How large are the neutrino mixing angles?

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We find that in a broad class of theories consistent with grand unification, the neutrino mixing angles are likely to be comparable to the corresponding quark mixing angles and may well be much larger in an interesting special case. This result holds for a wide range of mass ratios for the light-neutrino Majorana masses.

In the last few years, several authors[1] have noted that concepts from grand unified theories (GUT's) and from phenomenological quark analyses can be combined to provide some idea of possible masses and mixing angles in the neutral-lepton sector.[2] The general argument advanced by Gell-Mann, Ramond, and Slansky[3] is that the mass matrix will have $\Delta I^w = 0$ Dirac terms on the order of quark masses and must then have $\Delta I^w = 0$ Majorana mass terms which are very large, perhaps on the order of the grand unification mass. Such a mass matrix would then induce very small masses of a $\Delta I^w = 1$ Majorana character which would be associated with the neutrinos produced in ordinary weak interactions.[4]

In this paper, we explore further consequences of this idea and point out that, for various reasonable assumptions, it is natural to find rather large neutrino flavor mixings (in some cases, even larger than those observed in the quark sector) which may be or may soon be observable in oscillation experiments[5] or in nuclear double-β decay[6] if the overall neutrino mass scale is large enough. For this present work, we confine ourselves to the now "standard" model of three "generations" which are arranged in left-chiral doublets and right-chiral singlets under the weak interaction.[7]

We shall utilize the formalism introduced by Kobayashi and Maskawa[8] (KM) as it applies to GUT's such as[9] SU(5) and SO(10). We assume that the weak currents are defined by the mass eigenstates of the $Q = -\frac{1}{3}$ quarks and by their GUT partners, the charged leptons. Generation (or flavor) mixing is then described by the rotation from that basis to the basis of mass eigenstates for the $Q = +\frac{2}{3}$ quarks and for the neutral leptons. Since some of the parameters of the quark rotation matrix are not well known, and the top quark has not yet been directly encountered, we choose a parametrization of the undiagonalized quark mass matrix (KM matrix) that allows us to study the effects of varying these quantities. For modest mixing angles, it is convenient to follow Kolb and Goldman[2] and assume a rotation matrix of the form (neglecting CP violation for now)[10]

$$V = R_1(\theta_2)R_3(\theta_1)R_2(\theta_3), \tag{1}$$

where $R_i(\theta_j)$ is a 3×3 rotation matrix through angle θ_j with the ith axis as the rotation axis. We define $S_i = \sin\theta_i$. S_1 is the sine of the Cabibbo angle (as conventionally defined if θ_3 is not too large) and well fixed[11]; we take it to be 0.22. Furthermore, in this parametrization representation, S_3 may be very small[10] and so its precise value is unimportant. Only the parameter S_2 has a significantly large range $(0.1 < S_2 < 0.6)$ of possible values.[10] The KM mass matrix in the current representation is then

$$\underline{m}_q = \underline{V}\,\underline{m}_D\,\underline{V}^T, \tag{2}$$

where $\underline{m}_D$ is the diagonal matrix with with up-quark, charm-quark, and top-quark mass values in that order down the diagonal; m_t is an unknown parameter.

To see what this implies for the neutrino mass matrix, we generalize the discussion in Refs. 1 and 3 and write, in an obvious notation,

$$\underline{m}_\nu = \begin{pmatrix} \underline{0} & \underline{m}_q \\ \underline{m}_q & \underline{M} \end{pmatrix}, \tag{3}$$

where the M_{ij} are $\Delta I^w = 0$ Majorana masses with values (presumably) $\geq 10^9$ GeV. Clearly there remains a large degree of model freedom in this $\Delta I^w = 0$ sector. However, within the framework of GUT's,[3,9] whatever one chooses for $\underline{M}$, the off-diagonal 3×3 submatrix should be related to the KM Dirac quark-mass matrix defined in Eq. (2). We have set the $\Delta I^w = \frac{1}{2}$ neutrino masses to exactly the quark values. In some GUT's, these will differ by Clebsch-Gordan factors. We will return below to the effect of this on our results.

Now this off-diagonal submatrix has entries of order 1 GeV, whereas $\underline{M}$ is presumed to have entries on a scale of 10^9–10^{15} GeV. This disparity in scales is large enough that one may reasonably expect that a second-order perturbation theory

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may be used to understand the structure of the $\Delta I^{w}=1$ submatrix that is induced by diagonalizing the full 6×6 matrix. In this approximation

$$\underline{\tilde{U}}=\begin{pmatrix} \underline{U} & \underline{M}^{-1}\underline{m}_q\underline{U} \\ -\underline{M}^{-1}\underline{m}_q\underline{U} & \underline{U}\end{pmatrix} \tag{4}$$

is the unitary (to order m^2/M^2) matrix which block diagonalizes $\underline{m}_\nu$ in Eq. (3) to the form

$$\underline{m}_{\nu B}=\begin{pmatrix} -\underline{U}^T\underline{m}_q\underline{M}^{-1}\underline{m}_q\underline{U} & \underline{0} \\ \underline{0} & \underline{U}^T\underline{M}\,\underline{U}\end{pmatrix}, \tag{5}$$

where again the off-diagonal 3×3 blocks vanish only to order m^3/M^2. We denote the $\Delta I^{w}=1$ sector submatrix by $\underline{\mu}$. $\underline{U}$ is an as yet unspecified 3×3 unitary matrix.

For a variety of cases which may be of interest, $\underline{M}\propto(\underline{m}_q)^n$ where n is an integer. In these cases, $\underline{U}=\underline{V}$ in Eqs. (4) and (5) *completely* diagonalizes $\underline{m}_\nu$ provided $\underline{M}$ is not singular. For different values of n, the neutrino spectrum in $\underline{\mu}$ will vary dramatically, *but the mixing angles will remain identical to those in the quark sector*. For example, if $\underline{M}$ is proportional to the unit matrix ($n=0$) the neutrino mass ratios will vary as the square of the corresponding quark masses, while if $n=1$ the neutrino and quark-mass ratios will be the same. In either case, however, the mixing angles will be identical to the KM angles for the quarks. Should[12] $n=2$, the neutrinos will appear degenerate up to higher-order terms in the perturbation expansion or up to the generation-dependent effects of radiative corrections.[13] These could lead to arbitrarily *large* further mixing.

The form of $\underline{M}$ assumed for this argument may seem too restrictive. A perturbation argument expanding $\underline{M}$ about a linear combination of three of the above forms[14] with $n=0,1,2\cdots$ extends the result somewhat, but the range of validity is unclear (except for the exclusion of singular $\underline{M}$ about which more below). We have therefore studied numerically the range of validity of these results by diagonalizing 6×6 matrices of the form in Eq. (3), but where the entries of $\underline{M}$ are chosen *at random*, except that $\underline{M}$ is required to be symmetric and of sufficiently large overall scale.

Typical results of 10^5 trials are shown in Fig. 1. For virtually every case, S_1 stays within ± 0.03 of 0.22. The sine of the angle representing charm-top mixing undergoes somewhat greater variations, but throughout the allowed range[10] of S_2 values in the quark sector, the full width at half maximum (FWHM) of the induced neutrino mixing is 0.1 and approximately 80% or more of the events lie within ± 0.1 of the quark sector value of S_2. The results are similar for the sine of the third angle, although they may represent a much larger percentage effect as this angle is thought to be

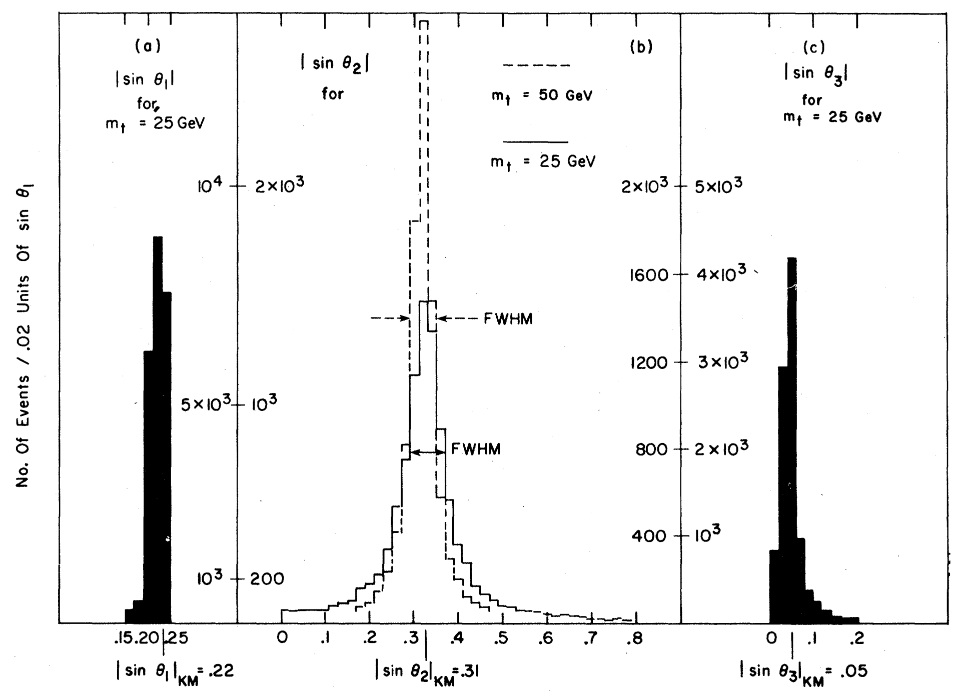

FIG. 1. Typical distributions of neutrino mixing angles induced by random Majorana mass matrices for the heavy neutrinos. The corresponding quark mixing angle is marked. (a) e-μ mixing; (b) μ-τ mixing; (c) e-τ mixing.

smaller than the others.[10] The neutrino mass spectrum varies widely, but except for the $n \geq 2$ cases, the mass hierarchy is maintained. [For $n \geq 3$, the hiearchy is reversed; when the scale is adjusted to obtain ~ 10 eV mass for the electron neutrino, the lightest right-handed neutrino acquires a mass ~ 3 TeV and mixes with ordinary (left-handed) neutrinos with an amplitude $\leq 10^{-6}$.] Of course, nature does not provide a statistical distribution for $\underline{M}$, but we believe these results demonstrate that unless $\underline{M}$ has peculiar properties, (eg., det $\underline{M}=0$) then large (Cabibbo-type) mixing angles may be expected in the neutrino sector. This result directly contradicts the generalization from the quark sector that the square of the tangent of the mixing angle equals the ratio of the fermion masses in the neutral-lepton sector also.

As an example of a case in which $\underline{M}$ is singular, we considered the interesting possibility that the $\Delta I^w=0$ sector is completely indifferent to generation, i.e.,

$$\underline{M}=M_0\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}. \tag{6}$$

In this case, two of the eigenvalues of $\underline{M}$ vanish, so Eq. (5) clearly cannot apply. In fact, this model results in a very light Majorana neutrino, a very massive ($3M_0$) Majorana neutrino, and two *Dirac* neutrinos with charm- and top-quark masses. The reason for this is simply that the small eigenvalues of $\underline{M}$ can undergo large mixing with those in the (initially) $\underline{\mu}=0$ sector. We believe this model is not viable. Nevertheless, it is interesting to note that even here the mixing of the light Majorana neutrino to the other generations is little changed from that of the corresponding quark.

Except for the case of singular $\underline{M}$, the light neutrinos we find are essentially pure Majorana particles with a Dirac admixture of order m/M. This, and our other results above, will continue to hold even if the number of generations is found to be larger than three.

Finally, we considered the possibility that the number of heavy Majorana neutrinos does not match the family structure.[15] For p additional heavy neutrinos, $\underline{M}$ must be generalized to a $3\times(p+3)$ matrix. For $p=1$, and the new, fourth column of $\underline{m}$ on the order of 1 GeV, another numerical study again showed very little effect on the mixing angles due to this change. We expect this result to continue to hold at least for modest values of p.

Of course, one may always choose a model in which the neutrino masses induced here are overwhelmed by direct, $\Delta I^w=1$ contributions to $\underline{\mu}$ from a new Higgs scalar [e.g., a 15-plet in SU(5)]. However, the new vacuum expectation value (VEV) distorts the usual relation between the W- and Z-boson masses and the Weinberg angle:

$$M_W{}^2M_Z{}^{-2}\cos^{-2}\theta_W=1-0.5\lambda^2\eta^{-2}, \tag{7}$$

where λ is the VEV of the new $I^w=1$ Higgs field and η is the VEV of the Weinberg doublet. Experimental results[16] then limit $\lambda/\eta \leq 0.25$ and the usual arguments which limit the mass of the Higgs scalar which survives from the Weinberg doublet put a correspondingly reduced upper limit on the new scalars, some of which carry electric charge [$Q=2$ in the SU(5) example]. This would seem reasonably amenable to stringent experimental test.

In some GUT's, the representation of Higgs scalars may differ in their quark and lepton couplings by Clebsch-Gordan factors. Thus, the angles which diagonalize the $\Delta I^w=\frac{1}{2}$ lepton mass matrix alone may differ by ratios of such factors from the corresponding angles in the quark sector. Nonetheless, we would still find that the $\Delta I^w=0$ lepton mass matrix usually induces only small additional effects. Our results may therefore be phrased more generally as showing that, except in the special cases discussed, the exact full lepton mixing is very similar to that obtained by diagonalizing the $\Delta I^w=\frac{1}{2}$ lepton mass matrix alone.

In summary, our investigation shows that in this class of models, the neutrino mixing angles may be expected to be comparable to, or even larger than, the corresponding quark mixing angles. If this is observed not to be the case, then one of the following is true. (1) The GUT ideas are inapplicable here. (2) $\underline{M}$ has a singular character (this may require extra neutrino degrees of freedom to avoid unacceptable Dirac-mass neutrinos). (3) Bare $\Delta I^w=1$ mass terms dominate the light-neutrino mass spectrum. (This is subject to the direct experimental test of the existence of a sufficiently light, charged Higgs scalar.) Although the first alternative seems to be undergoing a rapid decline in popularity, the other two offer interesting areas of study.

We conclude from our analysis that experiments which look for neutrino mixing should probably be designed with mixing angles similar to the quark mixing angles in mind. Information that these angles are much smaller than this could significantly affect theoretical considerations.

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Note added. After this work was completed, we learned of similar papers by K. Milton and K. Tanaka [Phys. Rev. D 23, 2087 (1981)] and by L. Wolfenstein [Carnegie-Mellon Report No. C00-3066-160 (unpublished)].

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[1]These ideas have been emphasized by P. Ramond, see, e.g., P. Ramond, in *Proceedings of the First Workshop on Grand Unification*, edited by P. Frampton, S. L. Glashow, and A. Yildiz (Math Science Press, New Hampshire, 1980); in *High Energy Physics—1980*, proceedings of the XXth International Conference, Madison, Wisconsin, edited by L. Durand and L. G. Pondrom (AIP, New York, 1981). They have also been discussed by E. Witten, Phys. Lett. 91B, 81 (1980); L. Wolfenstein, in Proceedings of the Neutrino Mass Miniconference, Telemark, Wisconsin, 1980, edited by V. Barger and D. Cline (unpublished). For a study of Dirac neutrino masses, see T. P. Cheng and L.-F. Li, Phys. Rev. D 17, 2375 (1978).

[2]For phenomenological analyses of these angles see, e.g., B. W. Lee *et al.*, Phys. Rev. Lett. 38, 937 (1977); B. W. Lee and R. Shrock, Phys. Rev. D 16, 1444 (1977); K. Sato and M. Kobayashi, Prog. Theor. Phys. 58, 1775 (1977); E. W. Kolb and T. Goldman, Phys. Rev. Lett. 43, 897 (1979); A. De Rújula *et al.*, Nucl. Phys. B168, 54 (1980); V. Barger *et al.*, Phys. Lett. 93B, 194 (1980).

[3]M. Gell-Mann, P. Ramond, and R. Slansky, in *Supergravity*, edited by P. van Nieuwenhuizen and D. Freedman (North-Holland, Amsterdam, 1979).

[4]If this, or some other complication, is not introduced, the neutrinos would be Dirac particles with masses similar to those of the quarks.

[5]We are aware of experiments underway by groups from the University of California, Irvine and from Caltech and Grenoble, and of proposals at the Los Alamos Meson Physics Facility by groups from Argonne National Laboratory, Los Alamos National Laboratory, University of Houston, Ohio State University, and Rice University.

[6]W. C. Haxton, G. J. Stephenson, Jr., and D. Strottman, Los Alamos Report No. LA-UR-81-1388 (unpublished); G. J. Stephenson, Jr., in Proceedings of the Neutrino Mass Miniconference, Telemark, Wisconsin, 1980, edited by V. Barger and D. Cline (University of Wisconsin, Madison, 1981).

[7]S. L. Glashow, Nucl. Phys. 22, 579 (1961); S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967); A. Salam, in *Elementary Particle Theory: Relativistic Groups and Analyticity (Nobel Symposium No. 8)*, edited by N. Svartholm (Wiley, New York, 1969).

[8]M. Kobayashi and K. Maskawa, Prog. Theor. Phys. 49, 652 (1973).

[9]H. Georgi and S. L. Glashow, Phys. Rev. Lett. 32, 438 (1974). For a survey of unified theories, see M. Gell-Mann, P. Ramond, and R. Slansky, Rev. Mod. Phys. 50, 721 (1978).

[10]We prefer this decomposition to that in Ref. 8 because in Ref. 8 $\theta_2+\theta_3$ is the angle that naturally describes mixing between the second and third quark generations, whereas here it is θ_2 itself. Furthermore, here θ_3 alone describes first- and third-generation mixing which seems to be small; see, e.g., Lee and Shrock (Ref. 2); J. Ellis *et al.*, Nucl. Phys. B131, 285 (1977); R. Shrock and L.-L. Wang, Phys. Rev. Lett. 41, 1692 (1978); R. Shrock, S. B. Treiman, and L.-L. Wang, *ibid.* 42, 1589 (1979); V. Barger, W. F. Long, and S. Pakvasa, *ibid.* 42, 1585 (1979). For experimental evidence that $\theta_3 < \theta_2$ in our parametrization, see D. Andrews *et al.*, Phys. Rev. Lett. 45, 219 (1980) and K. Berkelman, in *High Energy Physics—1980* (Ref. 1).

[11]See, for example, E. D. Commins, *Weak Interactions* (McGraw-Hill, New York, 1973); L. B. Okun, *Weak Interaction of Elementary Particles* (Pergamon, Oxford, 1965).

[12]We have found no attractive model to support this conjecture. In general, this pattern may occur if these $\Delta I^w=0$ masses arise from a higher-order-induced coupling of the symmetric product of $\Delta I^w=\frac{1}{2}$ vacuum expectation values.

[13]These differences are due to the mass differences of quarks and charged leptons in the virtual intermediate states.

[14]Only three such terms are needed as they span the space of these simultaneously diagonable matrices.

[15]A general approach to models with differing numbers of $I^w=0$ and $I^w=\frac{1}{2}$ neutrinos may be found in J. Schecter and J. Valle, Phys. Rev. D 22, 2227 (1980). Specific models and classes of models of these types are discussed in Ref. 3 and more recently by F. Gursey, P. Ramond, and P. Sikivie, Phys. Lett. B60, 177 (1975); A. Davidson, K. C. Wali, and P. Mannheim, Phys. Rev. Lett. 45, 1135 (1980).

[16]For example, as reported by K. H. Mess, in *High Energy Physics—1980* (Ref. 1.)