only of the three invariants x, y, and z defined by

$$\begin{aligned} x &= \mathrm{Tr}(MM^\sim) \ , \\ y &= 4\,\mathrm{Det}M \ , \\ z &= \tfrac{1}{2}[\mathrm{Tr}(MM^\sim)]^2 - \mathrm{Tr}(MM^\sim MM^\sim) \ . \end{aligned} \tag{26}$$

In fact $I(M) \equiv I(x,y,z)$ can be expressed as a Fourier-Mellin integral,

$$I(M) = I(x,y,z) = \frac{1}{2\pi i}\int_{s_0-i\infty}^{s_0+i\infty} ds\, e^s (s^4 - 2xs^2 - 2ys - 2z)^{-1/2} \ , \tag{27}$$

where s_0 has to be chosen in such a way that all singularities of the integrand are on the left-hand side of the integration contour. The derivation of (27) is shown in Appendix B. From (27) we get the power-series expansion

$$I(x,y,z) = \sum_{j,k,m=0}^{\infty} \frac{[2(j+k+m)-1]!!}{j!\,k!\,m!\,(1+2j+3k+4m)!} x^j y^k z^m \ . \tag{28}$$

The mathematical preliminaries have been given here because $I(M)$ and $\phi^l_{\mu\nu}(M)$ defined in terms of $I(M)$ through (22) play an important role in the coherent-state formalism to be developed in the following sections.

In Sec. II we construct three types of quasi-coherent states and derive their properties. In Sec. III we construct states which are simultanously eigenstates of $\vec{I}^2$, I_3, and the number operator N. In Secs. IV and V we treat a few simple examples.

II. QUASICOHERENT STATES FOR SU(2) GAUGE FIELDS

Consider now an isospin rotation $R=(R_{\alpha\beta})$ $(\alpha,\beta=1,2,3)$ acting on a $3N$-vector

$$\underline{x}=(\vec{x}_1,\ldots,\vec{x}_N)\ . \tag{29}$$

The result is

$$R\underline{x}=(R\vec{x}_1,\ldots,R\vec{x}_N)\ . \tag{30}$$

From the corresponding coherent state $|R\underline{x}\rangle$ we can project out a certain irreducible representation by using the D functions,

$$|{}^{l}_{\mu\nu};\underline{x}\rangle=e^{\underline{x}^*\cdot\underline{x}/2}(2l+1)^{1/2}\int d^3R\,D^l_{\mu\nu}(R)^*|R\underline{x}\rangle \tag{31}$$

The effect of a rotation on such a state is easily found,

$$\begin{aligned}U(R)|{}^{l}_{\mu\nu};\underline{x}\rangle&=(2l+1)^{1/2}e^{\underline{x}^*\cdot\underline{x}/2}\int d^3R'\,D^l_{\mu\nu}(R')^*|RR'\underline{x}\rangle\\&=D^l_{\mu\mu'}(R^{-1})^*(2l+1)^{1/2}e^{\underline{x}^*\cdot\underline{x}/2}\\&\quad\times\int d^3R'\,D^l_{\mu'\nu}(RR')^*|RR'\underline{x}\rangle\\&=D^l_{\mu'\mu}(R)|{}^{l}_{\mu'\nu};\underline{x}\rangle\ .\end{aligned} \tag{32}$$

The state (31) we shall call a *quasicoherent* state. It is an eigenstate of $\vec{I}^2$ and I_3 as well as of the isosinglet annihilation operators ω_{ij} and Ω_{ijk} defined by (11) and (12),

$$\begin{aligned}\vec{I}^2|{}^{l}_{\mu\nu};\underline{x}\rangle&=l(l+1)|{}^{l}_{\mu\nu};\underline{x}\rangle\ ,\\ I_3|{}^{l}_{\mu\nu};\underline{x}\rangle&=\mu|{}^{l}_{\mu\nu};\underline{x}\rangle\ ,\end{aligned} \tag{33}$$

$$\begin{aligned}\omega_{ij}|{}^{l}_{\mu\nu};\underline{x}\rangle&=\vec{x}_i\cdot\vec{x}_j|{}^{l}_{\mu\nu};\underline{x}\rangle\ ,\\ \Omega_{ijk}|{}^{l}_{\mu\nu};\underline{x}\rangle&=\vec{x}_i\cdot(\vec{x}_j\times\vec{x}_k)|{}^{l}_{\mu\nu};\underline{x}\rangle\ .\end{aligned} \tag{34}$$

To get a more explicit expression for $|{}^{l}_{\mu\nu};\underline{x}\rangle$ we use (3) and (20) in (31),

$$\begin{aligned}|{}^{l}_{\mu\nu};\underline{x}\rangle&=(2l+1)^{1/2}\int d^3R\,D^l_{\mu\nu}(R)^*e^{\mathrm{Tr}[R(\vec{x}_i\vec{a}_i^\dagger)]}|0\rangle\\&=(2l+1)^{1/2}\phi^l_{\mu\nu}(\vec{a}_i^\dagger\vec{x}_i)|0\rangle\ ,\end{aligned} \tag{35}$$

where summation of i over the N degrees of freedom is understood in $\vec{a}_i^\dagger\vec{x}_i$ which denotes the matrix (of linear combinations of creation operators) $(a^\dagger_{i\alpha}x_{i\beta})$.

Using (35) and (3) we easily get the scalar product between a coherent state and a quasicoherent state,

$$\langle\underline{y}|{}^{l}_{\mu\nu};\underline{x}\rangle=(2l+1)^{1/2}e^{-\underline{y}^*\cdot\underline{y}/2}\phi^l_{\mu\nu}(\vec{y}_i^*\vec{x}_i). \tag{36}$$

Like the coherent states, the quasicoherent states form an overcomplete basis. Their scalar product is easily obtained from (31) and (36) with the use of (21) and (16),

$$\begin{aligned}\langle{}^{l'}_{\mu'\nu'};\underline{y}|{}^{l}_{\mu\nu};\underline{x}\rangle&=(2l+1)\int d^3R\,D^{l'}_{\mu'\nu'}(R)\,\phi^l_{\mu\nu}(R\vec{y}_i^*\vec{x}_i)\\&=(2l+1)\int d^3R\,D^{l'}_{\mu'\nu'}(R)D^l_{\mu\mu''}(R)^*\phi^l_{\mu''\nu}(\vec{y}_i^*\vec{x}_i)=\delta_{ll'}\delta_{\mu\mu'}\phi_{\nu\nu'}(\vec{y}_i^*\vec{x}_i)\ .\end{aligned} \tag{37}$$

Starting from (6) and using (17) and (31) we can decompose the unit operators as follows:

$$\begin{aligned}1&=\iint d^3R\,d^3R'\,\delta^3(R;R')\int d^{6N}\underline{x}\,|R\underline{x}\rangle\langle R'\underline{x}|\\&=\sum_{l=0}^{\infty}(2l+1)\int d^{6N}\underline{x}\int d^3R\,D^l_{\mu\nu}(R)^*|R\underline{x}\rangle\int d^3R'\,D^l_{\mu\nu}(R')\langle R'\underline{x}|=\sum_{l=0}^{\infty}\int d^{6N}\underline{x}\,e^{-\underline{x}^*\cdot\underline{x}}|{}^{l}_{\mu\nu};\underline{x}\rangle\langle{}^{l}_{\mu\nu};\underline{x}|\ .\end{aligned} \tag{38}$$

We can also introduce a somewhat less overcomplete basis by choosing $\nu=0$ in $|{}^{l}_{\mu\nu};\underline{x}\rangle$. With a suitable normalization we define a *reduced* type of *quasicoherent states*,

$$\begin{aligned}|l,\mu;\underline{x}\rangle&=[\varphi_l(J(\underline{x}))]^{-1/2}|{}^{l}_{\mu0};\underline{x}\rangle\\&=\left(\frac{2l+1}{\varphi_l(J(\underline{x}))}\right)^{1/2}\phi^l_{\mu0}(\vec{a}_i^\dagger\vec{x}_i)|0\rangle\ ,\end{aligned} \tag{39}$$

where

$$\varphi_l(M)=\phi^l_{00}(M)=\phi^l_{00}(M^-) \tag{40}$$

and $J(\underline{x})$ is the tensor of inertia defined in (7).

Using (37) we get the scalar product for reduced coherent states,

$$\langle l',\mu';\underline{y}|l,\mu;\underline{x}\rangle=\delta_{ll'}\delta_{\mu'\mu}\frac{\varphi_l(\vec{y}_i^*\vec{x}_i)}{[\varphi_l(J(\underline{x}))\varphi_l(J(\underline{y}))]^{1/2}}\ , \tag{41}$$

which for $\underline{y}=\underline{x}$ reduces to the normalization

$$\langle l',\mu';\underline{x}|l,\mu;\underline{x}\rangle=\delta_{l'l}\delta_{\mu'\mu}\ . \tag{42}$$

The states (35) may be expressed in terms of the new states (39),

$$|{}^{l}_{\mu\nu};\underline{x}\rangle=(2l+1)\int d^3R\,D^l_{0\nu}(R)^*[\varphi_l(RJ(\underline{x})R^{-})]^{1/2}|l,\mu;R\underline{x}\rangle. \quad (43)$$

The completeness relation corresponding to (38) is

$$1=\sum_{l=0}^{\infty}\int d^{6N}\underline{x}\,e^{-\underline{x}^*\cdot\underline{x}}\varphi_l(J(\underline{x}))\,|l,\mu;\underline{x}\rangle\langle l,\mu;\underline{x}|. \quad (44)$$

The quasicoherent states (39) are closely related to the generalized coherent states as defined in Ref. 1. They are also a generalization of the states used by Botke *et al.*[5] for the case of one-particle wave functions that are parallel in the isospin space. The transformation property (32) under rotations and the eigenstate properties (33) and (34) clearly also hold for the reduced states $|l,\mu;\underline{x}\rangle$ as a special case.

A further reduction can be made. The SO(3) characters

$$X_l(R)=D^l_{\mu\mu}(R) \quad (45)$$

satisfy the orthonormality relation

$$\int d^3R\,X_{l'}(R)X_l(R)=\delta_{l'l} \quad (46)$$

which follows from (16). The further reduced quasicoherent states

$$|l;\underline{x}\rangle=\left(\frac{2l+1}{\chi_l(J(\underline{x}))}\right)^{1/2}\int d^3R\,X_l(R)e^{\underline{x}^*\cdot\underline{x}/2}|R\underline{x}\rangle$$
$$=(2l+1)^{1/2}[\chi_l(J(\underline{x}))]^{-1/2}\int d^3R\,X_l(R)e^{\mathrm{Tr}[R\vec{x}_i\vec{a}_i^\dagger]}|0\rangle=(2l+1)^{1/2}[\chi_l(J(\underline{x}))]^{-1/2}\chi_l(\vec{a}_i^\dagger\vec{x}_i)|0\rangle,\quad \chi_l(M)=\phi^l_{\mu\mu}(M) \quad (47)$$

transform as follows under rotations:

$$U(R)|l;\underline{x}\rangle=\frac{e^{\underline{x}^*\cdot\underline{x}/2}}{[\chi_l(J(\underline{x}))]^{1/2}}\int d^3R'\,X_l(R')|RR'\underline{x}\rangle$$
$$=\frac{e^{\underline{x}^*\cdot\underline{x}/2}}{[\chi_l(J(\underline{x}))]^{1/2}}\int d^3R'\,X_l(RR'R^{-1})|RR'R^{-1}R\underline{x}\rangle$$
$$=|l;R\underline{x}\rangle. \quad (48)$$

It is also an eigenstate of $\vec{I}^2$,

$$\vec{I}^2|l;\underline{x}\rangle=l(l+1)|l;\underline{x}\rangle, \quad (49)$$

but not of I_3.

The scalar product is easily found to be

$$\langle l';\underline{y}|l;\underline{x}\rangle=\delta_{ll'}\frac{\chi_l(\vec{y}_i^*\vec{x}_i)}{[\chi_l(J(\underline{x}))\chi_l(J(\underline{y}))]^{1/2}}, \quad (50)$$

which for $\underline{y}=\underline{x}$ reduces to

$$\langle l';\underline{x}|l;\underline{x}\rangle=\delta_{ll'}. \quad (51)$$

The completeness relation reads

$$1=\sum_{l=0}^{\infty}(2l+1)\int d^{6N}\underline{x}e^{-\underline{x}^*\cdot\underline{x}}\chi_l(J(\underline{x}))\,|l;\underline{x}\rangle\langle l;\underline{x}|. \quad (52)$$

It is also interesting to compute the scalar product between a Fock state (non-normalized)

$$|i_1\alpha_1;\ldots;i_n\alpha_n\rangle=a^\dagger_{i_1\alpha_1}\cdots a^\dagger_{i_n\alpha_n}|0\rangle \quad (53)$$

and a reduced quasicoherent state. Using (39) and (35) we then obtain

$$\langle i_1\alpha_1;\ldots;i_n\alpha_n|l,\mu;\underline{x}\rangle$$
$$=\frac{1}{[\varphi_l(J(\underline{x}))]^{1/2}}\left({}^{l}_{\mu 0}\Big|{}^{1}_{\alpha_1\beta_1}\cdots{}^{1}_{\alpha_n\beta_n}\right)x_{i_1\beta_1}\cdots x_{i_n\beta_n}, \quad (54)$$

where

$$\left({}^{l}_{\mu\nu}\Big|{}^{1}_{\alpha_1\beta_1}\cdots{}^{1}_{\alpha_n\beta_n}\right)$$
$$=(2l+1)^{1/2}\int d^3R\,D^l_{\mu\nu}(R)^*\prod_{m=1}^{n}R_{\alpha_m\beta_m} \quad (55)$$

is a generalized Clebsch-Gordan coefficient. The probability of having exactly n bosons in a quasicoherent state can now be expressed in terms of (54). Let Π_n be the n-particle state projector,

$$\Pi_n=\frac{1}{n!}|i_1\alpha_1;\ldots;i_n\alpha_n\rangle\langle i_1\alpha_1;\ldots;i_n\alpha_n|. \quad (56)$$

Then the corresponding probability can be expressed as follows:

$$\langle l,\mu';\underline{x}|\Pi_n|l,\mu;\underline{x}\rangle=\frac{\delta_{\mu\mu'}}{n!\,\varphi_l(J(\underline{x}))}(2l+1)\int d^3R\,x^*_{i_1\alpha_1}R_{\alpha_1\beta_1}x_{i_1\beta_1}\cdots x^*_{i_n\alpha_n}R_{\alpha_n\beta_n}x_{i_n\beta_n}D^l_{00}(R)^*$$
$$=\frac{(2l+1)^{1/2}}{n!\,\varphi_l(J(\underline{x}))}\left({}^{l}_{00}\Big|{}^{1}_{\alpha_1\beta_1}\cdots{}^{1}_{\alpha_n\beta_n}\right)J_{\alpha_1\beta_1}(\underline{x})\cdots J_{\alpha_n\beta_n}(\underline{x}). \quad (57)$$

For a coherent state the number operator

$$N=\underline{a}^\dagger\cdot\underline{a} \quad (58)$$

has the expectation value

$$\langle \underline{x}^* | N | \underline{x} \rangle = \underline{x}^* \cdot \underline{x} . \tag{59}$$

For a reduced quasicoherent state the corresponding value is independent of μ and can be obtained as the mean value

$$\langle N \rangle_{l;\underline{x}} \equiv \frac{1}{2l+1} \langle l, \mu; \underline{x} | N | l, \mu; \underline{x} \rangle = e^{\underline{x}^* \cdot \underline{x}} [\varphi_l(J(\underline{x}))]^{-1} \int\int d^3R\, d^3R'\, D^l_{\mu 0}(R') D^l_{\mu 0}(R)^* \langle R'\underline{x} | N | R\underline{x} \rangle$$

$$= e^{\underline{x}^* \cdot \underline{x}} [\varphi_l(J(x))]^{-1} \int d^3R\, D^l_{00}(R) \langle R\underline{x} | N | \underline{x} \rangle = [\varphi_l(J(\underline{x}))]^{-1} \frac{\partial}{\partial \lambda} \int d^3R\, D^l_{00}(R) e^{\mathrm{Tr}(R J(\underline{x}))\lambda} \Big|_{\lambda=1} .$$

Thus, because of (40),

$$\langle N \rangle_{l;\underline{x}} = \frac{\partial}{\partial \lambda} \ln \varphi_l(\lambda J(\underline{x})) \Big|_{\lambda=1} . \tag{60}$$

Similarly we can determine $[(\Delta N)_{l;\underline{x}}]^2 = \langle N^2 \rangle_{l;\underline{x}} - (\langle N \rangle_{l;\underline{x}})^2$ with the result

$$[(\Delta N)_{l;\underline{x}}]^2 = \left(\frac{\partial^2}{\partial \lambda^2} + \frac{\partial}{\partial \lambda} \right) \ln \varphi_l(\lambda J(\underline{x})) \Big|_{\lambda=1} . \tag{61}$$

III. NUMBER-OPERATOR EIGENSTATES

Following the discussion in Ref. 1 we may also introduce states $|n, l\mu; x\rangle$ which are eigenstates of $\vec{\mathrm{I}}^2$, I_3, and the number operator (58):

$$|n, l\mu; \underline{x}\rangle = \left(\frac{2l+1}{\varphi_{l;n}(J(\underline{x}))} \right)^{1/2} \int_0^{2\pi} \frac{d\psi}{2\pi} e^{-(n+l)i\psi} \int d^3R\, D^l_{\mu 0}(R)^* e^{-\underline{x}^* \cdot \underline{x}/2} | R e^{i\psi} \underline{x} \rangle . \tag{62}$$

Here we have introduced the notation [see Eqs. (3), (20), and (22)]

$$\phi^{l;n}_{\mu\nu}(M) = \int_0^{2\pi} \frac{d\psi}{2\pi} e^{-(n+l)i\psi} \phi^l_{\mu\nu}(e^{i\psi} M) = D^l_{\mu\nu}\left(\frac{\partial}{\partial M^*} \right)^* I_{n+2l}(M) \tag{63}$$

and

$$\varphi_{l;n}(M) = \phi^{l;n}_{00}(M) = D^l_{00}\left(\frac{\partial}{\partial M^*} \right)^* I_{n+2l}(M) , \tag{64}$$

where

$$I_n(M) = \int_0^{2\pi} \frac{d\psi}{2\pi} e^{-ni\psi} I(e^{i\psi} M) . \tag{65}$$

Using (28) we obtain $I_n(M)$ explicitly as the following polynomial in the invariants (26):

$$I_{2n}(M) = \frac{x^n}{(2n+1)!} \sum_{r=0}^{[n/2]} \sum_{s=0}^{[(n-2r)/3]} \frac{[2(n-r-s)-1]!!}{(n-2r-3s)!\, r!\, (2s)!} \left(\frac{z}{x^2} \right)^r \left(\frac{y^2}{x^3} \right)^s ,$$

$$I_{2n+1}(M) = \frac{x^{n-1} y}{(2n+2)!} \sum_{r=0}^{[(n-1)/2]} \sum_{s=0}^{[(n-1-2r)/3]} \frac{[2(n-r-s)-1]!!}{(n-1-2r-3s)!\, r!\, (2s+1)!} \left(\frac{z}{x^2} \right)^r \left(\frac{y^2}{x^3} \right)^s . \tag{66}$$

The number operator eigenstates (62) satisfy

$$\vec{\mathrm{I}}^2 | n, l\mu; \underline{x} \rangle = l(l+1) | n, l\mu; \underline{x} \rangle ,$$

$$I_3 | n, l\mu; \underline{x} \rangle = \mu | n, l\mu; \underline{x} \rangle , \tag{67}$$

$$N | n, l\mu; \underline{x} \rangle = (2n+l) | n, l\mu; \underline{x} \rangle .$$

The scalar product is given by

$$\langle n', l'\mu'; \underline{y} | n, l\mu; \underline{x} \rangle = \frac{\varphi_{l;n}(\vec{y}_i^* \vec{x}_i)}{[\varphi_{l;n}(J(\underline{x})) \varphi_{l;n}(J(\underline{y}))]^{1/2}} \delta_{nn'} \delta_{ll'} \delta_{\mu\mu'} . \tag{68}$$

For $\underline{x} = \underline{y}$ this reduces to

$$\langle n', l'\mu'; \underline{x} | n, l\mu; \underline{x} \rangle = \delta_{nn'} \delta_{ll'} \delta_{\mu\mu'} . \tag{69}$$

The overlap of a state (62) with a coherent state

$|\underline{y}\rangle$ is

$$\langle \underline{y}| n, l\mu;\underline{x}\rangle = \left(\frac{2l+1}{\varphi_{l;n}(J(\underline{x}))}\right)^{1/2} e^{-\underline{y}^*\cdot\underline{y}/2}\,\phi^{l;n}_{\mu 0}(\vec{y}_i^*\vec{x}_i)\ . \tag{70}$$

A completeness relation can also be derived for the number operator eigenstates in a way similar to the derivation of (52). One obtains

$$1=\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\int d^{6N}\underline{x}\ \varphi_{l;n}(J(\underline{x}))\,|n,l\mu;\underline{x}\rangle\langle n,l\mu;\underline{x}|\ . \tag{71}$$

IV. QUASICOHERENT STATES AND PARTICLE PRODUCTION

In discussions of multiple particle production one usually considers matrix elements of the scattering operator between Fock-space states. At high energies, when the number of produced particles is large, it may be useful instead to consider the corresponding matrix elements in terms of states which are not eigenstates of the number operator. In quantum electrodynamics (QED) it is well known (see Refs. 3 and 6) that coherent states are extremely useful for studying problems where the number of photons involved is large (or infinite). In physical processes of pion production at high energies a coherent-state basis may analogously be appropriate.[8] These states will not, however, have a definite isospin content. As suggested by Botke *et al.*[5] one could therefore generalize the concept of coherent states in order to get states transforming irreducibly under isospin transformations. Such an extension was indeed given by Botke *et al.*[5] We have now generalized their construction to a superposition of one-particle states which no longer have to be parallel in the isospin space. The quasicoherent states $|{}^{l}_{\mu\nu};\underline{x}\rangle$ (or $|l,m;\underline{x}\rangle$) which we are using constitute a complete basis in the Fock space. The model calculations of Botke *et al.*[5] could now be reconsidered in terms of quasicoherent states but we will not develop on this point further here. Here we notice that the number-operator eigenstate construction in Sec. III extends the isospin reduction of Ref. 8 to general n-pion states. It is of general interest, however, to consider in detail some properties of our quasicoherent states. For simplicity we will restrict ourselves to the isospin [SU(2)] singlet state but the extension to other representations is rather straightforward. We denote the singlet state by $|0;\underline{f}\rangle$. In the isosinglet case there is no difference between the quasicoherent states given by (35), (39), or (47). Using (35) and (40) we now get

$$|0;\underline{f}\rangle = \frac{e^{\underline{f}^*\cdot\underline{f}/2}}{[\varphi_0(\vec{f}_k^*\vec{f}_k)]^{1/2}}\int d^3R\,|R\underline{f}\rangle . \tag{72}$$

We shall now consider some specific choices of one-particle states. As our *first* example, let $\vec{n}$ denote a unit vector in isospin space and let $\underline{f}=\vec{n}f(\vec{k})$ be the one-particle state as a function of momentum. This choice of one-particle states corresponds to the construction given by Botke *et al.*[5] and to what is called identical pions in Ref. 8. The matrix J defined in (7) then takes the form

$$J_{\alpha\beta}=n_\alpha n_\beta c,\quad c=\int\frac{d^3k}{2\omega}f(\vec{k})^*f(\vec{k}), \tag{73}$$

where ω is the energy of the particle under consideration. The matrix $J_{\alpha\beta}$ given by (73) can easily be diagonalized. The group invariant $\varphi_0(J)$ can therefore be computed in closed form,

$$\varphi_0(J)=\frac{\sinh c}{c}\equiv\varphi_0(c)\ . \tag{74}$$

Using (60) we find the expectation value of the number operator

$$N_1\equiv\langle 0;\underline{f}|N|0;\underline{f}\rangle = c\coth c-1\ . \tag{75}$$

For a coherent state $|\underline{f}\rangle$ we would obtain [see Eq. (59)]

$$\langle\underline{f}|N|\underline{f}\rangle = c\ . \tag{76}$$

Comparison of (75) and (76) then leads to the conclusion that quasicoherent states are "less condensated" than the coherent states (see Fig. 1).

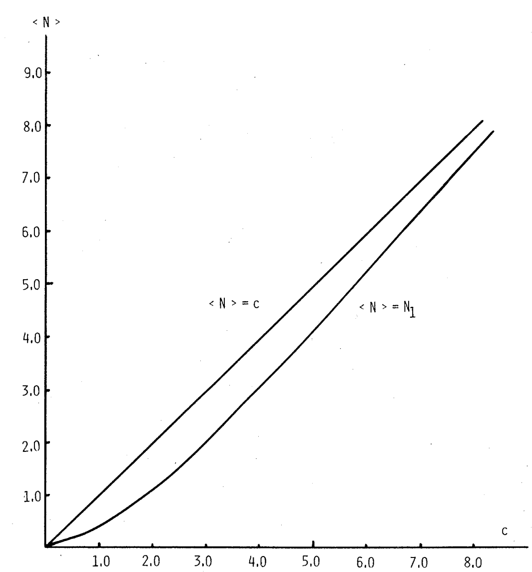

FIG. 1. The mean value of the number operator N in the state $\vec{n}f(\vec{k})$ as a function of c [Eq. (73)] for a coherent state and for a quasicoherent state. The two mean values approach the same value for large c.

This is an expected result.

The probability amplitude for finding n pions with momenta $\vec{k}_1, \ldots, \vec{k}_n$ and isospin indices $\mu_1, \ldots, \mu_n$ can be computed from the formula (57). If we let n_+, n_-, and n_0 denote the numbers of π^+, π^-, and π^0 particles, respectively, we find that $n_+ = n_-$ and that n_0 is even. The corresponding probability $P(n_+, n_-, n_0)$ is

$$P(n_+, n_-, n_0) = \frac{c^n}{\varphi_0(c)} \frac{1}{n_0!} \left[\frac{(n_0-1)!!}{(n+1)!!}\right]^2, \quad (77)$$

where $n = n_+ + n_- + n_0$ is the total number of pions. We notice that (74), (75), and (77) do not depend on the direction of the unit vector $\vec{n}$. This fact is obvious when we realize that $|0;R\underline{f}\rangle = |0;\underline{f}\rangle$ $R \in SO(3)$, according to the definition (72).

As our *second* example of one-particle states we consider the following form of the matrix J:

$$J_{\alpha\beta} = \int \frac{d^3k}{2\omega} f_\alpha(\vec{k})^* f_\beta(\vec{k}) = c\delta_{\alpha\beta} . \quad (78)$$

In this case it is convenient to use the integral representation (27) in order to compute the group invariant $\varphi_0(J)$. The corresponding inverse Laplace transform is elementary and the result is

$$\varphi_0(J) = e^c [I_0(2c) - I_1(2c)], \quad (79)$$

where $I_n(\ )$ is the nth-order modified Bessel function. The expectation value (60) of the number operator is in this case

$$N_2 \equiv \langle 0;\underline{f}|N|0;\underline{f}\rangle = c\, \frac{I_1(2c) - I_2(2c)}{I_0(2c) - I_1(2c)} . \quad (80)$$

For the corresponding coherent state $|\underline{f}\rangle$ we easily obtain

$$\langle \underline{f}|N|\underline{f}\rangle = 3c . \quad (81)$$

We find once again that the quasicoherent state $|0;\underline{f}\rangle$ contains less particles than the coherent state $|\underline{f}\rangle$ (see Fig. 2). We notice, however, that $N_2 > N_1$ (if $c \neq 0$) i.e., the orientation in isospin space of the one-particle state is essential for the physical properties of the quasicoherent states. The probability of finding n_i π^i mesons ($i = -0, +$) can be computed by integrating (57) over the momenta $\vec{k}_1, \ldots, \vec{k}_n$ ($n = n_+ + n_- + n_0$). We find that $n_+ = n_-$. The result of the computation is

$$P(n_+, n_-, n_0) = \frac{\delta_{n_+ n_-}}{\varphi_0(J)} \frac{2^{-2n_+-1} c^n}{(n_+!)^2 n_0!} \int_{-1}^{1} dx\,(1+x)^{2n_+} x^{n_0}, \quad (82)$$

where $\varphi_0(J)$ is given by (79). We notice that integrals of the form (82) occur in the statistical approach 9 to multiple particle production when isospin conservation is imposed.

V. GLUON BREMSSTRAHLUNG FROM A "CLASSICAL" QUARK LINE

We have constructed the analog of coherent states for [SU(2)] non-Abelian gauge fields. The corresponding construction simplifies if one considers fields carrying an Abelian charge. We refer the reader to Refs. 10 and 7 for a discussion of the one-particle and field-theoretical situations, respectively. The corresponding quasicoherent states can now be used, e.g., in a study[7] of soft emission of charged massless bosons in a scattering process. If one neglects self-interactions among the soft bosons and the quantum structure of their source, then the infrared divergences exponentiate and can be treated as in QED. Examples of the relevant Feynman diagrams are shown in Fig. 3. The emission of self-interacting bosons can, in principle, be investigated in detail by making use of functional techniques. A closed expression for the probability of soft boson emission up to a certain total energy can be written down[11] (also taking the quantum nature of the corresponding source into account). Below we will show that, under the assumptions mentioned above, the emission of soft [SU(2)] gluons from a classical quark current exponentiates in exactly the same way as in QED. Classical quark currents have been found to be useful in the study of the infrared structure of non-Abelian gauge theories[12] as well as in the study of quark and gluon jets in quantum chromodynamics (QCD),[13,14] Frautschi and Krzywicki have discussed the effect of con-

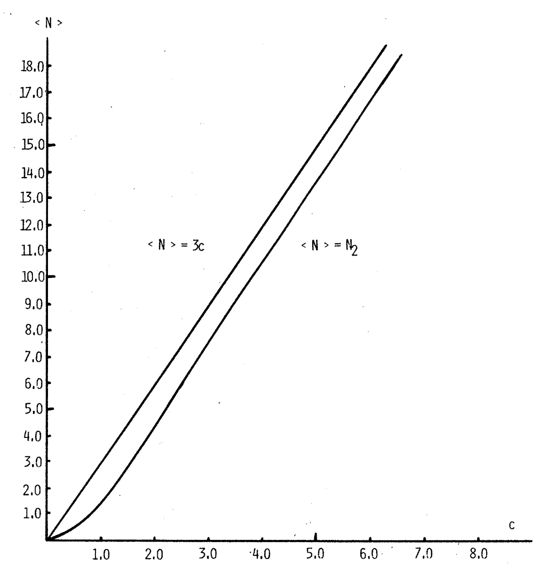

FIG. 2. The mean value of the number operator N is shown as a function of c in the state $f_\alpha(\vec{k})$ satisfying (78) for a coherent state and for a quasicoherent state. The two mean values approach the same value for large c.

finement on gluon bremsstrahlung in e^+e^- annihilation. They argued that the confinement mechanism in QCD may provide a natural infrared cutoff for the bremsstrahlung spectrum. In their analysis the gluon radiation emitted by a quark line is estimated by considering soft emission of photons from a classical current. As is well known, the infrared divergences in QED can be treated in terms of such a classical current.[6] The radiation field is then effectively described by a coherent state. In QCD the situation with regard to soft gluon emission is more complicated due to the self-coupling of the gluons and the color content of the sources. However, it has been shown[15] that properly defined transition probabilities are infrared finite order by order in a renormalized coupling constant. It is possible to develop a classical theory of Yang-Mills particles[16] interacting with non-Abelian fields.[17] The color content of a given particle can, formally, be described in terms of c numbers which after quantization are replaced by the Lie-algebra generators of the gauge group under consideration [SU(2) in our case].

Under the simplifying assumptions mentioned above, we can use the picture suggested by Frautschi and Krzywicki[14] to compute the probability for emission of soft gluons from a classical current,

$$j_\mu^\alpha(\vec{x},t)=j_\mu(\vec{x},t)I^\alpha\,, \tag{83}$$

where

$$\begin{aligned} j_0(\vec{x},t)&=g\delta^3(\vec{x}-\vec{v}t)\,,\\ \vec{j}(\vec{x},t)&=g\vec{v}\,\delta^3(\vec{x}-\vec{v}t) \end{aligned} \tag{84}$$

is the conventional form of a classical charged current ($\vec{v}$ is the velocity of the particle) and I^α denotes the classical color degrees of freedom. In general I^α will be time dependent and precess around the gluon field. For the case when the coherent radiation field generated by the "effective current" (83) is proportional to I^α, than we may consider I^α as effectively time independent. Let us consider the (perturbative) vacuum as the initial state. The final state of the soft gluons emitted will then be a coherent state. We have, however, to take color conservation into account. The final state will therefore be a quasicoherent isosinglet state

$$|0;\underline{j}\rangle=\frac{e^{\underline{j}^*\cdot\underline{j}/2}}{\varphi_0(\vec{j}_k^*\vec{j}_k)}\int d^3R\,|R\underline{j}\rangle\,, \tag{85}$$

where $\underline{j}$ is the classical current (83). Following the procedure of Ref. 7 we now compute the transition probability for the source (83) to emit soft gluons with a total energy not exceeding ΔE

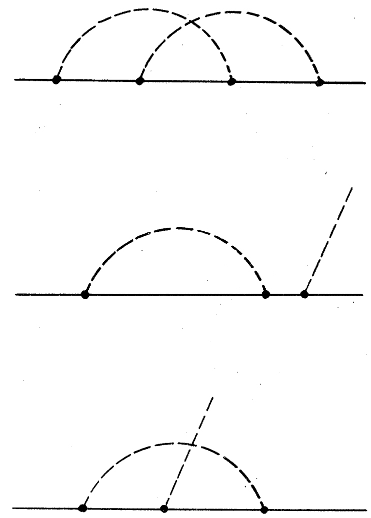

FIG. 3. Some typical Feynman diagrams contributing to the soft gluon emission from a classical quark current are exhibited. The vertices commute due to the classical nature of the current, and no gluon self-interactions are taken into account.

(other energy intervals can be treated analogously). The density operator describing the corresponding energy resolution is given by

$$\rho(\Delta E)=\int d^4p\,\theta(\Delta E-p_0)\delta^4(P-p)\,, \tag{86}$$

where P stands for the momentum operator and θ is the Heaviside step function. We observe that (85) is of the form described by the matrix (73). The transition probability $P(\Delta E)$ can then be evaluated with the result

$$\begin{aligned} P(\Delta E)&=\langle 0;\underline{j}\,|\,\rho(\Delta E)\,|\,0;\underline{j}\rangle\\ &=\frac{1}{2\pi}\int_0^{\Delta E}dp_0\int_{-\infty}^{\infty}dt\,e^{-ip_0t}\frac{\varphi_0(c(t))}{\varphi_0(c)}\,, \end{aligned} \tag{87}$$

where

$$c(t)=\vec{I}^2\int\frac{d^3k}{2\omega}e^{i\omega t}j_\mu(\vec{k})^*j_\mu(\vec{k}) \tag{88}$$

and

$$c=\vec{I}^2\int\frac{d^3k}{2\omega}j_\mu(\vec{k})^*j_\mu(\vec{k})\,. \tag{89}$$

In the analysis by Frautschi and Krzywicki, (88) and (89) are finite integrals (with an appropriate ultraviolet cutoff) due to the confinement mechanism. It is instructive to compare (87) with the corresponding result in QED (see, e.g., Ref. 7 and references cited therein). Then (88) and (89) are infrared divergent. By making use of (74) and its asymptotic form we then obtain

$$P(\Delta E)=\frac{1}{2\pi}\int_0^{\Delta E}dp_0\int_{-\infty}^{\infty}dt\,e^{-ip_0t}\exp\left[\vec{\mathrm{I}}^2\int\frac{d^3k}{2\omega}(e^{i\omega t}-1)j_\mu(\vec{k})^*j^\mu(\vec{k})\right] \tag{90}$$

which is finite and of the same form as the corresponding expression in QED.

In a more refined analysis one could (in the leading logarithm approximation) take self-interactions into account by replacing the coupling constant g in (84) by a running coupling constant as in Refs. 12 and 13.

VI. FINAL REMARKS

In Ref. 7 one of the present authors showed how quasicoherent states can be constructed in Abelian field theory where a conserved Abelian charge is present. Owing to superselection rules conventional coherent states are not appropriate as was noted by Bhaumik *et al.*[10] Here we have demonstrated a similar construction for [SU(2)] non-Abelian charges and for one-particle states (gauge bosons or pions) transforming under the adjoint representation of SU(2).

This is also a generalization of Ref. 1 which deals with the case of one single available kinematical state, and our construction also extends the work by Botke *et al.*[5]

From our presentation of the construction of quasicoherent states it is clear that our results can be extended to any compact group. Work on this extension is in progress.[18]

The construction in Sec. II of quasicoherent states can also easily be carried over to the case when the one-particle states transform according to the fundamental representation of SU(2) (appropriate for K mesons or in general two-level systems.[19,20] The corresponding complete set of states is by definition

$$|{}^{\,l}_{\mu\nu};\underline{x}\rangle=(2l+1)^{1/2}e^{\underline{x}^*\cdot\underline{x}/2}\int dg\,D^l_{\mu\nu}(g)^*\,|g\underline{x}\rangle\,. \tag{91}$$

The isospinor integral that corresponds to $I(M)$ in (22) and (28) for isovectors now depends only on one variable.[21] (See Appendix B.) Most of the results in Secs. II and III can now easily be carried over to the set of states given by (91).

We expect that quasicoherent states constructed in the present paper also may be a useful complete set of states with regard to physical applications. In Secs. IV and V we have indicated some properties of these states as well as some physical situations where they describe relevant properties of the system under consideration. The method can clearly be applied also to more complex situations.

With regard to the study of gluon condensates[22] and quark condensation[23] in QCD, quasicoherent states may simplify the analysis [when extended to the SU(3) case[18]]. We intend to study these questions elsewhere. Finally, it is amusing to notice that invariant integrals of the form (22) or (B9) frequently occur in the analysis of gauge-field theories in the Wilson lattice approach.[21]

ACKNOWLEDGMENTS

The authors wish to thank A. P. Balachandran for valuable discussions. One of the authors (B. S. S.) wishes also to thank R. Stora for several constructive remarks and discussions.

APPENDIX A

In this appendix we shall show how $D^l_{\mu\nu}(M)$ as homogeneous polynomials (23) are obtained.

Let

$$U(R)=e^{-i\vec{\mathrm{I}}\cdot\vec{\theta}} \tag{A1}$$

be a unitary rotation operator. The Winger D functions are defined by

$$D^l_{\mu\nu}(R)=\langle l\mu\,|U(R)|\,l\nu\rangle\,. \tag{A2}$$

Then[1]

$$Y^l_\mu(R\vec{e})=D^l_{\mu\nu}(R)^*Y^l_\nu(\vec{e})\,, \tag{A3}$$

i.e.,

$$D^l_{\mu\nu}(R)=\frac{1}{4\pi}\int d\Omega(\vec{e})Y^l_\mu(R\vec{e})^*Y^l_\nu(\vec{e})\,. \tag{A4}$$

Since $Y^l_\mu(e)$ is a homogeneous polynomial in $\vec{e}$ by equation (24) we obtain

$$D^l_{\mu\nu}(R)=\frac{1}{(l!)^2}(t^l_\mu)^*_{\alpha_1\cdots\alpha_l}(t^l_\nu)_{\beta_1\cdots\beta_l}R_{\alpha_1\alpha_1'}\cdots R_{\alpha_l\alpha_l'}\times\frac{1}{4\pi}\int d\Omega(\vec{e})e_{\alpha_1'}\ldots e_{\alpha_l'}e_{\beta_1}\ldots e_{\beta_l}\,. \tag{A5}$$

The integral in equation (A5) can easily be evaluated and the result is

$$D^l_{\mu\nu}(R)=\frac{2^l}{(2l+1)!}(t^l_\mu)^*_{\alpha_1\ldots\alpha_l}(t^l_\nu)_{\beta_1\cdots\beta_l}R_{\alpha_1\beta_1}\cdots R_{\alpha_l\beta_l}\,, \tag{A6}$$

which can now be extended to be valid for *any* 3×3 matrix.

APPENDIX B

In this appendix we shall derive an explicit expression for the invariant integral

$$I(M)=\int_{SO(3)}d^3R\,e^{\mathrm{Tr}(RM^\sim)} \tag{B1}$$

in terms of the three invariants

$$x = \mathrm{Tr}(MM^{\sim}), \quad y = 4\,\mathrm{Det}M\,,$$
$$z = \tfrac{1}{2}[\mathrm{Tr}(MM^{\sim})]^2 - \mathrm{Tr}(MM^{\sim}MM^{\sim})\,. \tag{B2}$$

Now $I(M)$ is an analytic function of M. We can therefore assume that M is real and *generic*, i.e., M is such that x, y, z are all nonzero. We therefore consider the case when M is a diagonal matrix:

$$M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}. \tag{B3}$$

SO(3) is the adjoint group of SU(2). Since, topologically, SU(2)$\approx S^3$ we can therefore rewrite the invariant integral (B1) as follows:

$$I(M) = \pi^{-2}\int d^4u\,\delta(u^2-1)\exp\left[\tfrac{1}{2}\sum_{i=1}^{3} m_i \mathrm{Tr}(\vec{\sigma}\cdot\vec{e}_i u\vec{\sigma}\cdot\vec{e}_i u^\dagger)\right]. \tag{B4}$$

Here $\{\vec{e}_i\}$ is an orthonormal basis in R^3 and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the three Pauli matrices; u is a general element of SU(2), i.e., $u = u_0 \cdot 1 + i\vec{\sigma}\cdot\vec{u}$ and $u^2 = u_0^2 + \vec{u}^2$. For the δ function we use the integral representation

$$\delta(u^2-1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{i\xi(u^2-1)}\,. \tag{B5}$$

The u integral in (B4) is then a Gaussian integral which is easily evaluated with the result $(s = i\xi)$

$$I(M) = \frac{1}{2\pi i}\int_{s_0-i\infty}^{s_0+i\infty} ds\, e^s f(s; m_1, m_2, m_3)\,, \tag{B6}$$

where

$$f(s; m_1, m_2, m_3) = [s^4 - 2(m_1^2 + m_2^2 + m_3^2)s^2 - 8m_1m_2m_3 s + m_1^4 + m_2^4 + m_3^4 - 2(m_2^2m_3^2 + m_3^2m_1^2 + m_1^2m_2^2)]^{-1/2} \tag{B7}$$

and s_0 has to be chosen in such a way that all singularities of $f(s, m_1, m_2, m_3)$ are to the left of the integration contour. Inserting (B7) into (B6) and expressing the polynomials in m_1, m_2, m_3 in terms of the invariants (B2) for the matrix (B3) we finally obtain

$$I(M) = \frac{1}{2\pi i}\int_{s_0-i\infty}^{s_0+i\infty} ds\, e^s(s^4 - 2xs^2 - 2ys - 2z)^{-1/2}\,. \tag{B8}$$

By analyticity (B8) is true for any 3×3 matrix M. Expanding (B8) in terms of x/s^2, y/s^3, and z/s^4 and evaluating the integral term by term we obtain the expansion (28).

For *isospinor* bosons the relevant matrix is a 2×2 matrix m and the integral corresponding to $I(M)$ in (22) and (28) is

$$K(m) = \int_{\mathrm{SU}(2)} dg\, e^{\mathrm{Tr}(gm^{\sim})}\,. \tag{B9}$$

By invariance arguments it may be shown that $K(m)$ is a function only of Det m. One obtains

$$K(m) = \frac{I_1(2(\mathrm{Det}M)^{1/2})}{(\mathrm{Det}M)^{1/2}} = \sum_{n=0}^{\infty}\frac{(\mathrm{Det}m)^n}{n!\,(n+1)!}\,. \tag{B10}$$

---

*Permanent address: Indian Institute of Science, Bangalore 560012, India.

†Present address: CERN, Geneva, Switzerland.

[1]K.-E. Eriksson and B.-S. Skagerstam, J. Phys. A 12, 2175 (1979); 14, 545(E) (1981).

[2]J. M. Radcliffe, J. Phys. A 4, 313 (1971); A. M. Perelemov, Commun. Math. Phys. 26, 222 (1972). See also M. M. Nieto, Phys. Rev. D 22, 391 (1980); M. M. Nieto and L. M. Simmons, Jr., *ibid.* 20, 1321 (1979); 20, 1332 (1979); 20, 1342 (1979); V. P. Gutschick and M. M. Nieto, *ibid.* 22, 403 (1980); 23, 922 (1981), and references therein; F. T. Hioe, J. Math. Phys. 15, 445 (1974).

[3]A detailed discussion is given in J. R. Klauder and E. C. G. Sudarshan, *Fundamentals in Quantum Optics* (Benjamin, New York, 1968) and in J. R. Klauder, Acta Phys. Austriaca, Suppl. XVII, 1 (1977). See also B. De Facio and C. L. Hammer, J. Math. Phys. 16, 267 (1976); C. L. Hammer *et al.*, Phys. Rev. D 18, 373 (1978); 19, 667 (1979); J. R. Klauder, *ibid.* 19, 2349 (1979).

[4]See, e.g., J. D. Talman, *Special Functions—A Group Theoretic Approach* (Benjamin, New York, 1968).

[5]J. C. Botke, D. J. Scalapino, and R. L. Sugar, Phys. Rev. D 9, 813 (1974).

[6]See, e.g., T. W. B. Kibble, in *Mathematical Methods in Theoretical Physics* (Gordon and Breach, New York, 1969), Vol. XID.

[7]B.-S. Skagerstam, Phys. Rev. D 19, 2471 (1979); 22, 534 (1980).

[8]D. Horn and R. Silver, Ann. Phys. (N.Y.) 66, 509 (1971).

[9]See, e.g., D. Horn and F. Zachariasen, *Hadron Physics at Very High Energies* (Benjamin, New York, 1973), Chap. 17 and references cited therein.

[10]D. Bhaumik, K. Bhaumik, and B. Dutta-Roy, J. Phys. A 9, 1507 (1976).

[11]K.-E. Eriksson and B.-S. Skagerstam, Phys. Rev. D 18, 3858 (1978).

[12]M. Greco, F. Palumbo, G. Pancheri-Srivastava, and Y. Srivastava, Phys. Lett. 77B, 282 (1978); G. Curci and M. Greco, *ibid.* 79B, 406 (1978).

[13]G. Curci, M. Greco, and Y. Srivastava, Nucl. Phys. B159, 451 (1979).

[14]S. Frautschi and A. Krzywicki, Z. Phys. C 1, 43

(1979).

[15]Y.-P. Yao, Phys. Rev. Lett. 36, 653 (1976); T. Appelquist, J. Carazzone, H. Kluberg-Stern, and M. Roth, *ibid.* 36, 768 (1976); T. Kinoshita and A. Ukawa, Phys. Rev. D 13, 1573 (1976); 13, 1977 (1976); 15, 1596 (1977); E. Poggio and H. Quinn, *ibid.* 14, 578 (1976).

[16]S. K. Wong, Nuovo Cimento 65A, 689 (1970).

[17]A. P. Balachandran, P. Salomonson, B.-S. Skagerstam, and J.-O. Winnberg, Phys. Rev. D 15, 2308 (1977); A. Barducci, R. Casalbuoni, and L. Lusanna, Nucl. Phys. B124, 93 (1977); A. P. Balachandran, S. Borchardt, and A. Stern, Phys. Rev. D 17, 3247 (1978).

[18]K.-E. Eriksson, N. Mukunda, and B.-S. Skagerstam, in preparation.

[19]F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, Phys. Rev. A 6, 2211 (1972); K. Hepp and E. H. Lieb, *ibid.* 8, 2517 (1973); L. M. Narducci, C. M. Bowden, V. Bluemel, G. P. Garrazana, and R. A. Tuft, *ibid.* 11, 973 (1975).

[20]R. Holtz and I. Hanus, J. Phys. A 7, L37 (1974); D. H. Feng, L. M. Narducci, and R. Gilmore, in Proceedings of the 1978 International Meeting on Frontiers of Physics, Singapore, edited by K. K. Phua, C. K. Chew, and Y. K. Lim (unpublished). In this context see also D. Janssen, Yad. Fiz. 25, 897 (1977) [Sov. J. Nucl. Phys. 25, 479 (1977)].

[21]See, e.g., M. Creutz, Rev. Mod. Phys. 50, 561 (1978); K.-E. Eriksson, N. Svartholm, and B.-S. Skagerstam, Report No. TH-2974-CERN (unpublished) and references cited therein.

[22]P. Houston and D. Pottinger, Z. Phys. C 3, 83 (1979) and references cited therein; A. B. Migdal, Zh. Eksp. Teor. Fiz. Pis'ma Red. 28, 37 (1978) [JETP Lett. 28, 35 (1978)].

[23]J. Finger, D. Horn, and J. E. Mandula, Phys. Rev. D 20, 3253 (1979).