PHYSICAL REVIEW D VOLUME 24, NUMBER 10 15 NOVEMBER 1981

# Scale covariance, nonrenormalizable interactions, and high-temperature expansions

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(Received 13 July 1981)

Modified (scale-covariant) Heisenberg operator equations of motion, derived on the assumption that local products are based on an operator-product expansion, are shown to exhibit a nonclassical, one-parameter degree of freedom in their functional integral formulations. By a proper choice of this freely adjustable, scale-covariance parameter the suitably normalized, connected (truncated) four-point functions can be changed from a bounded negative quantity to one that is positive and arbitrarily large. For nonrenormalizable, or possibly for nonasymptotically free renormalizable models, which when conventionally formulated as lattice theories violate hyperscaling and exhibit a trivial continuum limit, the previously stated property suggests that the scale-covariance parameter can be chosen to achieve nontrivial results for such models. This conclusion is supported by results of a high-temperature series analysis that incorporates the effects of the additional parameter.

## I. INTRODUCTION

### A. Operator heuristics

According to the arguments of scale-covariant quantum field theory[1-3] the usual Heisenberg operator equations of motion are modified whenever local products are defined by an operator-product expansion. In particular, for a $(\phi^p)_n$, $p$ even, scalar field theory with action functional

$$I=\int \{\tfrac{1}{2}[(\partial_\mu \phi)^2-m^2\phi^2]-\lambda\phi^p\}d^nx\,, \qquad (1)$$

the modified, scale-covariant, equation of motion is given by the formal expression

$$\phi\,\Box\,\phi+m^2\phi^2+p\lambda\phi^p=0\,. \qquad (2)$$

This equation arises from the stationarity of $I$ under infinitesimal scale transformations of the form $\delta\phi(x)=\delta S(x)\phi(x)$, $\delta S$ being a $c$ number. In (1) and (2) it is understood that local products are obtained from an operator-product expansion and not from any sort of normal ordering; as a consequence (2) cannot be interpreted simply as $\phi$ times the usual operator equation of motion—even in the special case that $\lambda\to 0$. The usual equation of motion arises from stationarity of $I$ under the $c$-number variations $\delta\phi(x)=\delta\Lambda(x)$ appropriate to normal-ordered local products, but such variations yield nothing if operator-product expansions are appropriate.[1,2]

### B. Functional formulation

Operator field equations such as (2) may be recast into coupled sets of Green's-function equations[4] or into other equivalent reformulations such as that provided by functional integrals. In the latter approach one may consider the formal expression for the Green's functional

$$Z'(h)=\mathfrak{N}\int \exp\left(i\int \{h\phi+\tfrac{1}{2}[(\partial_\mu\phi)^2-m^2\phi^2]-\lambda\phi^p\}d^nx\right)\mathfrak{D}'\phi\,, \qquad (3)$$

where $\mathfrak{N}$ is chosen so that $Z'(0)=1$ and $\mathfrak{D}'\phi$ is left unspecified for the moment. If we change the integration variables according to $\phi(x)\to S(x)\phi(x)$, $S(x)>0$, then

$$\begin{aligned}Z'(h)&=\mathfrak{N}\int \exp\left(i\int \{hS\phi+\tfrac{1}{2}[(\partial_\mu S\phi)^2-m^2S^2\phi^2]-\lambda S^p\phi^p\}d^nx\right)\mathfrak{D}'S\phi\\&=\mathfrak{N}_S\int \exp\left(i\int \{hS\phi+\tfrac{1}{2}[(\partial_\mu S\phi)^2-m^2S^2\phi^2]-\lambda S^p\phi^p\}d^nx\right)\mathfrak{D}'\phi\,,\end{aligned} \qquad (4)$$

where in the last line we have assumed that the formal measure fulfills a condition of scale covariance,

$$\mathfrak{D}'S\phi=F(S)\mathfrak{D}'\phi\,, \qquad (5)$$

and the factor $F(S)$ has been absorbed in $\mathfrak{N}_S$. Clearly an expression for $\mathfrak{N}_S$ is given by

$$\mathfrak{N}_S^{-1}=\int \exp\left(i\int \{\tfrac{1}{2}[(\partial_\mu S\phi)^2-m^2S^2\phi^2]-\lambda S^p\phi^p\}d^nx\right)\mathfrak{D}'\phi\,. \qquad (6)$$

The expression (4) [with (6)] for $Z'(h)$ seems to depend on $S$ but in fact does not, and so

$$\frac{\delta}{i\delta S(x)}Z'(h)\Big|_{S\equiv 1}\equiv 0\,, \qquad (7)$$

a relation that leads in the usual way to a functional differential equation[3,4] given by

$$\left[{}^{*}_{*}\frac{\delta}{i\delta h(x)}(\Box+m^2)\frac{\delta}{i\delta h(x)}+p\left(\frac{\delta}{i\delta h(x)}\right)^{p}{}^{*}_{*}-h(x)\frac{\delta}{i\delta h(x)}\right]Z'(h)=0. \quad (8)$$

In this expression we have set

$$
{}^{*}_{*}\left(\frac{\delta}{i\delta h(x)}\right)^{p}{}^{*}_{*}Z'(h)\equiv\left[\left(\frac{\delta}{i\delta h(x)}\right)^{p}-\left(\frac{\delta}{i\delta j(x)}\right)^{p}Z'(j)\bigg|_{j\equiv 0}\right]Z'(h), \quad (9)
$$

etc., with the subtractive term arising from the normalization factor $\mathfrak{N}_S$. Equation (8) may be recognized as the formal, functional differential equation equivalent to the formal operator field equation (2).

### C. Scale-covariance parameter

The Green's functional $Z'(h)$ provides a solution to the operator field equations for any choice of formal measure $\mathfrak{D}'\phi$ that satisfies (5). A little reflection shows that

$$\mathfrak{D}'\phi=\mathfrak{D}_B\phi\equiv\prod_x d\phi(x)/|\phi(x)|^{B} \quad (10)$$

satisfies (5) for any $B\leq 1$, and that there are no other homogeneous solutions (however, see Ref. 23). Thus there will be a complete family of Green's functionals of the form

$$Z_B(h)\equiv\mathfrak{N}_B\int\exp\left(i\int\{h\phi+\tfrac{1}{2}[(\partial_\mu\phi)^2-m^2\phi^2]-\lambda\phi^p\}d^nx\right)\mathfrak{D}_B\phi \quad (11)$$

each of which is a solution to the functional differential form of the scale-covariant operator field equation. We observe that (8) or its equivalent expression in terms of coupled Green's-function equations does not uniquely determine a solution, a point that has been emphasized by Nouri-Moghadam and Yoshimura,[5] by Ebbutt and Rivers,[6] and also in the simpler context of a discontinuous perturbation of a harmonic oscillator by Kay.[7]

In our previous work on this subject[8,3] we have chosen $B=1$ so that the measure $\mathfrak{D}'\phi$ is formally scale invariant. This choice was suggested by the fact that $B=1$ is the correct choice for the mathematically interesting but nonphysical independent-value models which are defined as in (11) except that all space-time gradients are dropped.[9] Since such models are highly nonrenormalizable and that problem is successfully overcome by choosing $B=1$, it was deemed appropriate to choose $B=1$ for other nonrenormalizable models. This hypothesis naturally led to the proposal to restore the discarded space-time gradients by a strong-coupling perturbation analysis about the independent-value model: and such an analysis was admirably studied by Kövesi-Domokos.[10]

In the following we will be led to give up the hypothesis that $B=1$ for covariant models and to choose $B$ on other grounds. In fact we shall be led to conclude that $B$ is not fixed at all but rather is a function of the parameters of the model such as $\lambda$, $p$, and $n$. The freedom to choose $B$ as we see fit exists because each quantum theory described by (11) corresponds to the same classical limit and satisfies the same basic operator equation of motion (8), which is just a reformulation of (2). Of course, for all $B\neq 0$ the formal measure $\mathfrak{D}_B\phi$ is not invariant under field translations, and consequently the associated field operators do not satisfy the canonical commutation relations. However this is to be expected for operators for which the operator-product expansion is appropriate to define local products, since if we assume that

$$[\phi(\vec{x}),\pi(\vec{y})]=i\delta(\vec{x}-\vec{y}), \quad (12)$$

then it follows for all $q\geq 2$ that

$$[\phi^q(\vec{x}),\pi(\vec{y})]=0, \quad (13)$$

a consequence which for any reasonable field operator contradicts the assumption (12).

### D. General remarks

The arguments presented above are meant to apply to any model for which normal-ordered local products are inappropriate, i.e., for models which are not asymptotically free. Typically such models are those generally classified as nonrenormalizable as well as some renormalizable models; super-renormalizable models are asymptotically free and may be conventionally defined as has been rigorously shown in several cases.[11] For the scalar field in question, renormalized covariant perturbation theory leads to the following conclusions: nonrenormalizable if $p>2n/(n-2)$; renormalizable if $p=2n/(n-2)$; super-renormalizable if $p<2n/(n-2)$. Our view of nonrenormalizable theories is that the interaction is a discontinuous perturbation,[12] and in the context of functional integration it acts partially as a hard core projecting out certain field histories that would otherwise be allowed. Such histories remain projected out even as the coupling of the nonrenormalizable interaction tends to zero, and so the limit of zero coupling of the interacting theory is not the free (unperturbed) theory but a so-called pseudofree theory. Such a hard-core picture suggests that a good place to begin the study of a nonrenormalizable model is with the pseudofree model, and idealized model studies indicate that large classes of nonrenormalizable models are continuously connected to the same pseudofree model.[3]

With the foregoing as motivation we are led to consider a class of models with Green's functional given by

$$Z_{0B}(h)=\mathfrak{N}_{0B}\int \exp\Big(i\int \{h\phi +\tfrac{1}{2}[(\partial_\mu\phi)^2-m^2\phi^2]\}d^nx\Big)\mathfrak{D}_B\phi \tag{14}$$

which for $B\neq 0$ describe nonfree theories. To this starting point we imagine introducing the interaction $\phi^p$ and renormalizing various parameters (including $B$) as needed. We shall study this problem in the context of a statistical-mechanical analog of the Euclidean-space, lattice-space version of the models.

## II. ANALOG SPIN PROBLEM

### A. Formulation

The Euclidean-space, lattice-space formulation of (14) may be taken as

$$Z_{0B}=N\int\cdots\int \exp\Big[\sum h_k\phi_k\Delta-\tfrac{1}{2}Z\sum(\phi_{k*}-\phi_k)^2\Delta/\epsilon^2 -\tfrac{1}{2}m_0^2Z\sum\phi_k^2\Delta\Big]\prod d\phi_k/|\phi_k|^B. \tag{15}$$

In this expression $k=(k_1,\ldots,k_n)$ denotes a point of a large but finite (hyper-) cubic lattice, $k^*$ denotes $\frac{1}{2}$ the nearest-neighbor points to $k$, $\epsilon$ is the lattice spacing, $\Delta\equiv\epsilon^n$ the cell volume, $N$ is a normalization factor, and $Z$ and $m_0^2$ are the usual parameters chosen to achieve consistency. Our goal is to seek a suitable continuum (and eventually a thermodynamic) limit of (15) as $\epsilon\to 0$, choosing $Z$, $m_0^2$, and $B$ as necessary—always bearing in mind that a $\phi^p$ term is to be added, or is even implicitly present with an arbitrarily small coefficient.

Let us first rearrange the exponent in (15) in the form

$$\sum h_k\phi_k\Delta+Z\sum\phi_{k*}\phi_k\Delta/\epsilon^2-Z(n\Delta/\epsilon^2+m_0^2\Delta/2)\sum\phi_k^2. \tag{16}$$

In terms of the new variables

$$S_k\equiv G\phi_k, \tag{17a}$$

$$K\equiv Z\Delta/(\epsilon G)^2, \tag{17b}$$

$$H_k\equiv h_k\Delta/G, \tag{17c}$$

$$G^2\equiv 2Z(n\Delta/\epsilon^2+m_0^2\Delta/2), \tag{17d}$$

it follows that (15) takes the form

$$Z_{0B}=N\int\cdots\int \exp\Big(\sum H_kS_k+K\sum S_{k*}S_k -\tfrac{1}{2}\sum S_k^2\Big)\prod|S_k|^{-B}dS_k, \tag{18}$$

with a suitable adjustment of $N$ as needed. In these variables the problem is equivalent to a continuous-spin statistical-mechanics model with nearest-neighbor ferromagnetic coupling ($K>0$) and a single-site spin distribution given by

$$d\mu=Me^{-S^2/2}|S|^{-B}dS, \tag{19}$$

$M$ being a normalization factor.

When a $\phi^p$ term is present then (18) is changed to

$$Z_B=N\int\cdots\int \exp\Big(\sum H_kS_k+K\sum S_{k*}S_k-\tfrac{1}{2}\sum S_k^2 -g\sum S_k^p\Big)\prod|S_k|^{-B}dS_k, \tag{20}$$

which is an analogous spin problem with a single-site distribution given by

$$d\mu=Me^{-S^2/2-gS^p}|S|^{-B}dS. \tag{21}$$

Here $g$ is a dimensionless coupling constant given by

$$g\equiv\lambda_0Z^{p/2}\Delta/G^p. \tag{22}$$

It is important to observe that the local product definition of an operator-product expansion which is being modeled here on a lattice does not entail the additional lower-order polynomial terms as are required for normal ordering.[3]

The continuum limit of (15) corresponds to going to the critical region of the statistical-mechanics problem (18). If an interaction is present, then one is interested in the critical region of the problem described by (20).

### B. Critical regime

The critical region is characterized by the divergence of the (dimensionless) correlation "length" $\xi$, which is defined by

$$\xi^2\equiv\mu_2/(2n\chi), \tag{23}$$

where (for $H_k\equiv 0$)

$$\chi\equiv\sum\langle S_0S_k\rangle, \tag{24}$$

$$\mu_2\equiv\sum K^2\langle S_0S_k\rangle. \tag{25}$$

One is also interested in higher-order correlation functions and their divergence, in particular (for $H_k\equiv 0$)

$$\chi^{(2)}\equiv\sum\langle S_0S_jS_kS_l\rangle_c, \tag{26}$$

where $c$ denotes the connected (truncated) part. A convenient dimensionless measure of this quantity is given by

$$\bar{g} \equiv -\chi^{(2)}/(\chi^2\xi^n) \tag{27}$$

which is invariant to spin-size and lattice-size scaling. The vanishing of $\bar{g}$ implies under suitable conditions that the theory is actually trivial (Gaussian).[13]

It is the usual assumption of the approach to criticality as $K$ increases (temperature decreases) toward the critical value of $K$ ($\equiv K_c$) that divergence occurs with characteristic critical exponents, namely,

$$\chi \sim \text{const}(K_c - K)^{-\gamma}, \tag{28}$$

$$\mu_2 \sim \text{const}(K_c - K)^{-\gamma-2\nu}, \tag{29}$$

$$\chi^{(2)} \sim \text{const}(K_c - K)^{-\gamma-2\Delta}, \tag{30}$$

and thus that

$$\xi \sim \text{const}(K_c - K)^{-\nu}, \tag{31}$$

$$\bar{g} \sim \text{const}(K_c - K)^{n\nu+\gamma-2\Delta}. \tag{32}$$

This last relation is particularly significant for if

$$\kappa \equiv n\nu + \gamma - 2\Delta > 0, \tag{33}$$

then $\bar{g} \to 0$ as $K \to K_c$ and the theory is potentially trivial. On the other hand if $\kappa \leqslant 0$, then the theory would be nontrivial in the continuum limit.

According to mean-field theory the classical exponents for polynomial continuous-spin models are given by $\gamma_c = 1$, $\nu_c = \frac{1}{2}$, $\Delta_c = \frac{3}{2}$, and so for these $\kappa_c = -2 + n/2$. For $n > 4$ mean-field theory suggests the triviality of the usual theory, and this has now been proved rigorously for pure $(\phi^4)_n$ models for $n > 4$ in the single-phase region.[14] As a next step we remark that the scaling hypotheses that enter the renormalization-group approach imply that $\kappa \equiv 0$; this is one of the so-called hyperscaling equations. However, the hyperscaling condition $\kappa = 0$ has only been rigorously proved for $n = 2$.[14] Finally we recall that for suitable spin distributions—e.g., of the $(\phi^4)_n$ type—which imply the Lebowitz inequality[15]

$$\langle S_i S_j S_k S_l \rangle_c \leqslant 0, \tag{34}$$

it follows with only minor assumptions[16] that

$$\gamma \geqslant 1, \tag{35}$$

$$\kappa \geqslant 0. \tag{36}$$

We note further that $2\nu \geqslant \gamma$ does not entail (34) and holds under fairly general conditions.

High-temperature expansions provide an important tool in the study of critical exponents, and there is evidence from the work of Baker and Kincaid[17] on conventional $(\phi^4)_n$ models that $\kappa = 0.028 \pm 0.003$ for $n = 3$ (for maximum coupling constant) and that $\kappa = 0.30 \pm 0.04$ for $n = 4$ (also for maximum coupling constant). On the other hand, high-temperature series analyses of Nickel[18] suggest that $\kappa$ is consistent with zero for $n = 3$. In any case, the particular hyperscaling violation $\kappa \neq 0$ seems to be possible for $n = 3$, quite likely for $n = 4$, and inescapable for $n \geqslant 5$. Our approach, which will include the additional variable $B$, will also make use of hyperscaling violation.

### C. Role of the scale-covariance parameter

We note first of all that the moments of the distribution (19) may readily be worked out, and are given by

$$\langle S^{2n} \rangle = (1-B)(3-B)\cdots(2n-1-B). \tag{37}$$

As a consequence

$$\langle S^4 \rangle_c = 2B(1-B), \tag{38}$$

and hence in the interval $0 < B < 1$ *this distribution will violate the Lebowitz inequality*, and measured in "natural units",

$$\langle S^4 \rangle_c / \langle S^2 \rangle^2 = 2B/(1-B), \tag{39}$$

*this violation can become arbitrarily large*. This is in sharp contrast to the situation for $B \leqslant 0$ for which

$$0 \geqslant \langle S^4 \rangle_c / \langle S^2 \rangle^2 \geqslant -2; \tag{40}$$

in fact, as $B \to -\infty$ this model effectively tends to the spin-$\frac{1}{2}$ Ising model.[19]

As a consequence, when $0 < B < 1$, it follows that $\gamma < 1$, $\nu < \frac{1}{2}$, and $\kappa < 0$ become possible, and our analysis of high-temperature series for this model suggest this is exactly what occurs. The results of our high-temperature analysis are summarized below, and some additional details are given elsewhere.[20]

The features outlined above for vanishing nonlinear term suggest, when we consider the measure (21) for a given interaction but arbitrary $g$, that we can choose $B = B(g)$ in order to obtain $\kappa = 0$. In other words, we shall choose $B$—which has the tendency to *lower* $\kappa$—to compensate the tendency of the interaction $\phi^p$—which has the tendency to *increase* $\kappa$—the desired *net result being* $\kappa = 0$.[21]

Our numerical results are neither as extensive nor as accurate as we would like, and so let us first indicate some qualitative properties that we expect may hold. For $n \geqslant 5$, where mean-field (classical) arguments (for $B = 0$) are basically correct and $\kappa_c > 0$, we expect in the limit that $g \to 0$ that $B(g) \to B_{(n)} > 0$, a number depending on the space-time dimension $n$, but not on the power $p$ of the interaction.[22] If this proposal is correct then for $n \geq 5$ the pseudofree theory would differ from the free theory as expected. For $n=4$, which is the expected lower limit for mean-field behavior and $\kappa_c=0$, we expect as $g \to 0$ that $B(g) \to B_{(4)}=0$. For $n=4$ this proposal suggests that the pseudofree theory coincides with the free theory.[23] If these ideas have any validity in $n=3$ then the situation is possibly similar to that for $n=4$. While we have predominantly stressed our approach for nonrenormalizable models, it is possible that the scale-covariant parameter $B$ may be successfully exploited to deal with nonasymptotically free renormalizable models. This view is suggested because it is the $(\phi^4)_4$ model for which Baker and Kincaid found strong evidence for hyperscaling violation $\kappa > 0$, and a trivial theory in the continuum limit. If this is true then perhaps $(\phi^4)_4$ is also a theory that could become nontrivial for the proper choice of $B$.

### D. High-temperature series

Baker and Kincaid have calculated high-temperature series expansions up to 10th order in $K$ for $\chi$, $\mu_2$, and $\chi^{(2)}$ for a variety of lattices and a general, even single-site spin distribution. We have particularly studied their series for the hypersimple cubic lattice in four dimensions. D log Padé approximants have been utilized to analyze these series and to estimate the critical exponents. We have concentrated on the single-site spin distribution (21) with an arbitrarily small coefficient of the interaction term $g$. This has the feature that the moments may be taken as those of the distribution (19), namely the simple quantities in (37), but that the model will be defined according to (20) for arbitrary values of $K$. The main results of this analysis are summarized in Table I.

Several points regarding Table I are worth noting. First, the appearance of nonzero $\kappa$ values supports the contention that hyperscaling is violated for $n=4$. Second, the appearance of negative $\kappa$ values—albeit with less apparent accuracy than that for positive $\kappa$ values—arises from the violation of the Lebowitz inequality. Nonpositive $\kappa$ values are those that lead to nontrivial continuum theories. To the accuracy available, it is consistent that $\kappa=0$ occurs for $B=0$ ($n=4$), as expected. Observe for $B>0$ that $\gamma<1$ and $\nu<\frac{1}{2}$ (yet $\gamma \leq 2\nu$) supporting the contention that both $\gamma$ and $\nu$ decrease if the Lebowitz inequality fails.

Unfortunately, the apparent error for the critical exponents (and for $K_c$ as well) increases due to poorer convergence of the Padé approximants as $B$ increases toward 1. In fact it is not clear that the simple asymptotic behavior of (28)–(30) is appropriate as $B$ approaches 1, and perhaps confluent singularities are present. We intend to study this point further by analyzing the 21-term series for $\chi$ and $\mu_2$ (for body-centered cubic lattices) kindly provided by Nickel.

Preliminary evidence of the analysis of high-temperature series based on a perturbation with a large value of $p$ suggests that by choosing $B$ suitably close to unity the value of $\kappa$ can indeed be lowered through zero. These data support the contention that by exploiting the classically-invisible, scale-covariance parameter $B$ we can obtain nontrivial behavior for general nonrenormalizable theories and possibly for certain (nonasymptotically free) renormalizable theories as well.

TABLE I. Critical temperature and critical exponents for various $B$. Apparent error estimates for the critical exponent $\kappa$ range from $\pm 0.2$ for $B=\frac{9}{10}$ to $\pm 0.1$ for $B=-9$; for the other exponents these error estimates are to be halved as they range from $B=\frac{9}{10}$ to $B=-9$.

| $B$ | $K_c$ | $\gamma$ | $\nu$ | $\Delta$ | $\kappa$[a] |
|---|---|---|---|---|---|
| $\frac{9}{10}$ | 0.40 | 0.14 | 0.084 | 0.36 | −0.25 |
| $\frac{2}{3}$ | 0.26 | 0.52 | 0.26 | 1.06 | −0.58 |
| $\frac{1}{2}$ | 0.21 | 0.74 | 0.37 | 1.20 | −0.16 |
| $\frac{1}{10}$ | 0.136 | 0.98 | 0.49 | 1.41 | 0.12 |
| $-\frac{1}{10}$ | 0.116 | 1.01 | 0.51 | 1.45 | 0.14 |
| −1 | 0.0686 | 1.07 | 0.51 | 1.49 | 0.27 |
| −2 | 0.0472 | 1.09 | 0.55 | 1.49 | 0.33 |
| −9 | 0.0147 | 1.10 | 0.58 | 1.50 | 0.42 |

[a] For $n=4$, $\kappa \equiv 4\nu+\gamma-2\Delta$.

### ACKNOWLEDGMENTS

We are indebted to the Laboratory for Computer Science at MIT for allowing us the use of MACSYMA to perform algebraic manipulations. Thanks are expressed to F. Cooper and C.M. Bender for several helpful discussions regarding their use of MACSYMA to develop strong-coupling perturbation expansions. We are especially grateful to B. G. Nickel for the interest he has shown in our problem and for his illuminating correspondence.

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[21]It is of course conceivable that the desired net result for the particular dimensionless coupling constant $\bar{g}$ is $\kappa \neq 0$, in which case the precise value of $B$ will be set by the finite nonzero value of some other dimensionless coupling constant for which, for some suitable $\kappa'$, the criterion $\kappa' = 0$ fixes $B$. We shall ignore that alternative possibility here.

[22]It is suggestive that as $n \to \infty$, $B_{(N)} \to 1$, since for increasing space-time dimensionality the models approach the independent-value models. This behavior is suggested by the fact that the criterion for nonrenormalizability, $p > 2n/(n-2)$, tends to $p > 2$ as $n \to \infty$, which is exactly the criterion for the independent-value models. The viewpoint of field theory as noise theory [J. R. Klauder, Phys. Lett. 56B, 93 (1975)] provides strong additional arguments for this conclusion.

[23]There are solutions (10) that are best not written in that form, namely

$$D_{B,b'}\phi = \exp\left[-b'\int \ln|\phi(x)|d^n x\right]\prod_x d\phi(x)/|\phi(x)|^B ,$$

where $b'$ is an arbitrary constant with dimension $(\text{length})^{-n}$. Even if we suspect $B_{(4)} = 0$, we cannot rule out the appearance of a term with $b' \neq 0$. Actually such a term cannot be ruled out for any $n$. Indeed, just such a situation effectively happens at $n = \infty$ (independent-value model) where $B = 1$ and $b' = -2b$, $b > 0$; $b$ is the parameter that plays such an important role for these models (cf. Ref. 3).