PHYSICAL REVIEW B VOLUME 24, NUMBER 10 15 NOVEMBER 1981

Static dust sphere in Einstein-Cartan theory

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(Received 15 April 1981)

A class of solutions of the Einstein-Cartan theory for the static spherical dust distribution is obtained. The equilibrium of the distribution is found to be due to the balance of the gravitational attraction and the repulsion due to the spin and torsion effect.

I. INTRODUCTION

Following the work of Trautman,[1] the Einstein-Cartan equations with special reference to a perfect-fluid distribution were studied by Prasanna.[2] He obtained some interesting solutions adopting Hehl's[3] approach and Tolman's[4] technique. If one assumes that the equation of hydrostatic equilibrium holds as in general relativity, a space-time metric similar to the Schwarzschild interior solution satisfies the Einstein-Cartan equations. However, the metric no longer represents a homogeneous fluid sphere, and at the boundary of the fluid sphere the hydrostatic pressure p is discontinuous. Further the general relativistic boundary condition that the metric potential are C^1 is not satisfied.

In this paper we first make an investigation of the role of spin and torsion in the case of a purely static distribution of incoherent dust and are led to the conclusion that the incoherent dust distribution may exist in a static condition under the combined effect of gravitation, spin, and torsion when the metric potentials are no longer C^1. We here make a study of a pure static incoherent spherical dust distribution.

II. EINSTEIN-CARTAN EQUATIONS

The Einstein-Cartan equations are

$$R^i{}_j - \tfrac{1}{2}\delta^i{}_j R = -\kappa t^i{}_j \,, \qquad (2.1)$$

$$Q^i{}_{jk} - \delta^i{}_j Q^l_{lk} - \delta^i{}_k Q^l_{jl} = -\kappa S^i{}_{jk} \,, \qquad (2.2)$$

where $t^i{}_j$ for an incoherent dust is given by

$$t^i{}_j = [\rho u^i - g^{ik} u^l \nabla_m (u^m S_{lk})] u_j \,, \qquad (2.3)$$

ρ being the matter density and u^i the velocity-four vector, with $u^i u_i = 1$, ∇_m the usual covariant derivative, and S_{ij} the intrinsic angular momentum density. The classical description of spin is then defined by the relation

$$S^i{}_{jk} = u^i S_{jk} \quad \text{with } u^i S_{ik} = 0 \,. \qquad (2.4)$$

From the Bianchi identities we have

$$\nabla_k [\rho u^k - g^{ki} u^l \nabla_m (u^m S_{li})] = 0 \qquad (2.5)$$

and

$$[\rho u^k - g^{ki} u^l \nabla_m (u^m S_{li})] \nabla_k u_j = -\nabla_l (u^l u_j) + u^k S_{jm} R^m{}_k - \tfrac{1}{2} u^k S_{lm} R^{lm}{}_{jk} \,. \qquad (2.6)$$

III. SPHERICAL DISTRIBUTIONS

We consider a spherically symmetric dust distribution represented by the space-time metric

$$ds^2 = e^{2\nu} dt^2 - e^{2\mu} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \,. \qquad (3.1)$$

We consider further the simplest case where the spins of the individual particles composing the dust distributions are all aligned in the radial directions. However, such an alignment of spins is completely artificial. Then from Eqs. (2.4) we get for the spin tensor the only surviving component S_{23}. Since the distribution is supposed to be static, we have the velocity four-vector $u^i = \delta^i_0 (g_{00})^{-1/2}$. Thus the only surviving components of $S^i{}_{jk}$ are

$$S^0_{23} = -S^0_{32} = k (g_{00})^{-1/2} \,. \qquad (3.2)$$

Hence from Eqs. (2.2) we get for $Q^i{}_{jk}$ the components

$$Q^0_{23} = -Q^0_{32} = -\kappa k (g_{00})^{-1/2} \qquad (3.3)$$

The field equations (2.1) now may be written, using (3.1) and (3.4), setting

$$\kappa = -\frac{8\pi G}{c^4} \quad \text{with } G = c = 1 \qquad (3.4)$$

as

$$-\frac{1}{r^2} + e^{-2\mu}\left(\frac{2\nu'}{r} + \frac{1}{r^2}\right) = 8\pi \tilde{p} \,, \qquad (3.5)$$

$$e^{-2\mu}\left[\nu'' + \nu'^2 - \mu'\nu' + \frac{1}{r}(\nu' - \mu')\right] = 8\pi \tilde{p} \,, \qquad (3.6)$$

$$\frac{1}{r^2} + e^{-2\mu}\left(\frac{2\mu'}{r} - \frac{1}{r^2}\right) = 8\pi \tilde{\rho} \,, \qquad (3.7)$$

where $\tilde{\rho} = \rho - 2\pi k^2$ is the effective density of the distribution and $\tilde{p} = -2\pi k^2$ is the effective pressure.[5] The equation of continuity (2.6) becomes

$$\frac{d\tilde{p}}{dr} + (\tilde{\rho} + \tilde{p})\nu' = 0 \,. \qquad (3.8)$$

The set of Eqs. (3.5)–(3.8) describes a system of classical spin fluid with zero pressure. Although the effective pressure $\tilde{p}$ is found to be negative, we find that the effective pressure supplies the necessary repulsion against the gravitational attraction for the fluid. Equation (3.8) admits solution corresponding to $\rho > 0$ and $k^2 > 0$. Of the four Eqs. (3.5)–(3.8), only three are independent, the last equation being a consequence of the Bianchi identities. To consider solutions of Einstein-Cartan equations, we use (3.5), (3.6), and (3.7). As there are four unknowns, we now have in principle a completely determined system if any one of the unknowns is arbitrarily specified.

We consider a simple case where $\tilde{\rho} = (\rho - 2\pi k^2) = \rho_0 r^2/a^2$, $\rho_0 = \text{constant}$. Such a distribution seems to be reasonable in the sense that at the origin, if the spin density vanishes, the matter density, which is intrinsically related with the spin density, also vanishes. Equation (3.7) can be integrated to give

$$e^{-2\mu} = 1 - Ar^4 , \tag{3.9}$$

where

$$A = \frac{8\pi\rho_0}{5a^2} . \tag{3.10}$$

To obtain the differential equation for e^ν, we write the derivatives of ν in the combination of Eqs. (3.5) and (3.6) in terms of derivatives with respect to $x = r^2$:

$$4xe^{-2\mu}(e^\nu)_{xx} + 2x(e^{-2\mu})_x(e^\nu)_x + \left[\frac{1}{x} - \frac{e^{-2\mu}}{x} + (e^{-2\mu})_x\right]e^\nu = 0 , \tag{3.11}$$

where the subscript x denotes the differentiation with respect to x. On substitution of (3.9) into (3.11) we obtain

$$(1 - Ax^2)(e^\nu)_{xx} - Ax(e^\nu)_x - \tfrac{1}{4}Ae^\nu = 0 . \tag{3.12}$$

Now making a change of variable $\sqrt{A}\,\xi = \sin^{-1}\sqrt{A}x$, we get for (3.12)

$$[e^\nu(\xi)]_{\xi\xi} = \tfrac{1}{4}Ae^\nu(\xi) , \tag{3.13}$$

where the subscript ξ represents the differentiation with respect to ξ. The Eq. (3.13) can be easily integrated to give

$$e^\nu = B\cosh\tfrac{1}{2}\sin^{-1}\sqrt{A}r^2 , \tag{3.14}$$

where B is an arbitrary constant of integration. The other constant of integration has been set equal to zero to ensure

$$\nu' = 0 \quad \text{at } r = 0 .$$

From Eq. (3.5) we have, on substitution of ν' from (3.14) and (3.9),

$$-16\pi^2k^2 = 2\sqrt{A}\tanh(\tfrac{1}{2}\sin^{-1}\sqrt{A}r^2)(1 - Ar^4)^{1/2} - Ar^2 . \tag{3.15}$$

From (3.15) it is evident that at $r = 0$, $k^2 = 0$. Further, to guarantee the real values of k, we impose the condition

$$\frac{Aa^4}{1 - Aa^4} - 4\tanh^2\tfrac{1}{2}\sin^{-1}\sqrt{A}a^2 \geq 0 . \tag{3.16}$$

The equality sign holds for the case $k^2 = 0$ at $r = a$.

However, the constants A and B can be determined from the boundary conditions. Assuming that the sphere has a finite radius $r = a$, since the torsion does not contribute for $r > a$, the metric is represented by the Schwarzschild solution

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2 , \tag{3.17}$$

where m is the Schwarzschild mass. With this we use the boundary condition

$$[e^{-2\mu}]_{r=a} = [e^{2\nu}]_{r=a} = \left(1 - \frac{2m}{a}\right) . \tag{3.18}$$

From (3.18) we have

$$B = \left(1 - \frac{2m}{a}\right)^{1/2} \Big/ \cosh\tfrac{1}{2}\sin^{-1}\sqrt{A}a^2 .$$

Incidentally, for this particular choice of $\tilde{\rho}$ one obtains also the continuity of $\nu'_{r=a}$ at the boundary suitably choosing the constant ρ_0. The continuity of $g_{rr} = -e^{2\mu}$ is satisfied by $m = \frac{4}{5}\pi\rho_0 a^3$, but its derivative is discontinuous. A discontinuity in the derivative of radial gravitational field is usually encountered on the boundary of systems containing diffuse distribution of matter.[6,7] However, in our case, the discontinuity in the derivative of $e^{2\mu}$ is due to the curvature coordinates and is not incompatible with the set of Eqs. (3.5)–(3.7).

IV. DISCUSSION

Following the scheme of Hehl *et al.*, we obtain a class of solutions that represents a static incoherent spherical distribution of dust in equilibrium under the influences of torsion and spin. From (3.7) we find that the contribution of the classical spin density to T^0_0 is $-2\pi k^2$. This negative contribution of the spin density is an important feature of the Einstein-Cartan theory. Equation (3.8) gives the condition of balance. It behaves as an effective repulsive force.[5] We may think of the effect of spin similar to that of some unconventional matter that produces a repulsive field giving rise to negative energy density unlike the electrostatic field due to the charged dust. The exact solution obtained here is based upon an initial assumption of $\tilde{\rho}=\rho_0 r^2/a^2$, which leads to $\rho=0$ at the center of the sphere. Although such a distribution is not regarded as physical, it seems useful in understanding certain aspects of the Einstein-Cartan theory.

From (3.16) one finds that in the case of the equality of sign, the effective pressure vanishes at the surface of the sphere. The density ρ then increases from zero value at the origin to a maximum at some value of the radial coordinate and then decreases to ρ_0 at $r=a$.

ACKNOWLEDGMENTS

It is a pleasure to thank Dr. A. Banerjee with whom we had many useful discussions. We also express our thanks to CNPq and FINEP of Brazil for financial support.

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