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Phase-integral calculation of the energy levels of a quantal anharmonic oscillator

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The phase-integral method developed by N. Fröman and P. O. Fröman is used for solving the quantal eigenvalue problem of an anharmonic oscillator with quartic anharmonicity. The generalized Bohr-Sommerfeld quantization condition up to the seventh-order phase-integral approximation is expressed explicitly in terms of complete elliptic integrals. Solving this quantization condition numerically, and comparing the results with recent very accurate numerical results obtained by Banerjee, we present curves exhibiting in a general way the accuracies of various orders of the phase-integral approximations. These curves clearly illustrate the utility of higher-order phase-integral approximations for the treatment of anharmonic oscillators.

I. INTRODUCTION

The phase-integral method developed by N. Fröman and P. O. Fröman (see Refs. 1–4, pp. 126–131 in Ref. 5, and Refs. 6–9) has in many important cases proved to be a very accurate approach for solving physical problems that can be reduced to the solution of one-dimensional, linear, second-order differential equations of the Schrödinger type. The phase-integral approximations used in this method, which for the higher orders have a considerably simpler analytic structure and greater generality than the corresponding higher-order JWKB approximations (cf. Refs. 3, 8, and 10), have been calculated explicitly by Campbell[11,12] up to very high orders. To a large extent one uses in principle exact solutions in the phase-integral method in question, and hence strict upper bounds for the errors can often be given. Furthermore, with the aid of this method the connection formulas for the first-order JWKB approximation can be generalized to the arbitrary-order phase-integral approximations in a rather straightforward way.[3] The investigations on a variety of models already made exhibit clearly the accuracy of the method, and its usefulness in various physical problems is well established. In view of these successes, it is of interest to apply the method in question to the solution of the eigenvalue problem of quantal anharmonic oscillators, which play an important role both in quantum field theory[13,14] and in chemical physics.[15-17] The phase-integral method to be used is very systematic, and therefore one can push the calculations on the anharmonic-oscillator models up to rather high orders of approximation, which is important in view of the interest to obtain very accurate results for such models.

In the present paper we shall treat the eigenvalue problem of a quantal anharmonic oscillator with quartic anharmonicity by means of phase-integral approximations up to the seventh order and express the generalized Bohr-Sommerfeld quantization condition in terms of complete elliptic integrals. With the aid of the formulas thus obtained the eigenvalues can be evaluated numerically. General information on the dependence of the errors of the eigenvalues thus calculated on the parameter values and on the order of approximation is then obtained by comparison with the recent very accurate numerical results of Banerjee.[18]

The plan of the paper is as follows. In Sec. II the generalized Bohr-Sommerfeld quantization condition for the eigenvalue problem of a quantal particle in a smooth single-well potential is expressed in convenient form up to the seventh-order phase-integral approximation, and the use of the solutions of the classical equation of motion for evaluating the integrals in the quantization condition is discussed. In Sec. III we express, for the orders 1, 3, 5, and 7, the quantization condition for an anharmonic oscillator analytically in terms of complete elliptic integrals. The numerical analysis of the quantization condition is dealt with in Sec. IV. The accuracy of the eigenvalues for various values of the parameters of the anharmonic oscillator and various orders of the phase-integral approximations is calculated by comparison with the numerical results of Banerjee.[18] The general information thus obtained on the accuracy of the quantization condition is displayed by means of suitable plots. The essential formulas and relations for elliptic functions and elliptic integrals which are needed in the present work are collected in Appendix A, which also contains a derivation of a general decomposition formula involving certain elliptic functions which is needed for the evaluation of the integrals in the quantization condition. Some other decomposition formulas are given in Appendix B. In Appendix C we give the results for the limiting cases of a simple harmonic oscillator and of a pure quartic oscillator.

II. THE GENERALIZED BOHR-SOMMERFELD QUANTIZATION CONDITION FOR A QUANTAL PARTICLE MOVING IN A SMOOTH SINGLE-WELL POTENTIAL

Consider the one-dimensional Schrödinger equation

$$\frac{d^2\psi}{dz^2}+Q^2(z)\psi=0\ , \tag{1}$$

where

$$Q^2(z)=\frac{2m}{\hbar^2}[\mathcal{E}-V(z)] \tag{2}$$

with obvious notations. When phase-integral approximations of the order $2N+1$ are used, the quantization condition for the bound states in a smooth single-well potential is

$$\tfrac{1}{2}\int_\Gamma \sum_{n=0}^{N} Z_{2n}d\zeta=(n'+\tfrac{1}{2})\pi\ ,\quad n'=0,1,2,\ldots\ , \tag{3}$$

where, if the function $Q_{\text{mod}}(z)$ defined in Refs. 4 and 5 is chosen to be equal to $Q(z)$,

$$\zeta=\int^z Q(z)dz \tag{4}$$

and Z_{2n}, for $2n\leqslant 8$, can be obtained from Eqs. (9a)–(9e) and (10) in Ref. 7 and Eq. (11″) in Ref. 5. The contour Γ is a closed loop in the complex z plane encircling both classical turning points but no other zeros or singularities of $Q^2(z)$. Choosing, as just mentioned, $Q_{\text{mod}}(z)=Q(z)$, and introducing the quantities

$$\chi_0=\frac{1}{4Q^3(z)}\frac{dQ^2(z)}{dz} \tag{5a}$$

and

$$\chi_\nu=\frac{d^\nu\chi_0}{d\zeta^\nu},\quad \nu=1,2,\ldots\ , \tag{5b}$$

we can write the expressions for $Z_0,\ldots,Z_6$ as

$$Z_0=1\ , \tag{6a}$$

$$Z_2=-\tfrac{1}{2}\chi_0^2-\frac{1}{2}\frac{d\chi_0}{d\zeta}, \tag{6b}$$

$$Z_4=-\tfrac{1}{8}(\chi_0^4+\chi_1^2)-\frac{1}{12}\frac{d}{d\zeta}(\chi_0^3)\ , \tag{6c}$$

$$Z_6=-\tfrac{1}{32}(2\chi_0^6+10\chi_0^2\chi_1^2+\chi_2^2)-\frac{1}{80}\frac{d}{d\zeta}(3\chi_0^5+5\chi_0\chi_1^2)\ . \tag{6d}$$

Using the fact that the path of integration Γ in the quantization condition (3) is a closed loop in the complex z plane, and that the quantities χ_ν are single-valued functions of z on this path of integration, we obtain from (6a)–(6d) the formulas

$$\tfrac{1}{2}\int_\Gamma Z_0 d\zeta=\tfrac{1}{2}\int_\Gamma d\zeta\ , \tag{7a}$$

$$\tfrac{1}{2}\int_\Gamma Z_2 d\zeta=-\tfrac{1}{2}\int_\Gamma \tfrac{1}{2}\chi_0^2 d\zeta\ , \tag{7b}$$

$$\tfrac{1}{2}\int_\Gamma Z_4 d\zeta=-\tfrac{1}{2}\int_\Gamma \tfrac{1}{8}(\chi_0^4+\chi_1^2)d\zeta\ , \tag{7c}$$

$$\tfrac{1}{2}\int_\Gamma Z_6 d\zeta=-\tfrac{1}{2}\int_\Gamma \tfrac{1}{32}(2\chi_0^6+10\chi_0^2\chi_1^2+\chi_2^2)d\zeta\ . \tag{7d}$$

Defining

$$\tau=\int^z\frac{dz}{Q(z)}, \tag{8}$$

and noting that according to (4) and (8)

$$d\zeta=Q^2 d\tau\ , \tag{9}$$

we can express the first few of the quantities defined by (5a)–(5b) as

$$\chi_0=\frac{1}{2Q^3}\frac{dQ}{d\tau}, \tag{10a}$$

$$\chi_1=\frac{1}{2Q^5}\frac{d^2Q}{d\tau^2}-\frac{3}{2Q^6}\left(\frac{dQ}{d\tau}\right)^2, \tag{10b}$$

$$\chi_2=\frac{1}{2Q^7}\frac{d^3Q}{d\tau^3}-\frac{11}{2Q^8}\frac{dQ}{d\tau}\frac{d^2Q}{d\tau^2}+\frac{9}{Q^9}\left(\frac{dQ}{d\tau}\right)^3. \tag{10c}$$

Substituting (10a)–(10c) into (7a)–(7d) and using (9), we obtain after some partial integrations

$$\tfrac{1}{2}\int_\Gamma Z_0 d\zeta=\tfrac{1}{2}\int_\Gamma Q^2 d\tau\ , \tag{11a}$$

$$\tfrac{1}{2}\int_\Gamma Z_2 d\zeta=-\tfrac{1}{48}\int_\Gamma \frac{1}{Q^3}\frac{d^2Q}{d\tau^2}d\tau\ , \tag{11b}$$

$$\tfrac{1}{2}\int_\Gamma Z_4 d\zeta=\tfrac{1}{768}\int_\Gamma\left[\frac{35}{Q^9}\left(\frac{dQ}{d\tau}\right)^2\frac{d^2Q}{d\tau^2}-\frac{12}{Q^8}\left(\frac{d^2Q}{d\tau^2}\right)^2\right]d\tau\ , \tag{11c}$$

$$\tfrac{1}{2}\int_\Gamma Z_6 d\zeta=-\tfrac{1}{6144}\int_\Gamma\left[\frac{5005}{Q^{15}}\left(\frac{dQ}{d\tau}\right)^4\frac{d^2Q}{d\tau^2}+\frac{372}{Q^{14}}\left(\frac{dQ}{d\tau}\right)^2\left(\frac{d^2Q}{d\tau^2}\right)^2-\frac{528}{Q^{13}}\frac{dQ}{d\tau}\frac{d^2Q}{d\tau^2}\frac{d^3Q}{d\tau^3}+\frac{24}{Q^{12}}\left(\frac{d^3Q}{d\tau^3}\right)^2\right]d\tau\ . \tag{11d}$$

The integrals in (11b)–(11d) are those that give the simplest expressions for $\frac{1}{2}\int_\Gamma Z_{2p}\,d\zeta$. Further simplification is not obtained by more partial integrations. For $2N+1=7$ we have thus expressed the integrals in the left-hand member of the quantization condition (3) in terms of integrals involving the variable τ. The evaluation of these integrals can be related to the solution of the classical equation of motion[19,20] as we shall now further elucidate.

The connection between the variables z and τ is given by Eq. (8), from which it follows that

$$\frac{dz}{d\tau}=Q\,. \tag{12}$$

From this equation in turn it easily follows that

$$\frac{d^2z}{d\tau^2}=\frac{1}{2}\frac{d(Q^2)}{dz}\,. \tag{13}$$

Using (2) and defining

$$t=\frac{m\tau}{\hbar}\,, \tag{14}$$

we see that (13) is formally equivalent to the classical equation of motion

$$m\frac{d^2z}{dt^2}=-\frac{dV}{dz}\,. \tag{15}$$

Suppose now that the solutions of the classical equation of motion are known, which, according to the above discussion, means that the solutions of the differential equation (13) are known. Every such solution satisfies the differential equation

$$\frac{d}{d\tau}\left[\left(\frac{dz}{d\tau}\right)^2-Q^2\right]=0\,, \tag{16}$$

which one easily sees by multiplying (13) by $2\,dz/d\tau$. From (16) it is obvious that $(dz/d\tau)^2-Q^2$ is constant for every one of the solutions considered. In order that (12) be satisfied $(dz/d\tau)^2-Q^2$ must be equal to zero, which means that

$$dz/d\tau=0 \quad \text{when } Q^2=0\,, \tag{17}$$

i.e., that $dz/d\tau$ is equal to zero at the classical turning points. It is therefore precisely the classical solution of the problem which gives the appropriate connection between the variables z and τ. The solutions of (15) give all possible classical orbits without specifying the energy, but (12) selects the orbit with the actual energy.

III. THE GENERALIZED BOHR-SOMMERFELD QUANTIZATION CONDITION FOR AN ANHARMONIC OSCILLATOR

We shall now consider the quantal anharmonic oscillator described by the potential

$$V(z)=\tfrac{1}{2}m\omega^2z^2+\tfrac{1}{4}\lambda z^4\,, \tag{18}$$

where ω and λ are positive constants. Inserting (18) into (2), we get

$$Q^2(z)=a-bz^2-cz^4\,, \tag{19}$$

where

$$a=2m\mathcal{E}/\hbar^2\,, \tag{20a}$$

$$b=m^2\omega^2/\hbar^2\,, \tag{20b}$$

$$c=m\lambda/(2\hbar^2)\,. \tag{20c}$$

The zeros of the equation $Q^2(z)=0$ are the classical turning points

$$z=\pm(\{[b/(2c)]^2+a/c\}^{1/2}-b/(2c))^{1/2}$$

and the purely imaginary transition points

$$z=\pm i(\{[b/(2c)]^2+a/c\}^{1/2}+b/(2c))^{1/2}\,.$$

The general solution of "the classical equation of motion" (13), when $Q^2(z)$ is given by (19), can be expressed in terms of a Jacobian elliptic function (see Appendix A for some useful formulas)[19,20]

$$z=A\,\mathrm{cn}u\,, \tag{21}$$

where

$$u=\gamma\tau+\delta\,. \tag{22}$$

Here A is a constant amplitude, and δ is a constant phase. In order that (21) with (22) will satisfy (13), when $Q^2(z)$ is given by (19), the "frequency" γ and the square of the modulus of the elliptic function k^2 will be given by the formulas

$$\gamma=(b+2cA^2)^{1/2} \tag{23a}$$

and

$$k^2=\frac{cA^2}{b+2cA^2}=\frac{cA^2}{\gamma^2}\,, \tag{23b}$$

respectively. In order that (17) be fulfilled, the amplitude A will satisfy the equation

$$Q^2(\pm A)\equiv a-bA^2-cA^4=0\,. \tag{24}$$

From this equation we obtain

$$A^2=\frac{1}{2c}[(b^2+4ac)^{1/2}-b]=\frac{2a}{(b^2+4ac)^{1/2}+b}\,. \tag{25}$$

Inserting (25) into (23a) and (23b) we get

$$\gamma=(b^2+4ac)^{1/4} \tag{26a}$$

and

$$k^2=\frac{1}{2}\left(1-\frac{b}{(b^2+4ac)^{1/2}}\right), \tag{26b}$$

respectively. The complementary modulus k' is given by

$$k'^2=1-k^2=\frac{1}{2}\left(1+\frac{b}{(b^2+4ac)^{1/2}}\right). \tag{26c}$$

By means of (12), (21), (22), and formulas in Appendix A we obtain

$$Q=\frac{dz}{d\tau}=\frac{d}{d\tau}(A\,\mathrm{cn}u)=-A\gamma\,\mathrm{sn}u\,\mathrm{dn}u\,, \tag{27a}$$

$$\frac{dQ}{d\tau}=-A\gamma^2(\mathrm{dn}^2u-k^2\,\mathrm{sn}^2u)\,\mathrm{cn}u\,, \tag{27b}$$

$$\frac{d^2Q}{d\tau^2}=A\gamma^3(1+4k^2-6k^2\,\mathrm{sn}^2u)\mathrm{sn}u\,\mathrm{dn}u\,, \tag{27c}$$

$$\frac{d^3Q}{d\tau^3}=A\gamma^4[1+4k^2+4k^2(1-2k^2)\mathrm{sn}^2u-24k^2\,\mathrm{sn}^2u\,\mathrm{dn}^2u]\,\mathrm{cn}u\,. \tag{27d}$$

The path Γ in the quantization condition (3) is a closed curve in the complex z plane which encloses the two classical turning points but none of the two purely imaginary transition points [see Fig. 1(a)]. The corresponding path of integration in the complex u plane is shown in Fig. 1(b). It shall proceed from a point u_0 [the point a' in Fig. 1(b), which can to a large extent be chosen arbitrarily] to the point u_0+4K [the point a'' in Fig. 1(b)] by proceeding above the points in the complex u plane which correspond to the classical turning points in the z plane, but below the points in the complex u plane which correspond to the purely imaginary transition points in the z plane. Here K is the complete elliptic integral of the first kind.

Inserting (27a)–(27d) into (11a)–(11d) and using (22), we get

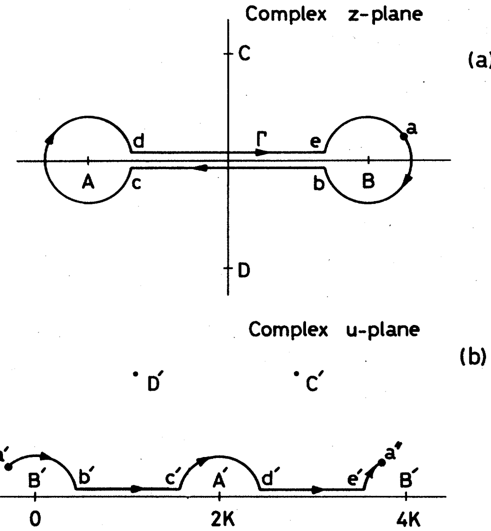

FIG. 1. (a) The complex z plane with the classical turning points A, B, and the purely imaginary transition points C, D. The closed loop Γ is the path of integration in the complex z plane, and a, b, c, d, and e are certain points on this path. (b) The complex u plane [u being connected with z according to (21)] with the points A', B', which correspond to the classical turning points A, B, and the points C', D', which correspond to the purely imaginary transition points C, D. The points a', b', c', d', e', and a'' on the path of integration in the complex u plane correspond to the points a, b, c, d, e, and a, respectively, on the path Γ in the complex z plane [see Fig. 1(a)].

$$\tfrac{1}{2}\int_\Gamma Z_0\,d\zeta=\frac{\kappa}{8}\int_{u_0}^{u_0+4K}\mathrm{sn}^2u\,\mathrm{dn}^2u\,du\,, \tag{28a}$$

$$\tfrac{1}{2}\int_\Gamma Z_2\,d\zeta=\frac{1}{12\kappa}\int_{u_0}^{u_0+4K}\frac{1+4k^2-6k^2\,\mathrm{sn}^2u}{\mathrm{sn}^2u\,\mathrm{dn}^2u}\,du\,, \tag{28b}$$

$$\tfrac{1}{2}\int_\Gamma Z_4\,d\zeta=-\frac{1}{12\kappa^3}\int_{u_0}^{u_0+4K}\left(\frac{35(\mathrm{dn}^2u-k^2\,\mathrm{sn}^2u)^2(1+4k^2-6k^2\,\mathrm{sn}^2u)\mathrm{cn}^2u}{\mathrm{sn}^8u\,\mathrm{dn}^8u}+\frac{12(1+4k^2-6k^2\,\mathrm{sn}^2u)^2}{\mathrm{sn}^6u\,\mathrm{dn}^6u}\right)du\,, \tag{28c}$$

$$\begin{aligned}\tfrac{1}{2}\int_\Gamma Z_6\,d\zeta=-\frac{1}{6\kappa^5}\int_{u_0}^{u_0+4K}\Bigg(&-\frac{5005(\mathrm{dn}^2u-k^2\,\mathrm{sn}^2u)^4(1+4k^2-6k^2\,\mathrm{sn}^2u)\,\mathrm{cn}^4u}{\mathrm{sn}^{14}u\,\mathrm{dn}^{14}u}\\&+\{372(\mathrm{dn}^2u-k^2\,\mathrm{sn}^2u)^2(1+4k^2-6k^2\,\mathrm{sn}^2u)^2-528(\mathrm{dn}^2u-k^2\,\mathrm{sn}^2u)\\&\quad\times(1+4k^2-6k^2\,\mathrm{sn}^2u)[1+4k^2+4k^2(1-2k^2)\,\mathrm{sn}^2u-24k^2\,\mathrm{sn}^2u\,\mathrm{dn}^2u]\\&\quad+24[1+4k^2+4k^2(1-2k^2)\,\mathrm{sn}^2u-24k^2\,\mathrm{sn}^2u\,\mathrm{dn}^2u]^2\}\frac{\mathrm{cn}^2u}{\mathrm{sn}^{12}u\,\mathrm{dn}^{12}u}\Bigg)du\,,\end{aligned} \tag{28d}$$

where

$$\kappa=4\gamma A^2\,, \tag{29a}$$

i.e. [cf. (25) and (26a)],

$$\kappa=\frac{2}{c}(b^2+4ac)^{1/4}[(b^2+4ac)^{1/2}-b]=\frac{8a(b^2+4ac)^{1/4}}{(b^2+4ac)^{1/2}+b}\,. \tag{29b}$$

Using (A5) and (A6a)–(A6d), and decomposing $\mathrm{sn}^{-2n}u\,\mathrm{dn}^{-2m}u$ according to (A13), we can write (28a)–(28d) as (see Appendix B)

$$\begin{aligned}\tfrac{1}{2}\int_\Gamma Z_{2p}\,d\zeta&=d_{2p}\int_{u_0}^{u_0+4K}\left(\sum_\nu\frac{f_{2p,2\nu}}{\mathrm{sn}^{2\nu}u}+\sum_\mu\frac{g_{2p,2\mu}}{\mathrm{dn}^{2\mu}u}\right)du\\&=d_{2p}\left(\sum_\nu f_{2p,2\nu}\int_{u_0}^{u_0+4K}\frac{du}{\mathrm{sn}^{2\nu}u}+\sum_\mu g_{2p,2\mu}\int_{u_0}^{u_0+4K}\frac{du}{\mathrm{dn}^{2\mu}u}\right),\end{aligned}$$

$$2p=0,2,4,6, \tag{30}$$

where

$$d_0 = \kappa/8\,, \tag{31a}$$

$$d_2 = 1/(12\kappa)\,, \tag{31b}$$

$$d_4 = -1/(12\kappa^3)\,, \tag{31c}$$

$$d_6 = -1/(6\kappa^5)\,, \tag{31d}$$

and the nonvanishing ones of the quantities $f_{2p,2\nu}$ and $g_{2p,2\mu}$ are

$$f_{0,-4} = -k^2\,, \tag{32a$_1$}$$

$$f_{0,-2} = 1\,, \tag{32a$_2$}$$

$$f_{2,2} = 1 + 4k^2,\quad g_{2,2} = -k^2(1+4k'^2)\,, \tag{32b}$$

$$f_{4,2} = 2k^4(71 - 296k^2 + 296k^4),\quad g_{4,2} = 2k^6(71 - 296k'^2 + 296k'^4)\,, \tag{32c$_1$}$$

$$f_{4,4} = 2k^2(51 - 179k^2 + 148k^4),\quad g_{4,4} = 2k^6(51 - 179k'^2 + 148k'^4)\,, \tag{32c$_2$}$$

$$f_{4,6} = -23 - 254k^2 + 192k^4,\quad g_{4,6} = k^6(-23 - 254k'^2 + 192k'^4)\,, \tag{32c$_3$}$$

$$f_{4,8} = 35(1+4k^2),\quad g_{4,8} = 35k^6k'^2(1+4k'^2)\,, \tag{32c$_4$}$$

$$f_{6,2} = -2k^8(5725 + 113\,906k^2 - 376\,068k^4 + 250\,712k^6)\,,$$
$$g_{6,2} = 2k^{10}(5725 + 113\,906k'^2 - 376\,068k'^4 + 250\,712k'^6), \tag{32d$_1$}$$

$$f_{6,4} = -2k^6(15\,826 - 47\,950k^2 - 83\,131k^4 + 125\,356k^6),$$
$$g_{6,4} = 2k^{10}(15\,826 - 47\,950k'^2 - 83\,131k'^4 + 125\,356k'^6), \tag{32d$_2$}$$

$$f_{6,6} = -2k^4(-2765 - 65\,548k^2 + 81\,209k^4 + 15\,796k^6),$$
$$g_{6,6} = 2k^{10}(-2765 - 65\,548k'^2 + 81\,209k'^4 + 15\,796k'^6), \tag{32d$_3$}$$

$$f_{6,8} = k^2(27\,955 - 2764k^2 - 175\,300k^4 + 77\,968k^6),$$
$$g_{6,8} = -k^{10}(27\,955 - 2764k'^2 - 175\,300k'^4 + 77\,968k'^6), \tag{32d$_4$}$$

$$f_{6,10} = -4873 - 81\,954k^2 - 20\,674k^4 + 75\,856k^6,$$
$$g_{6,10} = -k^{10}(-4873 - 81\,954k'^2 - 20\,674k'^4 + 75\,856k'^6), \tag{32d$_5$}$$

$$f_{6,12} = 9878 + 74\,019k^2 + 17\,908k^4,$$
$$g_{6,12} = -k^{10}k'^2(9878 + 74\,019k'^2 + 17\,908k'^4)\,, \tag{32d$_6$}$$

$$f_{6,14} = -5005(1+4k^2),\quad g_{6,14} = 5005k^{10}k'^4(1+4k'^2). \tag{32d$_7$}$$

Formula (30) can be written

$$\tfrac{1}{2}\int_\Gamma Z_{2p}\,d\zeta = d_{2p}\Big(\sum_\nu f_{2p,2\nu}\hat{B}_{2\nu} + \sum_\mu g_{2p,2\mu}\hat{I}_{2\mu}\Big)\,, \tag{33}$$

where we have introduced the quantities

$$\hat{B}_{2n} = \int_{u_0}^{u_0+4K}\frac{du}{\mathrm{sn}^{2n}u}\,,$$

$$n = \text{integer (not necessarily positive)}\,, \tag{34}$$

$$\hat{I}_{2m} = \int_{u_0}^{u_0+4K}\frac{du}{\mathrm{dn}^{2m}u}\,,$$

$$m = \text{integer (not necessarily positive)}\,, \tag{35}$$

obeying the recurrence relations [cf. (A9) and (A11) in Appendix A]

$$\hat{B}_{2n+2} = \frac{2n(1+k^2)}{2n+1}\hat{B}_{2n} - \frac{(2n-1)k^2}{2n+1}\hat{B}_{2n-2}\,, \tag{36}$$

$$\hat{I}_{2m+2} = \frac{2m(1+k'^2)}{(2m+1)k'^2}\hat{I}_{2m} - \frac{2m-1}{(2m+1)k'^2}\hat{I}_{2m-2}\,. \tag{37}$$

Using (A8b)–(A8c) and (A10b)–(A10c), we can express B_0, B_2, I_0, and $\hat{I}_2$ in terms of the complete elliptic integrals K and E of the first and second kind, respectively. By means of (36) and (37) we can then obtain the expressions for those quantities $\hat{B}_{2n}$ and $\hat{I}_{2m}$ which are needed in our calculations. The results thus obtained are

$$\hat{B}_{-4} = \frac{4}{3k^4}\,[(2+k^2)K - 2(1+k^2)E]\,, \tag{38a}$$

$$\hat{B}_{-2} = \frac{4}{k^2}(K-E)\,, \tag{38b}$$

$$\hat{B}_0 = 4K\,,\quad \hat{I}_0 = 4K\,, \tag{38c}$$

$$\hat{B}_2 = 4(K-E)\,,\quad \hat{I}_2 = \frac{4}{k'^2}E\,, \tag{38d}$$

$$\hat{B}_4 = \tfrac{4}{3}[(2+k^2)K - 2(1+k^2)E]\,,\quad \hat{I}_4 = \frac{4}{3k'^4}[2(1+k'^2)E - k'^2K]\,, \tag{38e}$$

$$\hat{B}_6=\tfrac{4}{15}\left[(8+3k^2+4k^4)K-(8+7k^2+8k^4)E\right],$$

$$\hat{I}_6=\frac{4}{15k'^6}\left[(8+7k'^2+8k'^4)E-4k'^2(1+k'^2)K\right],\tag{38f}$$

$$\hat{B}_8=\tfrac{4}{105}\left[(48+16k^2+17k^4+24k^6)K-(48+40k^2+40k^4+48k^6)E\right],$$

$$\hat{I}_8=\frac{4}{105k'^8}\left[(48+40k'^2+40k'^4+48k'^6)E-k'^2(24+23k'^2+24k'^4)K\right],\tag{38g}$$

$$\hat{B}_{10}=\tfrac{4}{945}\left[(384+120k^2+117k^4+132k^6+192k^8)K-(384+312k^2+297k^4+312k^6+384k^8)E\right],$$

$$\hat{I}_{10}=\frac{4}{945k'^{10}}\left[(384+312k'^2+297k'^4+312k'^6+384k'^8)E-k'^2(192+180k'^2+180k'^4+192k'^6)K\right],\tag{38h}$$

$$\hat{B}_{12}=\tfrac{4}{10395}\left[(3840+1152k^2+1074k^4+1113k^6+1296k^8+1920k^{10})K\right.$$
$$\left.-(3840+3072k^2+2850k^4+2850k^6+3072k^8+3840k^{10})E\right],$$

$$\hat{I}_{12}=\frac{4}{10395k'^{12}}\left[(3840+3072k'^2+2850k'^4+2850k'^6+3072k'^8+3840k'^{10})E\right.$$
$$\left.-k'^2(1920+1776k'^2+1737k'^4+1776k'^6+1920k'^8)K\right],\tag{38i}$$

$$\hat{B}_{14}=\tfrac{4}{135135}\left[(46\,080+13\,440k^2+12\,192k^4+12\,087k^6+12\,936k^8+15\,360k^{10}+23\,040k^{12})K\right.$$
$$\left.-(46\,080+36\,480k^2+33\,312k^4+32\,463k^6+33\,312k^8+36\,480k^{10}+46\,080k^{12})E\right],$$

$$\hat{I}_{14}=\frac{4}{135135k'^{14}}\left[(46\,080+36\,480k'^2+33\,312k'^4+32\,463k'^6+33\,312k'^8+36\,480k'^{10}+46\,080k'^{12})E\right.$$
$$\left.-k'^2(23\,040+21\,120k'^2+20\,376k'^4+20\,376k'^6+21\,120k'^8+23\,040k'^{10})K\right].\tag{38j}$$

Using (31a)–(31d), ($32a_1$)–($32d_7$), (38a)–(38j), and (A5), we obtain from (33), after some lengthy calculations, the following explicit formulas:

$$\tfrac{1}{2}\int_\Gamma Z_0\,d\zeta=\frac{\kappa k'^2}{6}\left(\frac{K-E}{k^2}+\frac{E}{k'^2}\right),\tag{39a}$$

$$\tfrac{1}{2}\int_\Gamma Z_2\,d\zeta=\frac{k^2}{3\kappa}\left[(1+4k^2)\frac{K-E}{k^2}-(1+4k'^2)\frac{E}{k'^2}\right],\tag{39b}$$

$$\tfrac{1}{2}\int_\Gamma Z_4\,d\zeta=-\frac{k^2}{45\kappa^3k'^4}\left[(56-153k^2+285k^4-9320k^6+32\,400k^8-37\,632k^{10}+14\,336k^{12})\frac{K-E}{k^2}\right.$$
$$\left.+(56-153k'^2+285k'^4-9320k'^6+32\,400k'^8-37\,632k'^{10}+14\,336k'^{12})\frac{E}{k'^2}\right],\tag{39c}$$

$$\tfrac{1}{2}\int_\Gamma Z_6\,d\zeta=\frac{2k^2}{315\kappa^5k'^8}\left[(3968-12\,952k^2+19\,393k^4+4342k^6-222\,227k^8+17\,667\,524k^{10}-141\,913\,296k^{12}\right.$$
$$+459\,879\,744k^{14}-766\,823\,424k^{16}+699\,572\,224k^{18}-333\,185\,024k^{20}$$
$$+65\,011\,712k^{22})\frac{K-E}{k^2}-(3968-12\,952k'^2+19\,393k'^4+4342k'^6-222\,227k'^8$$
$$+17\,667\,524k'^{10}-141\,913\,296k'^{12}+459\,879\,744k'^{14}-766\,823\,424k'^{16}$$
$$\left.+699\,572\,224k'^{18}-333\,185\,024k'^{20}+65\,011\,712k'^{22})\frac{E}{k'^2}\right].\tag{39d}$$

We recall that k^2, k'^2, and κ are given by (26b), (26c), and (29b), respectively. Formulas (39a)–(39d) are written in such a way that there appear no singular expressions when one approaches the harmonic-oscillator limit ($b \neq 0$, $c \to 0$), $(K-E)/k^2$ being finite as $k \to 0$. This limiting case, as well as the limiting case when one approaches the quartic oscillator ($b \to 0$, $c \neq 0$), is discussed in Appendix C with the aid of formulas (A2c)–(A2d) and (A3a)–(A3b) given in Appendix A.

Using (39a)–(39d), we can write the quantization condition (3) in explicit, analytical form up to the seventh-order phase-integral approximation. Its left-hand side is a function of the parameters k, k', and κ and hence, according to (26b), (26c), and (29b), of the parameters a, b, and c, which appear in expression (19) for $Q^2(z)$. For given values of b and c one can thus determine the eigenvalue a by solving the quantization condition numerically for various orders of the phase-integral approximations. The energy eigenvalue $\mathcal{E}$ can then be calculated from formula (20a), which gives

$$\mathcal{E}=\frac{\hbar^2}{2m}a\,. \tag{40}$$

We remark that $\mathcal{E}$ is also given by the formula

$$\mathcal{E}=\tfrac{1}{2}m\omega^2A^2+\tfrac{1}{4}\lambda A^4\,, \tag{41}$$

which one obtains by inserting (20a), (20b), and (20c) into (24), and that A^2 is given by (25).

IV. THE DEPENDENCE OF THE ACCURACY OF THE QUANTIZATION CONDITION ON THE PARAMETERS OF THE ANHARMONIC OSCILLATOR AND ON THE ORDER OF THE PHASE-INTEGRAL APPROXIMATION

In the manner described at the end of Sec. III we have evaluated the eigenvalues of the anharmonic oscillator numerically in the first-, third-, fifth-, and seventh-order phase-integral approximations. The calculations were carried out on an IBM 370 computer in double precision, which gives an accuracy of 15–16 digits. The time required for calculating one single eigenvalue was about 10–20 msec, and this time was almost independent of the order of approximation used. For evaluating the complete elliptic integrals appearing in (39a)–(39d) standard library routines were used. The nonlinear equation for the eigenvalue a, given by the quantization condition (3) with (39a)–(39d), was then solved by means of a standard iterative routine.

It is worth mentioning that in the first- and third-order approximations the above-mentioned calculations can be carried out on a small programmable calculator, e.g., Hewlett-Packard HP 67/97. The accuracy of the evaluations is then limited to at the most ten digits, and the time required for evaluating one eigenvalue is generally 1–2 min.

We shall in this section show in a simple way how the accuracy of the eigenvalues obtained by means of the generalized Bohr-Sommerfeld quantization condition depends on the parameters of the anharmonic oscillator and on the order of the phase-integral approximation. To this purpose we compare the numerical values of the eigenvalues $\mathcal{E}_{\rm PI}$ determined from the quantization condition, given by (3) and (39a)–(39d), with corresponding eigenvalues $\mathcal{E}_{\rm B}$ with 15 digits obtained by Banerjee[18] in very accurate numerical calculations. As a measure of the accuracy of $\mathcal{E}_{\rm PI}$ we introduce the quantity

$$D=-\log_{10}\frac{|\mathcal{E}_{\rm PI}-\mathcal{E}_{\rm B}|}{\mathcal{E}_{\rm B}}\,, \tag{42}$$

which can be considered as a rigorous definition of the number of significant figures in $\mathcal{E}_{\rm PI}$, valid also when D is not an integer. By plotting (for each order of approximation and each eigenvalue) D against the parameter describing the degree of anharmonicity of the oscillator we obtain curves displaying the accuracy of the phase-integral eigenvalues.

Since the parameters describing the degree of anharmonicity used by us differ from those used by Banerjee[18] as well as from those used in earlier well-known papers by Chan, Stelman, and Thompson[15] and by Hioe and Montroll,[21] we shall now show how these parameters are related to each other. When different authors have used the same letter for denoting a parameter, we shall indicate the names of the authors by appropriate subscripts. Banerjee[18] uses the following expression for the function $Q^2(z)$ appearing in our differential equation (1):

$$Q^2(z)=E_{\rm B}-z^2-\lambda_{\rm B}z^4\,. \tag{43a}$$

To relate Banerjee's parameters to those used by other authors, we introduce in their differential equation for the anharmonic oscillator a new independent variable (proportional to the original one) such that the coefficient of the square of the independent variable becomes equal to -1. Alternatively one can use the scaling relations (2) in Ref. 19. The results are presented below.

The present work:

$$Q^2(z)=a-bz^2-cz^4\,,\quad \lambda_{\rm B}=\frac{c}{b^{3/2}}\,. \tag{43b}$$

Hioe and Montroll[21]:

$$Q^2(z)=2E_{\rm HM}-z^2-2\lambda_{\rm HM}z^4\,,\quad \lambda_{\rm B}=2\lambda_{\rm HM}\,. \tag{43c}$$

Chan, Stelman, and Thompson[15]:

$$Q^2(z)=\tfrac{1}{4}[E_{\rm CST}-(1-\alpha)z^2-\alpha^{3/2}z^4],$$
$$\lambda_{\rm B}=2\left(\frac{\alpha}{1-\alpha}\right)^{3/2}. \qquad (43d)$$

In Figs. 2(a)–2(d) we have plotted the quantity D, defined in (42), as a function of the parameter ($\lambda_{\rm B}=c/b^{3/2}$, $\lambda_{\rm HM}$, or α), describing the degree of anharmonicity of the anharmonic oscillator, for various values of the quantum number n' and for the orders $2N+1=1$, 3, 5, and 7 of the phase-integral approximations. As we have already explained, the parameters $\lambda_{\rm B}$, $\lambda_{\rm HM}$, and α are those used in Refs. 18, 21, and 15, respectively. From the definition (42) it is obvious that the larger the value of D, the better is the accuracy of the phase-integral eigenvalue. From the curves in Figs. 2(a)–2(d) it is seen that, for a given value of n' and a given order of the phase-integral approximation, the quantity D is almost constant when $\lambda_{\rm B}\gtrsim 1$ (for large values of n' even for much smaller values of $\lambda_{\rm B}$) but increases rapidly (especially for the higher-order approximations) when $\lambda_{\rm B}$ decreases towards $\lambda_{\rm B}=0$. It is also seen that D in general increases (although not very rapidly) when n' increases, but the increase of D with increasing order of the phase-integral approximation is more pronounced. Already in the first-order approximation the "number of significant figures" D is reasonable even for $\lambda_{\rm B}\gtrsim 1$, if one disregards the few lowest-lying energy levels. When one proceeds from the first- to the third-order phase-integral approximation, the accuracy increases considerably for the excited states, D being larger than 4.5 for every value of $\lambda_{\rm B}$ when $n'>1$ [see Fig. 2(b)]. The crossing of the curves for $n'=2$ and $n'=3$ in Fig. 2(b) seems to be accidental and we do

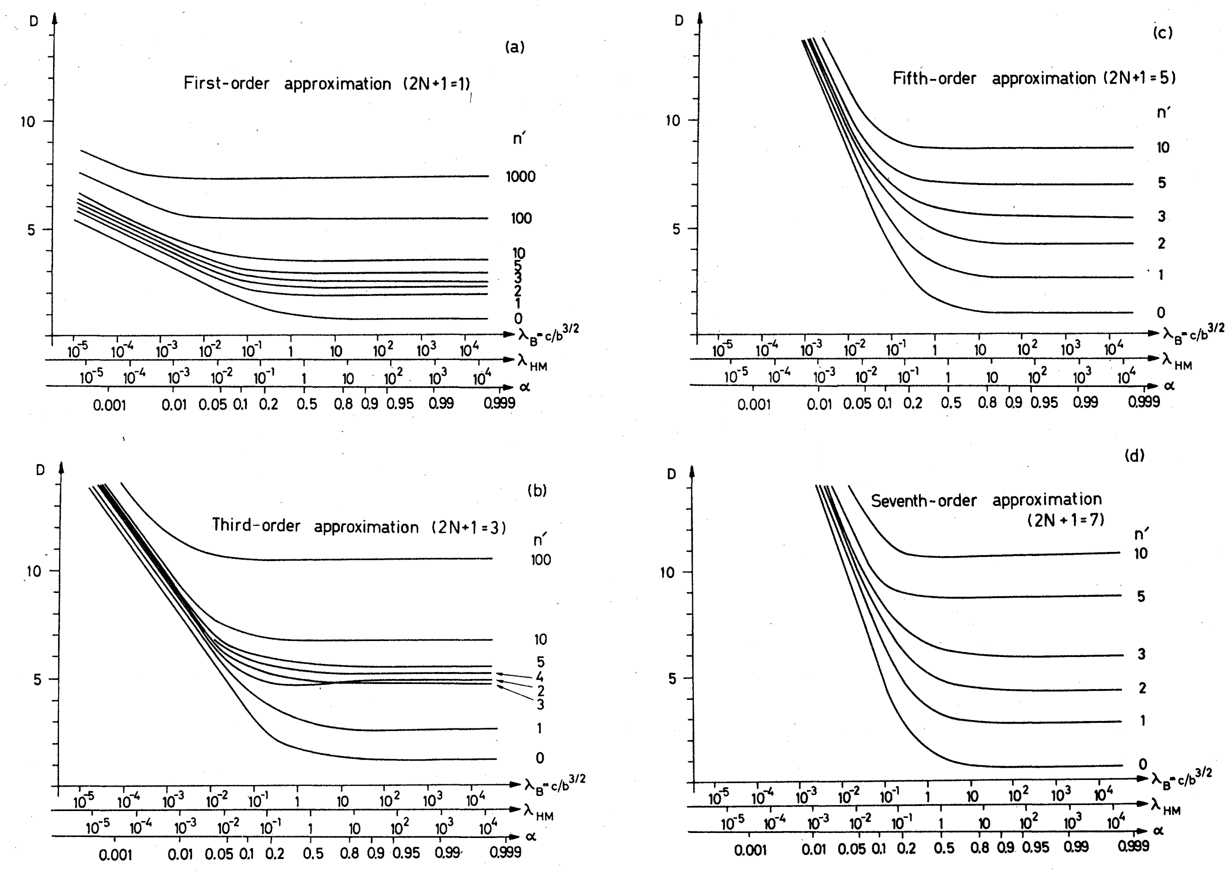

FIG. 2. (a)–(d) The "number of significant figures" D, defined as $D=-\log_{10}|(\mathcal{E}_{\rm PI}-\mathcal{E}_{\rm B})/\mathcal{E}_{\rm B}|$, in the energy eigenvalue $\mathcal{E}_{\rm PI}$ obtained from the phase-integral quantization condition (3) with (39a)–(39d) is plotted as a function of the parameter describing the degree of anharmonicity of the oscillator for various quantum numbers n' and the orders $2N+1=1,3,5,7$ of the phase-integral approximations. As the parameter which describes the degree of anharmonicity (and increases with increasing anharmonicity), one can choose the parameter λ used by Banerjee (Ref. 18), which we denote by $\lambda_{\rm B}$, or the parameter λ used by Hioe and Montroll (Ref. 21), which we denote by $\lambda_{\rm HM}$, or the parameter α used by Chan *et al.* (Ref. 15). Figures 2(a), 2(b), 2(c), and 2(d) refer to the phase-integral orders $2N+1=1$, 3, 5, and 7, respectively.

not attach any particular significance to it. When one proceeds to the fifth- and seventh-order approximations the accuracy increases further, except for the few lowest-lying energy levels; see Figs. 2(c)–2(d). From Figs. 2(a)–2(d) it is seen that when $\lambda_B \gtrsim 1$, the accuracy of the eigenvalue of the ground state ($n'=0$), as well as of the next lowest state ($n'=1$), is practically independent of the order of the phase-integral approximation used (for $2N+1 \leq 7$). This is not surprising in view of the asymptotic nature of the phase-integral approximations.

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APPENDIX A: SOME USEFUL RELATIONS AND FORMULAS INVOLVING ELLIPTIC FUNCTIONS AND ELLIPTIC INTEGRALS

In this appendix we first collect the relevant formulas involving elliptic functions and elliptic integrals which are needed in the present paper. Then we derive a decomposition formula for the function $\mathrm{sn}^{-2n}u\,\mathrm{dn}^{-2m}u$ (n and m being positive integers), which is used to evaluate the integrals (28b)–(28d) in Sec. III.

From Refs. 22 and 23 we collect the following formulas:

$$u \equiv F(\varphi, k) = \int_0^{\varphi} \frac{d\vartheta}{(1-k^2\sin^2\vartheta)^{1/2}}, \tag{A1a}$$

$$\begin{aligned} K \equiv F(\pi/2, k) &= \int_0^{\pi/2} \frac{d\vartheta}{(1-k^2\sin^2\vartheta)^{1/2}} \\ &= \frac{\pi}{2}\left[1 + (\tfrac{1}{2})^2 k^2 + \left(\frac{1\times 3}{2\times 4}\right)^2 k^4 + \cdots\right] \\ &= \frac{\pi}{2} F(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2), \end{aligned} \tag{A1b}$$

$$K = \pi/2 \quad \text{when } k=0, \tag{A1c}$$

$$K = \frac{[\Gamma(1/4)]^2}{4\sqrt{\pi}} \quad \text{when } k = 1/\sqrt{2}, \tag{A1d}$$

$$\begin{aligned} E(u) \equiv E(\varphi, k) &= \int_0^{\varphi} (1-k^2\sin^2\vartheta)^{1/2} d\vartheta \\ &= \int_0^u \mathrm{dn}^2 u\, du, \end{aligned} \tag{A2a}$$

$$\begin{aligned} E \equiv E(K) \equiv E(\pi/2, k) &= \int_0^{\pi/2} (1-k^2\sin^2\vartheta)^{1/2} d\vartheta \\ &= \int_0^K \mathrm{dn}^2 u\, du \\ &= \frac{\pi}{2}\left[1 - (\tfrac{1}{2})^2 k^2 - \frac{1}{3}\left(\frac{1\times 3}{2\times 4}\right)^2 k^4 - \cdots\right] \\ &= \frac{\pi}{2} F(-\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2). \end{aligned} \tag{A2b}$$

Note that Bowman[23] (Example 1, p. 24) gives this expansion incorrectly.

$$E = \pi/2 \quad \text{when } k=0, \tag{A2c}$$

$$\begin{aligned} E &= \frac{K}{2} + \frac{\pi}{4K} \\ &= \frac{[\Gamma(1/4)]^2}{8\sqrt{\pi}} + \frac{\pi\sqrt{\pi}}{[\Gamma(1/4)]^2} \quad \text{when } k = 1/\sqrt{2}, \end{aligned} \tag{A2d}$$

$$\frac{K-E}{k^2} = \frac{\pi}{4}\left[1 + \tfrac{3}{2}(\tfrac{1}{2})^2 k^2 + \frac{5}{3}\left(\frac{1\times 3}{2\times 4}\right)^2 k^4 + \cdots\right], \tag{A3a}$$

$$\frac{K-E}{k^2} \to \frac{\pi}{4} \quad \text{as } k \to 0, \tag{A3b}$$

$$\mathrm{sn}u = \sin\varphi, \tag{A4a}$$

$$\mathrm{cn}u = \cos\varphi, \tag{A4b}$$

$$\mathrm{dn}u = (1-k^2\sin^2\varphi)^{1/2}, \tag{A4c}$$

$$k' = (1-k^2)^{1/2}, \tag{A5}$$

$$\mathrm{sn}^2u + \mathrm{cn}^2u = 1, \tag{A6a}$$

$$k^2\,\mathrm{sn}^2u + \mathrm{dn}^2u = 1, \tag{A6b}$$

$$\mathrm{dn}^2u - k^2\,\mathrm{cn}^2u = k'^2, \tag{A6c}$$

$$k'^2\,\mathrm{sn}^2u + \mathrm{cn}^2u = \mathrm{dn}^2u, \tag{A6d}$$

$$\frac{d}{du}(\mathrm{sn}u) = \mathrm{cn}u\,\mathrm{dn}u, \tag{A7a}$$

$$\frac{d}{du}(\mathrm{cn}u) = -\,\mathrm{sn}u\,\mathrm{dn}u, \tag{A7b}$$

$$\frac{d}{du}(\mathrm{dn}u) = -\,k^2\,\mathrm{sn}u\,\mathrm{cn}u. \tag{A7c}$$

For m positive, negative, or zero we introduce the definition

$$B_m = \int \frac{du}{\mathrm{sn}^m u} \tag{A8a}$$

by extending the corresponding definition used by Byrd and Friedman,[23] p. 192. From (A8a) we obtain the particular formulas

$$B_0 = u\,, \tag{A8b}$$

$$B_2 = \int \frac{1}{\mathrm{sn}^2 u}\, du = u - E(u) - \frac{\mathrm{cn}u\,\mathrm{dn}u}{\mathrm{sn}u}\,, \tag{A8c}$$

and the recurrence formula

$$B_{2m+2} = \frac{2m(1+k^2)}{2m+1} B_{2m} - \frac{(2m-1)k^2}{2m+1} B_{2m-2} - \frac{\mathrm{cn}u\,\mathrm{dn}u}{(2m+1)\mathrm{sn}^{2m+1}u}. \tag{A9}$$

Following Byrd and Friedman[23] p. 194, we use the definition

$$I_m = \int \frac{du}{\mathrm{dn}^m u} \tag{A10a}$$

from which we obtain the particular formulas

$$I_0 = u\,, \tag{A10b}$$

$$I_2 = \int \frac{1}{\mathrm{dn}^2 u}\, du = \frac{1}{k'^2}\left[E(u) - \frac{k^2 \mathrm{sn}u\,\mathrm{cn}u}{\mathrm{dn}u}\right], \tag{A10c}$$

and the recurrence formula

$$I_{2m+2} = \frac{2m(1+k'^2)}{(2m+1)k'^2} I_{2m} - \frac{2m-1}{(2m+1)k'^2} I_{2m-2} - \frac{k^2\,\mathrm{sn}u\,\mathrm{cn}u}{(2m+1)k'^2\,\mathrm{dn}^{2m+1}u}. \tag{A11}$$

We shall now decompose the expression $\mathrm{sn}^{-2n}u\,\mathrm{dn}^{-2m}u$ $(n \geq 1,\ m \geq 1)$ into terms containing $\mathrm{sn}^{-2\nu}u$ $(1 \leq \nu \leq n)$ and $\mathrm{dn}^{-2\mu}u$ $(1 \leq \mu \leq m)$ separately. Using (A6b), we obtain the formula

$$\frac{1}{\mathrm{sn}^2 u\,\mathrm{dn}^2 u} = \frac{1}{\mathrm{sn}^2 u} + \frac{k^2}{\mathrm{dn}^2 u} \tag{A12a}$$

by means of which one can easily prove the more general formulas

$$\frac{1}{\mathrm{sn}^{2n}u\,\mathrm{dn}^2 u} = \sum_{\nu=1}^{n} \frac{k^{2(n-\nu)}}{\mathrm{sn}^{2\nu}u} + \frac{k^{2n}}{\mathrm{dn}^2 u}\,, \quad n \geq 1 \tag{A12b}$$

and

$$\frac{1}{\mathrm{sn}^2 u\,\mathrm{dn}^{2m}u} = \frac{1}{\mathrm{sn}^2 u} + \sum_{\mu=1}^{m} \frac{k^2}{\mathrm{dn}^{2\mu}u}\,, \quad m \geq 1 \tag{A12c}$$

by complete induction. The formulas (A12a)–(A12c) are particular cases of the formula

$$\frac{1}{\mathrm{sn}^{2n}u\,\mathrm{dn}^{2m}u} = \sum_{\nu=1}^{n} \frac{(m-1+n-\nu)!}{(m-1)!(n-\nu)!} \frac{k^{2(n-\nu)}}{\mathrm{sn}^{2\nu}u} + \sum_{\mu=1}^{m} \frac{(n-1+m-\mu)!}{(n-1)!(m-\mu)!} \frac{k^{2n}}{\mathrm{dn}^{2\mu}u}\,, \quad n \geq 1\,,\ m \geq 1\,, \tag{A13}$$

which can be proved by complete induction. To prove that if (A13) is valid, the formula obtained by replacing n by $n+1$ in (A13) is also valid, one uses (A12c), with m replaced by μ and μ replaced by μ', and the formula

$$\sum_{\mu=\lambda}^{m} \frac{(n-1+m-\mu)!}{(n-1)!(m-\mu)!} = \sum_{\alpha=0}^{m-\lambda} \frac{(n-1+\alpha)!}{(n-1)!\alpha!} = \frac{(m-\lambda+n)!}{(m-\lambda)!n!}\,, \quad 1 \leq \lambda \leq m\,. \tag{A14a}$$

To prove that if (A13) is valid, the formula obtained by replacing m by $m+1$ in (A13) is also valid, one uses (A12b), with n replaced by ν and ν replaced by ν', and the formula

$$\sum_{\nu=\lambda}^{n} \frac{(m-1+n-\nu)!}{(m-1)!(n-\nu)!} = \sum_{\alpha=0}^{n-\lambda} \frac{(m-1+\alpha)!}{(m-1)!\alpha!} = \frac{(n-\lambda+m)!}{(n-\lambda)!m!}\,, \quad 1 \leq \lambda \leq n\,, \tag{A14b}$$

which is the same formula as (A14a), although different notations are used.

APPENDIX B: DECOMPOSITIONS OF THE EXPRESSIONS IN THE INTEGRANDS ON THE RIGHT-HAND SIDES OF (28a)–(28d)

Since most of the expressions in this appendix require large space, we shall here use s, c, and d as short-hand notations for $\mathrm{sn}u$, $\mathrm{cn}u$, and $\mathrm{dn}u$, respectively.

Using (A6b), we can write the integrand in the right-hand member of (28a) as

$$s^2 d^2 = s^2 - k^2 s^4\,. \tag{B1a}$$

Using (A12a), (A6b), and (A5), we can decompose the integrand in the right-hand member of (28b) as follows:

$$\frac{1+4k^2-6k^2s^2}{s^2d^2} = \frac{1+4k^2}{s^2} - \frac{k^2(1+4k'^2)}{d^2}\,. \tag{B1b}$$

The decompositions of the expressions which appear in the integrand in the right-hand member of (28c), obtained by means of (A13) and (A5), are

$$\frac{(d^2-k^2s^2)^2(1+4k^2-6k^2s^2)c^2}{s^8d^8} = \frac{1+4k^2}{s^8} - \frac{1+10k^2}{s^6} + \frac{2k^2(3-k^2-4k^4)}{s^4} + \frac{2k^4(1+8k^2-8k^4)}{s^2} + \frac{2k^6(1+8k'^2-8k'^4)}{d^2} + \frac{2k^6(3-k'^2-4k'^4)}{d^4} - \frac{k^6(1+10k'^2)}{d^6} + \frac{k^6k'^2(1+4k'^2)}{d^8} \tag{B1c}$$

$$\frac{(1+4k^2-6k^2s^2)^2}{s^6d^6}=\frac{(1+4k^2)^2}{s^6}+\frac{3k^2(1+4k^2)(1-4k'^2)}{s^4}+\frac{6k^4(1-16k^2+16k^4)}{s^2}$$
$$+\frac{6k^6(1-16k^2+16k^4)}{d^2}+\frac{3k^6(1+4k'^2)(1-4k^2)}{d^4}+\frac{(1+4k'^2)^2k^6}{d^6}. \qquad \text{(B1c}'\text{)}$$

Similarly we obtain the decompositions of the expressions which appear in the integrand in the right-hand member of (28d),

$$\frac{(d^2-k^2s^2)^4(1+4k^2-6k^2s^2)c^4}{s^{14}d^{14}}=\frac{1+4k^2}{s^{14}}-\frac{2+15k^2+4k^4}{s^{12}}+\frac{1+18k^2+10k^4-16k^6}{s^{10}}$$
$$-\frac{k^2(7+8k^2-52k^4+16k^6)}{s^8}+\frac{2k^4(1-28k^2+29k^4+4k^6)}{s^6}$$
$$-\frac{2k^6(-10+34k^2+7k^4-28k^6)}{s^4}+\frac{2k^8(13+2k^2-84k^4+56k^6)}{s^2}$$
$$-\frac{2k^{10}(13+2k'^2-84k'^4+56k'^6)}{d^2}+\frac{2k^{10}(-10+34k'^2+7k'^4-28k'^6)}{d^4}$$
$$-\frac{2k^{10}(1-28k'^2+29k'^4+4k'^6)}{d^6}+\frac{k^{10}(7+8k'^2-52k'^4+16k'^6)}{d^8}$$
$$-\frac{k^{10}(1+18k'^2+10k'^4-16k'^6)}{d^{10}}+\frac{k^{10}k'^2(2+15k'^2+4k'^4)}{d^{12}}-\frac{k^{10}k'^4(1+4k'^2)}{d^{14}}, \qquad \text{(B1d)}$$

$$\frac{(d^2-k^2s^2)^2(1+4k^2-6k^2s^2)^2c^2}{s^{12}d^{12}}=\frac{(1+4k^2)^2}{s^{12}}-\frac{(1+14k^2-8k^4)(1+4k^2)}{s^{10}}$$
$$+\frac{k^2(10+45k^2-120k^4+16k^6)}{s^8}-\frac{k^4(13-144k^2+96k^4+64k^6)}{s^6}$$
$$+\frac{2k^6(-28+75k^2+72k^4-112k^6)}{s^4}-\frac{2k^8(35+42k^2-336k^4+224k^6)}{s^2}$$
$$+\frac{2k^{10}(35+42k'^2-336k'^4+224k'^6)}{d^2}-\frac{2k^{10}(-28+75k'^2+72k'^4-112k'^6)}{d^4}$$
$$+\frac{k^{10}(13-144k'^2+96k'^4+64k'^6)}{d^6}-\frac{k^{10}(10+45k'^2-120k'^4+16k'^6)}{d^8}$$
$$+\frac{k^{10}(1+4k'^2)(1+14k'^2-8k'^4)}{d^{10}}-\frac{k^{10}k'^2(1+4k'^2)^2}{d^{12}}, \qquad \text{(B1d}'\text{)}$$

$$\frac{(d^2-k^2s^2)(1+4k^2-6k^2s^2)[1+4k^2+4k^2(1-2k^2)s^2-24k^2s^2d^2]c^2}{s^{12}d^{12}}$$
$$=\frac{(1+4k^2)^2}{s^{12}}-\frac{(1+4k^2)(1+26k^2-8k^4)}{s^{10}}+\frac{k^2(22+129k^2-264k^4+16k^6)}{s^8}$$
$$-\frac{k^4(49-444k^2+336k^4+64k^6)}{s^6}+\frac{2k^6(-106+345k^2-48k^4-112k^6)}{s^4}$$
$$+\frac{2k^8(1-2k^2)(-185-112k^2+112k^4)}{s^2}-\frac{2k^{10}(1-2k'^2)(-185-112k'^2+112k'^4)}{d^2}$$
$$-\frac{2k^{10}(-106+345k'^2-48k'^4-112k'^6)}{d^4}+\frac{k^{10}(49-444k'^2+336k'^4+64k'^6)}{d^6}$$
$$-\frac{k^{10}(22+129k'^2-264k'^4+16k'^6)}{d^8}+\frac{k^{10}(1+4k'^2)(1+26k'^2-8k'^4)}{d^{10}}-\frac{k^{10}k'^2(1+4k'^2)^2}{d^{12}}, \qquad \text{(B1d}''\text{)}$$

$$\frac{[1+4k^2+4k^2(1-2k^2)s^2-24k^2s^2d^2]^2c^2}{s^{12}d^{12}}$$
$$=\frac{(1+4k^2)^2}{s^{12}}-\frac{(1+4k^2)(1+38k^2-8k^4)}{s^{10}}+\frac{k^2(34+357k^2-408k^4+16k^6)}{s^8}$$
$$-\frac{k^4(229-1320k^2+576k^4+64k^6)}{s^6}+\frac{2k^6(-472+1335k^2-168k^4-112k^6)}{s^4}$$
$$-\frac{2k^8(1055-1998k^2-336k^4+224k^6)}{s^2}+\frac{2k^{10}(1055-1998k'^2-336k'^4+224k'^6)}{d^2}$$
$$-\frac{2k^{10}(-472+1335k'^2-168k'^4-112k'^6)}{d^4}+\frac{k^{10}(229-1320k'^2+576k'^4+64k'^6)}{d^6}$$
$$-\frac{k^{10}(34+357k'^2-408k'^4+16k'^6)}{d^8}+\frac{k^{10}(1+4k'^2)(1+38k'^2-8k'^4)}{d^{10}}-\frac{k^{10}k'^2(1+4k'^2)^2}{d^{12}}. \quad \text{(B1d}'''\text{)}$$

APPENDIX C: THE LIMITING CASES OF THE SIMPLE HARMONIC OSCILLATOR AND THE PURE QUARTIC OSCILLATOR

The harmonic-oscillator limit: The harmonic-oscillator limit is attained when $\lambda\to 0$, i.e., $c\to 0$, which according to (26b) and (29b) implies that $k^2\to 0$ and $\kappa\to 4a/\sqrt{b}$. Then $k'\to 1$, $E\to\pi/2$, and $(K-E)/k^2\to\pi/4$, and from (39a) we obtain

$$\lim_{k\to 0}\tfrac{1}{2}\int_\Gamma Z_0 d\zeta=\frac{\pi\kappa}{8}=\frac{\pi a}{2\sqrt{b}}. \quad \text{(C1a)}$$

The right-hand side of each one of the formulas (39b)–(39d) is the product of k^2 and a factor that remains finite as $k^2\to 0$, and therefore we obtain

$$\lim_{k\to 0}\tfrac{1}{2}\int_\Gamma Z_2 d\zeta=0, \quad \text{(C1b)}$$

$$\lim_{k\to 0}\tfrac{1}{2}\int_\Gamma Z_4 d\zeta=0, \quad \text{(C1c)}$$

$$\lim_{k\to 0}\tfrac{1}{2}\int_\Gamma Z_6 d\zeta=0. \quad \text{(C1d)}$$

In the harmonic-oscillator limit we thus see that the quantization condition (3) is the same in every order of approximation and is given by

$$\frac{\pi a}{2\sqrt{b}}=(n'+\tfrac{1}{2})\pi, \quad n'=0,1,2,\ldots. \quad \text{(C2)}$$

Inserting here the expressions (20a)–(20b) for a and b, we get the well-known result

$$\mathcal{E}=(n'+\tfrac{1}{2})\hbar\omega, \quad n'=0,1,2,\ldots. \quad \text{(C3)}$$

The quartic-oscillator limit: The quartic-oscillator limit is attained when $\omega\to 0$, i.e., $b\to 0$, which according to (26b), (A5), and (29b) implies that $k\to k'\to 1/\sqrt{2}$ and $\kappa\to 4(4a^3/c)^{1/4}$. Using the values of K and E obtained from (A1d) and (A2d), we obtain from (39a)–(39d) in the limit of the quartic oscillator ($b\to 0$)

$$\tfrac{1}{2}\int_\Gamma Z_0 d\zeta=\frac{[\Gamma(1/4)]^2\kappa}{24\sqrt{\pi}}=\frac{[\Gamma(1/4)]^2}{6\sqrt{\pi}}\left(\frac{4a^3}{c}\right)^{1/4}, \quad \text{(C4a)}$$

$$\tfrac{1}{2}\int_\Gamma Z_2 d\zeta=-\frac{2\pi\sqrt{\pi}}{[\Gamma(1/4)]^2\kappa}=-\frac{\pi\sqrt{\pi}}{2[\Gamma(1/4)]^2}\left(\frac{4a^3}{c}\right)^{-1/4}, \quad \text{(C4b)}$$

$$\tfrac{1}{2}\int_\Gamma Z_4 d\zeta=\frac{11[\Gamma(1/4)]^2}{12\sqrt{\pi}\kappa^3}=\frac{11[\Gamma(1/4)]^2}{768\sqrt{\pi}}\left(\frac{4a^3}{c}\right)^{-3/4}, \quad \text{(C4c)}$$

$$\tfrac{1}{2}\int_\Gamma Z_6 d\zeta=\frac{18788\pi\sqrt{\pi}}{15[\Gamma(1/4)]^2\kappa^5}=\frac{4697\pi\sqrt{\pi}}{3840[\Gamma(1/4)]^2}\left(\frac{4a^3}{c}\right)^{-5/4}. \quad \text{(C4d)}$$

Krieger *et al.*[24] have studied the pure $x^{2\nu}$ oscillator and given a quantization condition [their Eq. (6)], which, when specialized to $2\nu=4$, is equivalent to the fifth-order quantization condition obtained from our formulas (C4a)–(C4c) and (3) with $N=2$. When (C4a)–(C4d) are inserted into (3) with $N=3$, the resulting seventh-order quantization condition is equivalent to the one obtained by truncating Eq. (14) in Ref. 25 after the term corresponding to $n=3$.

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