Beam-beam luminosity limitation in electron-positron colliding rings

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To account for observed luminosity limitations in electron-positron colliding rings we identify parametric pumping of vertical betatron oscillations by horizontal oscillations as the leading effect, solve the nonlinear single-particle equation exactly, obtain the strong-beam–strong-beam equilibrium by numerical simulation, calculate the luminosity, and identify regions of bad beam lifetime.

Existing e^+-e^- colliding rings have luminosities less than design values by factors of 10 or more. For sufficiently large beam current I, the luminosity fails to increase proportional to I^2, as it would if the beam shapes remained constant. Also, beyond a current $I_{\max}$, the beam lifetimes become unacceptably short. These effects are due to the "beam-beam" interaction; that is, the electrostatic and magnetostatic forces on the particles in one beam as they pass through the other.

In this paper we identify "parametric pumping" of vertical betatron oscillations by horizontal oscillations as the effect leading to the observed behavior and we describe the solution of the equations governing the situation. The results conform with observations. We also give machine parameters expected to yield good and bad luminosity.

This beam-beam interaction has attracted rather broad interest.[1] A reason for this is that it suggests the possibility of experimental investigation of the onset of stochastic behavior in classical mechanics. Questions first raised in celestial mechanics[2] can, it is hoped, be studied in accelerators. But the presence of strong fluctuations and damping in electron rings (the only case considered here) reduces the characteristic number of revolutions to, say, 10^4 instead of, say, 10^{10} relevant for protons. As a result, recent rigorous mathematical studies of stability[3] do not enter our discussion.

Our theoretical investigations have proceeded at two independent levels: (a) An exact analytic solution of the single-particle equation of motion in the presence of a strong beam and accounting for all resonances, and (b) numerical simulation of the entire strong-beam–strong-beam system, including self-consistent relaxation to equilibrium in both transverse directions. The analytic solution at level (a) identifies the important resonances controlling the situation and gives estimates of the beam currents at which they become important. The numerical simulation at level (b) is necessary for more quantitative predictions. We feel it is obligatory to develop parallel intuitive understanding at levels (a) and (b) to produce reliable quantitative results.

The single-particle equation of motion in the presence of the other beam is a nonlinear difference equation potentially exhibiting arbitrarily many resonances. The vertical component y_m on the mth passage satisfies[4]

$$y_{m+1} - 2y_m \cos\omega_{y_0} + y_{m-1} = \sin\omega_{y_0} \Delta y'(x_m, y_m)\,, \quad (1)$$

where $\omega_{y_0}/2\pi$ is called the vertical single-beam betatron tune ν_{y_0} and similarly for x. $\Delta y'$ is the vertical angular deflection on the mth passage through the other beam. It can be obtained by an electrostatic and magnetostatic calculation from the charge density $\rho(x_m, y_m)$ of the other beam. Since we assume a ribbon beam, as is typical for most e^+-e^- machines, the transverse standard deviations satisfy $\sigma_y \ll \sigma_x$. In that case $\Delta y'$ can be obtained approximately using Gauss's theorem. For Gaussian profiles this yields[5]

$$\Delta y'(x_m, y_m) = -4\pi\left(\frac{\pi}{2}\right)^{1/2} \xi_V e^{-x_m^2/2\sigma_x^2} \sigma_y \,\mathrm{erf}\left(\frac{y_m}{\sqrt{2}\sigma_y}\right). \quad (2)$$

Here ξ_V is the customary vertical "linear tune shift" parameter specifying the strength of the beam-beam interaction. For small values of y_m, a leading term in (2) proportional to y_m can be grouped with the second term in (1) leading to a tune shift ξ_V. For sufficiently large values of ξ_V no real tune would exist, corresponding to exponential growth that would occur except for nonlinearity (such as the next term, proportional to y_m^3, which causes the tune to depend on amplitude).

In practice, controlled vertical beam growth normally sets in at a much lower value of ξ_V than is suggested by the previous paragraph. We understand this as being due to the modulation of $\Delta y'$ due to periodic variation of x_m. At calculational level (a) we assume that x_m is given inexorably by

$$x_m = a_x \cos m\omega_{x_0} \quad (3)$$

Parametric amplification of vertical oscillations can occur through terms in (2) such as $y_m a_x^2 \cos^2(m\omega_{x_0})$, due to horizontal betatron (or synchrotron) oscillations. That is, the vertical oscillations are parametrically pumped by horizontal oscillations in much the way a garden swing is pumped by the systematic shortening and lengthening of the pendulum length, and hence the natural frequency.[6]

We now describe an iterative procedure for solving (1) with x_m given by (3). Assume a double Fourier-series expansion

$$y_m = a_y \cos(m\omega_y) + \sum_{r,s=1}^{\infty} a_{rs} \begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(rm\omega_y)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(sm\omega_{x_0}) + \cdots, \tag{4}$$

where the sum is extended over all combinations of sin and cos. The analysis truncates this series to a finite number of terms. It is not obvious that such an expansion should exist. Damping would usually, but not always, rule it out as it remains finite at large m. Empirically, for relevant values of ξ_V, we have always succeeded in finding such an expansion. The difference of ω_y from ω_{y_0} is due to the perturbation.

When (4) is substituted into (2) a similar expansion for $\Delta y'$ can be made since $\Delta y'$ is periodic in $m\omega_y$ and $m\omega_{x_0}$. The coefficients can be found by finite (fast) Fourier transform (FFT). The eigenfunctions of the linear difference operator on the left-hand side of (1) are linear sinusoids. Expanding the Fourier transforms of y_m and $\Delta y'$ into sums of linear sinusoids (all possible sum and difference frequencies appear) enables the iterative production of new a_{rs}'s from old. An instability threshold ξ_{max} is reasonably well defined as that value at which the number of harmonics necessary for convergence proliferates. Allowing more harmonics, say 32 instead of 16, or more iterations usually makes little difference to the threshold. Normally one or two harmonics are especially large owing to the "resonance denominators" appearing in the iterative scheme.

For values of ξ_V roughly equal to ξ_{max} and higher, phase-space plots of the motion become very contorted and orbits of sufficient amplitude no longer spiral into the origin when damping is turned on. They damp instead to stable limit cycles reminiscent of those of Ref. 6.

At level (b) we have developed a strong-strong numerical simulation which can be used to make quantitative comparison with observations at the Cornell Electron Storage Ring (CESR). Many particles ($\gtrsim 100$) in each of the two beams are tracked for many turns ($\gtrsim 3000$) in six-dimensional phase space around the linear lattice of a storage ring which, like CESR, has two crossing points and one rf cavity. Once per turn the horizontal and vertical betatron angles are damped at the rf cavity by a factor $\exp(-2/\tau)$. The damping time τ is typically 10^3 turns. Energy oscillations are damped at twice this rate. Natural beam sizes are maintained by the competing process of quantum excitation simulated once per turn by a step in six dimensions which is random in direction and amplitude.

Starting from Gaussian distributions with fields such as (2), the distributions and their associated fields are allowed to evolve towards self-consistency during time increments of 0.3τ during which one beam is held rigid while the other relaxes. Distributions are recorded during each increment, in, typically, 300 (turns) $\times$ 100 (particles) $\times$ 2 (crossing points) instances. From these, smooth horizontal and vertical fields are calculated. Here the x and y dependence is assumed to remain factorized as in (2). For the next iteration the roles of rigid and relaxing bunches are reversed. After about 3τ equilibrium has been adequately established. It is possible for a particle to be "lost" if it strikes a mask (typically set at $\pm 10\sigma_x$ and $\pm 10\sigma_y$), and the lifetime is declared bad if it is less than (typically) 10 seconds. Otherwise the luminosity is calculated from the equilibrium bunch distributions which are usually identical, within statistics, at small ξ_V values but which may be quite different at high values, consistent with observations in existing storage rings.[7]

We now turn to the results, first establishing in Fig. 1 that the parametric-pumping model and the simulation yield qualitatively equivalent results. Both calculations were performed at every point on a uniform grid in the tune plane. The granularity of this grid can be inferred from the crosses which mark points of bad lifetime as defined previously, with $\xi_V = 0.08$.[8] At all other points with the same value for ξ_V the luminosity was calculated from the simulation and then the contours of constant relative luminosity were calculated. Regions of very low luminosity are shown shaded. They correspond closely to regions of bad lifetime. Straight lines in the tune plane with integer coefficients represent possible resonances. The lines drawn in Fig. 1 represent those main resonances usually causing instability thresholds as calculated analytically. As ξ_V in (2) increases from zero the perturbed vertical tune ν_y increases until it is near one of these lines and then the motion usually becomes unstable. If ξ_V is increased from 0.08, in the simulation, new valleys of bad lifetime appear for example along the diagonal running from the lower right corner to the upper left corner.

The valleys of bad lifetime and low luminosity

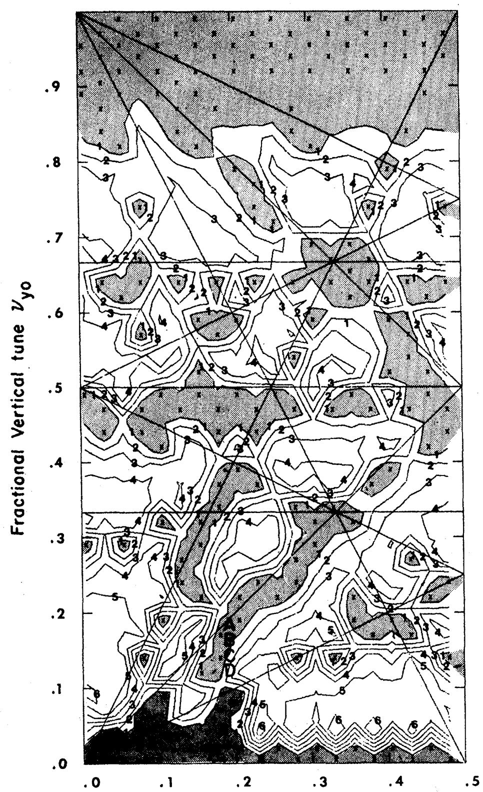

FIG. 1. The tune plane as studied analytically [the straight lines indicate instability thresholds of (1)] and numerically (the contours are curves of constant relative luminosity, very roughly in units of 10^{30} cm^{-2} sec^{-1} for typical parameters). The crosses indicate bad lifetime and the shaded regions have very low luminosity. Points labeled A, B, C, and D identify lattice points for an investigation described in the text.

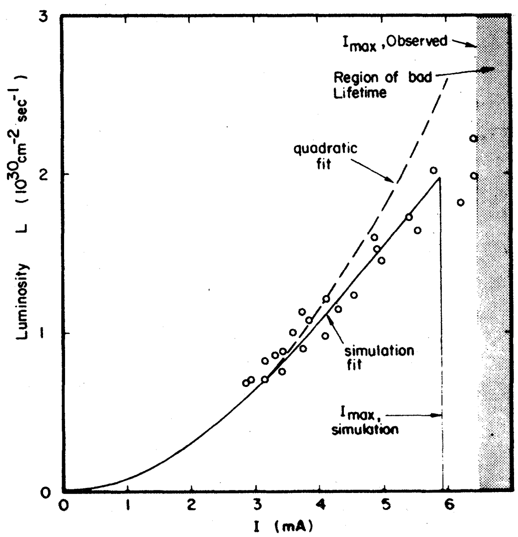

FIG. 2. Dependence of luminosity on current. Comparison between experiment and theory at CESR. Limits imposed by bad beam lifetime are shown. For these data the tune parameters were 9.415 for ν_{x0} and 9.345 for ν_{y0}.

are strikingly parallel to the analytically calculated resonance lines. They are shifted to slightly lower values of ν_y as expected due to the linear tune shift.

In Fig. 1 various effects have been neglected for easier comparison between the two calculations, the main ones being that ξ_H was taken as zero and energy dispersion was assumed to be zero. Including them does not change the qualitative picture, though with energy dispersion new valleys of bad lifetime appear which are consistent with being due to "synchrobetatron" resonances. The results are also insensitive to the presence or absence of noise in the horizontal motion provided the horizontal profile is kept unchanged.

In Fig. 2 a comparison between the simulation, with horizontal and longitudinal effects treated realistically, and observation at CESR is shown. There is good agreement on the dependence of L on I and approximate agreement on the value I_{max} beyond which the lifetime is bad. In this plot the parameters have been adjusted to fit the data in the lower-I region where $L \propto I^2$. This is largely a matter of convenience as the expected simple relation between measured beam profiles and I and L is approximately satisfied at small I. The only other arbitrariness in Fig. 2 relates to the location of I_{max} for which the lifetime is 10 seconds (much less than is acceptable in practice) with masks at $\pm 10\sigma$ (much smaller than is achievable in practice). Actual apertures, especially vertical, are not well known. When the vertical mask was raised to $\pm 12\sigma$ and the lifetime to 100 seconds, I_{max} was found to be almost unchanged.

Another corroboration of the theory can be obtained, semiquantitatively, by comparison with experience of the Stanford facility SPEAR at the four lattice points labeled A, B, C, and D in Fig. 1. They found[7] that the maximum luminosity increased steadily by a factor of about 5 in proceeding from A to D, but they could not cross the line

$$\nu_{x_0} = 2\nu_{y_0}$$

with two beams in spite of the fact that the presence of this line was undetectable with single beams. These observations are quite consistent with Fig. 1.

Finally, it is of interest to find optimal running parameters according to the theory. It is plausible, and tends to be borne out by the simulation that energy dispersion is harmful. Also from Fig. 1 the region around $\nu_{x_0}=0.4$, $\nu_{y_0}\lesssim 0.1$ appears to be the most promising, but we have not expended enough computer time to prove this. On the other hand, many unambiguously bad regions have been identified which should be avoided in practice.

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[1]M. Month and J. C. Herrera, in *Nonlinear Dynamics and the Beam-Beam Interaction*, proceedings of the Symposium, Brookhaven National Laboratory, 1979, edited by M. Month and J. C. Herrera (AIP, New York, 1979), p. 57; H. S. Uhm and C. S. Lin, Phys. Rev. Lett. 43, 914 (1979). They attacked the same problem but the picture presented in our paper is totally different and more detailed.

[2]H. Poincaré, *Les Methodes Nouvelles de la Mecanique Celeste III* (Gautiers-Villars, Paris, 1892); G. D. Birkhoff, *Dynamical Systems*, revised edition (Amer. Math. Soc., Providence, Rhode Island, 1966).

[3]J. Moser, in *Nonlinear Dynamics and the Beam-Beam Interaction* (Ref. 1). Numerous other references are given here.

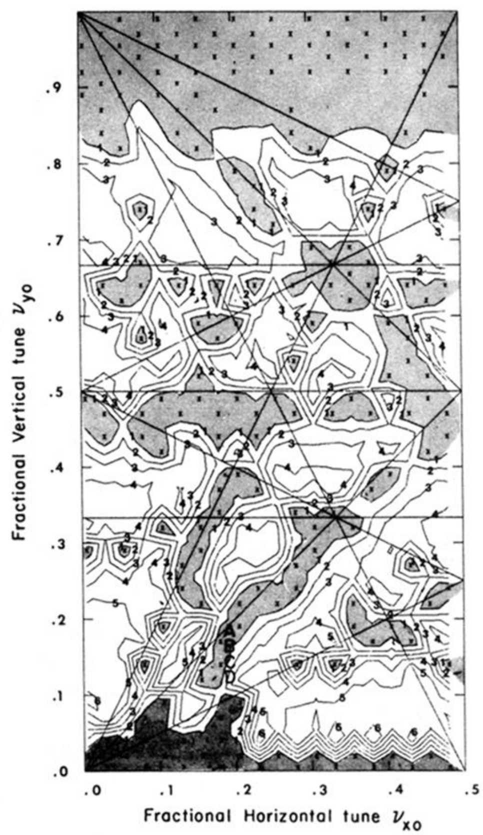
[4]R. H. G. Helleman, Ref. 1.; R. Talman, Cornell internal report, 1976 (unpublished).

[5]An exact analytical expression for Δ' can be obtained from M. Bassetti and G. A. Erskine, Report No. CERN-ISR-TH/80-06 (unpublished), in terms of the complex error function. Rational approximations for the evaluation of this expression are contained in Y. Okamoto and R. Talman, Cornell Report No. CBN 80-13 (unpublished).

[6]K. Magnus, *Vibrations* (Blackie, Glasgow, 1965); N. N. Bogoliubov and Y. A. Mitropolsky, *Asymptotic Methods in the Theory of Non-Linear Oscillators* (Gordon and Breach, New York, 1961).

[7]H. Wiedemann, Ref. 1.

[8]As used here ξ_V is the "unrenormalized" tune-shift parameter which would be observed if the beams maintained their single-beam profiles. The measured or "renormalized" value would be less by, typically, a factor of 2.



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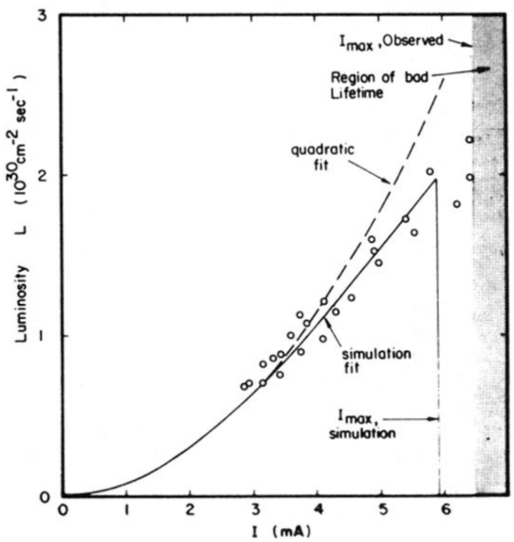

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