Stokes's theorems for non-Abelian fields

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We study Wilson loops in quantum chromodynamics to discuss two versions of a non-Abelian Stokes theorem, one of which is stated as an expansion in local operators of the theory.

I. INTRODUCTION

Manifestly gauge-invariant formulations of quantum chromodynamics (QCD) are of great interest. Attempts have been made[1] to formulate the theory in terms of the so-called Wilson loop operators $W(C)$ (Ref. 2). These are defined as

$$W(C) \equiv \mathrm{Tr}\,O(C) = \mathrm{Tr}\,P\exp\left(\oint_C dz_\mu A_\mu(z)\right). \quad (1.1)$$

The $A_\mu(z)$ are $N\times N$ matrix-valued fields [for SU(N)] defined in terms of the generalized Gell-Mann matrices λ^a,

$$A_\mu(z) = \tfrac{1}{2} g \sum_{a=1}^{N^2-1} \lambda^a A_\mu^a(z)\,, \quad (1.2)$$

and P represents the color-space path ordering. By this we mean that if the continuous, closed, smooth curve C is parametrized by s, so that

$$z_\mu = z_\mu(s), \quad 0 \leq s \leq 1$$
$$z_\mu(0) = z_\mu(1)\,,$$

then

$$O(C) = \sum_{n=0}^{\infty} \int_0^1 ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n \frac{dz_{\mu_1}(s_1)}{ds_1} \cdots \frac{dz_{\mu_n}(s_n)}{ds_n} A_{\mu_1}(s_1)\cdots A_{\mu_n}(s_n)\,. \quad (1.3)$$

Because of the color trace operation, $W(C)$ is independent of the starting point $z_\mu(0)$.

There have also been programs[3] that deal with gauge-invariant local operators acting on the physical vacuum state. In studying the connection between these methods we were led to look for a non-Abelian generalization[4,5] of the Stokes theorem. This paper deals with our approach to this problem. After this work was completed, we became aware of a rigorous proof[6] of the theorem. Our treatment is more heuristic and thus somewhat simpler. In any case we concentrate on aspects that do not appear in the literature. We discuss two methods: One is based on a differential approach and leads to identification of $W(C)$ with the trace of the exponential of an integral over an area that spans the loop. The second is based on generalized Baker-Hausdorff techniques and leads to a connection between the Wilson loop operators and an expansion in terms of local operators. In this expansion gauge-covariant operators acting on the non-Abelian field tensor appear where ordinary derivatives acting on the curl of the field appear in the usual Abelian Stokes theorem. The local operator expansion is equivalent to the first form of the non-Abelian Stokes theorem.

II. DIFFERENTIAL-EQUATION APPROACH

Consider the operator $O(C)$ defined in (1.1) for a given smooth closed contour C. Consider some surface Σ spanning the contour (Fig. 1). Let the contour C' consist of C deformed so as to include an infinitesimal surface element $d\sigma_{\mu\nu}(y)$ in Σ. In compact notation we have

$$O(C) = P(e^{\int_A^B}) \quad (2.1)$$

and

$$O(C') = P(e^{\int_D^A})P(e^{\oint})P(e^{\int_B^C})P(e^{\int_A^B})\,, \quad (2.2)$$

where $P(e^{\oint})$ is the ordered integral around the infinitesimal area $d\sigma_{\mu\nu}(y)$. It is easy to show that (see Sec. III)

$$P(e^{\oint dz_\mu A_\mu(z)}) \cong 1 + d\sigma_{\mu\nu}(y) G_{\mu\nu}(y)\,, \quad (2.3)$$

where

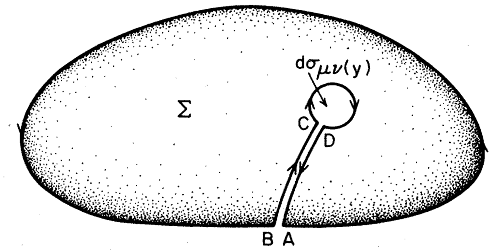

FIG. 1. Surface Σ spanning the Wilson loop contour C, with infinitesimal surface element $d\sigma_{\mu\nu}$ at point y.

$$G_{\mu\nu}(y)=\frac{\partial}{\partial y_\mu}A_\nu-\frac{\partial}{\partial y_\nu}A_\mu-[A_\mu(y),A_\nu(y)] \qquad (2.4)$$

is the non-Abelian field tensor. Thus

$$O(C')=P(e^{\int_D^A})[1+d\sigma_{\mu\nu}(y)G_{\mu\nu}(y)]\,P(e^{\int_C^B})O(C)\,. \qquad (2.5)$$

If we write

$$P(e^{\int_D^A})\equiv U_{AD} \qquad (2.6)$$

and note that

$$P(e^{\int_C^B})=U_{BC}=U_{DA}=U_{AD}^{-1}\,, \qquad (2.7)$$

we get

$$O(C')=O(C)+U_{AD}d\sigma_{\mu\nu}(y)G_{\mu\nu}(y)U_{DA}O(C)\,. \qquad (2.8)$$

Thus

$$\delta O=U_{Ay}G_{\mu\nu}(y)U_{yA}d\sigma_{\mu\nu}(y)O\,. \qquad (2.9)$$

The sign corresponds to going around the infinitesimal area in a counterclockwise direction. The integration of this equation requires that an ordering be specified, since the integrand $U_{Ay}G_{\mu\nu}(y)U_{yA}$ is matrix valued. There is an ordering that appears natural, in that it reproduces the way in which the Stokes theorem is proved for an Abelian theory. Consider a square consisting of four infinitesimal squares as shown in Fig. 2. (We limit our discussion to surfaces that can be mapped onto squares.) We write [see Fig. 2(a)]

$$P\exp\left(\oint A_\mu(x)dx_\mu\right)=U_{01}U_{12}U_{23}U_{34}U_{45}U_{56}U_{67}U_{70} \qquad (2.10)$$

in the form

$$[U_{01}U_{18}U_{87}U_{70}]U_{07}U_{78}[U_{81}U_{12}U_{23}U_{38}]U_{87}U_{70}\vdots\, U_{07}U_{78}$$
$$\times U_{83}U_{34}U_{45}U_{58}]U_{87}U_{70}\vdots\, U_{07}[U_{78}U_{85}U_{56}U_{67}]U_{70}\,. \qquad (2.11)$$

The terms enclosed in the square brackets are loops around the elementary squares, and the remaining U's take us from the starting point 0 to the loop and back. The above may be written in

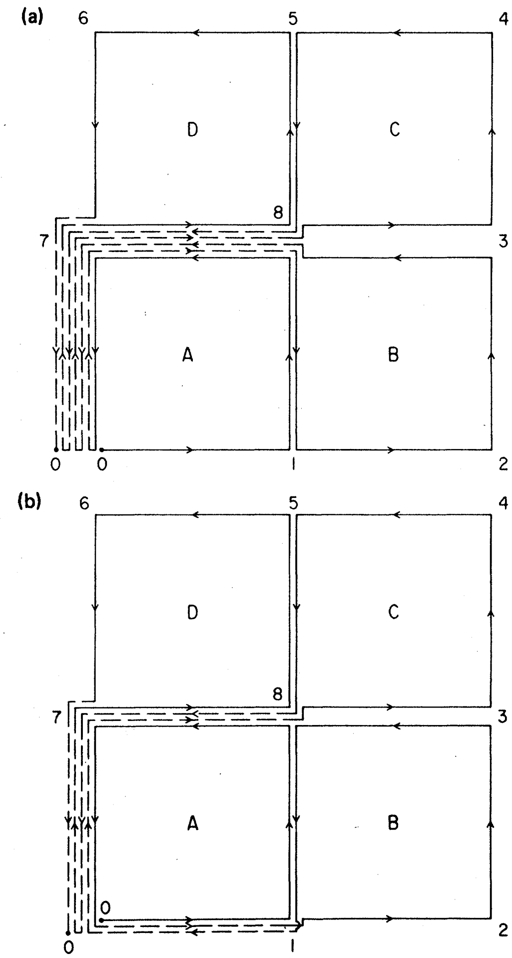

FIG. 2. Paths for loop integral 0→2→4→6→0 [Eq. (2.10)] as succession of subpaths UGU^{-1}. Dashed lines give U and U^{-1}, solid lines give G. (a) A "good" path [Eq. (2.11)]. (b) A "bad" path, which would not give the loop integral. Note that the only difference between (a) and (b) is in the approach and return from subsquare B.

the form

$$(1+d\sigma_{\mu\nu}G_{\mu\nu})_A\{U_{01}U_{14}(1+d\sigma_{\mu\nu}G_{\mu\nu})_D U_{41}U_{10}\}$$
$$\times\{U_{01}U_{14}(1+d\sigma_{\mu\nu}G_{\mu\nu})_C U_{41}U_{10}\}$$
$$\times\{U_{01}(1+d\sigma_{\mu\nu}G_{\mu\nu})_B U_{10}\}\,,$$

which is a sum of contributions of the type

$$U_{0y}(1+d\sigma_{\mu\nu}G_{\mu\nu})_y U_{y0}\,.$$

The ordering is such that a U_{ij} is always adjacent to a U_{ji} after the relation $U_{kl}U_{lk}=1$ is taken into account. In an Abelian theory a U_{ij} will cancel a U_{ji} no matter where it appears in the sequence. Thus in Fig. 2(b) we show a path which will *not* lead to a Stokes theorem for the non-Abelian case, although it is as good as the Fig. 2(a) path for the

Abelian case. The above procedure is easily generalized to a collection of four 2×2 squares and so on, so that with this ordering understood we have

$$O(C)=P\exp\left(\int d\sigma_{\mu\nu}(y)U_{Ay}G_{\mu\nu}(y)U_{yA}\right) \tag{2.12}$$

and the gauge invariant

$$W(C)=\mathrm{Tr}P\exp\left(\int d\sigma_{\mu\nu}(y)U_{Ay}G_{\mu\nu}(y)U_{yA}\right). \tag{2.13}$$

For an Abelian theory this reduces to the usual Stokes theorem written in the form

$$\exp\left(\int_c A_\mu dz_\mu\right)=\exp\left(\int d\sigma_{\mu\nu}F_{\mu\nu}\right).$$

The difference between the two forms, aside from the P orderings, lies in the replacement of the electromagnetic field tensor

$$F_{\mu\nu}(y)=\frac{\partial}{\partial y_\mu}A_\nu-\frac{\partial}{\partial y_\nu}A_\mu \tag{2.14}$$

by $G_{\mu\nu}(y)$, and in the presence of the U factors. These ensure that the surface integral is independent of the choice of surface. To see this, consider an element $d\sigma_{\mu\nu}(y)$ on Σ and deform it, keeping the boundary fixed, so that the new surface element and the old one form a closed surface that encloses a volume δV. Gauss's theorem states that

$$\int_{\substack{\text{closed}\\ \text{surface}}} d\sigma_{\mu\nu}(UG_{\mu\nu}U^{-1})=\int_{\delta V} dV_\mu e_{\mu\nu\rho\sigma}\frac{\partial}{\partial y_\nu}(UG_{\rho\sigma}U^{-1}). \tag{2.15}$$

Now

$$\begin{aligned}\frac{\partial}{\partial y_\mu}(U_{Ay}G_{\rho\sigma}(y)U_{yA}) &= -U_{Ay}A_\mu(y)G_{\rho\sigma}(y)U_{yA}+U_{Ay}\frac{\partial}{\partial y_\mu}G_{\rho\sigma}(y)U_{yA}+U_{Ay}G_{\rho\sigma}(y)A_\mu(y)U_{yA}\\ &= U_{Ay}\left(\frac{\partial}{\partial y_\mu}G_{\rho\sigma}(y)-[A_\mu(y),G_{\rho\sigma}(y)]\right)U_{yA}=U_{Ay}D_\mu G_{\rho\sigma}U_{yA}.\end{aligned} \tag{2.16}$$

Hence the right-hand side of (2.15) is equal to

$$\int dV_\mu U_{Ay}(e_{\mu\nu\rho\sigma}D_\nu G_{\rho\sigma})U_{yA}=\int dV_\mu U_{Ay}D_\nu G^*_{\mu\nu}U_{yA},$$

which vanishes because of the Bianchi identity

$$\tfrac{1}{2}e_{\mu\nu\rho\sigma}D_\nu G_{\rho\sigma}=D_\nu G^*_{\mu\nu}=0. \tag{2.17}$$

The left-hand side of (2.15) suggests that $UG_{\mu\nu}U^{-1}$ be viewed as a flux. It should be noted however that this quantity depends on the choice of path in the string operator U_{Ay} and is thus not uniquely defined. This arises because the color field is itself a source of color flux lines.

III. EXPANSION IN TERMS OF LOCAL OPERATORS

The approach that we initially used to arrive at the form of the Stokes theorem was to use generalized Baker-Hausdorff identities. Let us for definiteness consider a rectangular loop in the (12) plane (Fig. 3). We have

$$O_\lambda(C)=e^{\lambda L_4}e^{\lambda L_3}e^{\lambda L_2}e^{\lambda L_1} \tag{3.1}$$

with

$$L_1=\int_{x_1}^{x_1+a}dx_1'A_1(x_1',x_2), \tag{3.2a}$$

$$L_2=\int_{x_2}^{x_2+b}dx_2'A_2(x_1+a,x_2'), \tag{3.2b}$$

$$L_3=\int_{x_1+a}^{x_1}dx_1'A_1(x_1',x_2+b), \tag{3.2c}$$

$$L_4=\int_{x_2+b}^{x_2}dx_2'A_2(x_1,x_2'). \tag{3.2d}$$

We are suppressing the dependence on the remaining coordinates and will set $\lambda=1$ at the end of the calculation.

We now write

$$O_\lambda(C)\equiv\exp[\lambda\Sigma+F(\lambda)]\equiv\exp[H(\lambda)], \tag{3.3}$$

where

$$\Sigma=L_1+L_2+L_3+L_4. \tag{3.4}$$

The factor $F(\lambda)$ is present because of the non-Abelian character of the A_μ. Let us write

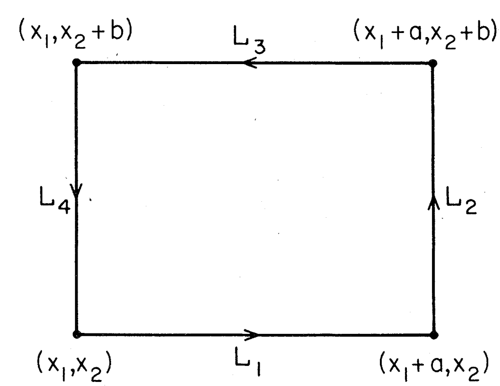

FIG. 3. Rectangular loop in the (12) plane, with individual legs L_i labeled.

$$F(\lambda)=\sum_{n=2}^{\infty}\lambda^n F_n \tag{3.5}$$

and obtain recursion relations for the F_n by differentiating both sides of

$$e^{\lambda L_4}e^{\lambda L_3}e^{\lambda L_2}e^{\lambda L_1}=e^{\lambda\Sigma+F(\lambda)} \tag{3.6}$$

with respect to λ and identifying powers of λ. On the left side we get

$$\frac{d}{d\lambda}O_\lambda(C)=(L_4+e^{\lambda L_4}L_3e^{-\lambda L_4}+e^{\lambda L_4}e^{\lambda L_3}L_2e^{-\lambda L_3}e^{-\lambda L_4}+e^{\lambda L_4}e^{\lambda L_3}e^{\lambda L_2}L_1e^{-\lambda L_2}e^{-\lambda L_3}e^{-\lambda L_4})O_\lambda(C)\,. \tag{3.7}$$

Expanding in powers of λ we get, after a little algebra,

$$\frac{d}{d\lambda}O_\lambda(C)=\Sigma+\lambda\{[L_4L_3]+[(L_4+L_3)L_2]+[(L_4+L_3+L_2)L_1]\}+\cdots+\frac{1}{n!}\lambda^n\{[L_4^nL_3]+[(L_4+L_3)^nL_2]+[(L_4+L_3+L_2)^nL_1]\}+\cdots\,. \tag{3.8}$$

We have introduced the notation

$$[A_1\ldots A_n]=[A_1,[A_2,\ldots[A_{n-1},A_n]\ldots]] \tag{3.9}$$

and it is understood that in the above expression the order (4321) is to be preserved in every term of the multiple commutator. Thus, for example,

$$\begin{aligned}[(L_4+L_3)^2L_2]&=[L_4^2L_2]+2[L_4L_3L_2]+[L_3^2L_2]\\&=[L_4,[L_4,L_2]]+2[L_4,[L_3,L_2]]\\&\quad+[L_3,[L_3,L_2]]\,.\end{aligned}$$

On the right-hand side of the equation we use the identity

$$\frac{d}{d\lambda}e^{H(\lambda)}=\left(\int_0^1 d\sigma e^{\sigma H(\lambda)}\frac{dH}{d\lambda}e^{-\sigma H(\lambda)}\right)e^{H(\lambda)} \tag{3.10}$$

$$\begin{aligned}&=\int_0^1 d\sigma\left(\frac{dH}{d\lambda}+\sigma\left[H,\frac{dH}{d\lambda}\right]+\frac{1}{2!}\sigma^2\left[H,\left[H,\frac{dH}{d\lambda}\right]\right]+\cdots\right)e^{H(\lambda)}\\&=\sum_{n=1}^{\infty}\frac{1}{n!}\left[H^{n-1}\frac{dH}{d\lambda}\right]e^{H(\lambda)}\,,\end{aligned} \tag{3.11}$$

where in the last line we use our notation for the multiple commutator. Inserting

$$H(\lambda)=\lambda\Sigma+\sum_{n=2}^{\infty}\lambda^nF_n\,, \tag{3.12}$$

we get

$$\frac{d}{d\lambda}O_\lambda(C)=\left\{\Sigma+2\lambda F_2+\lambda^2(3F_3+\tfrac{1}{2}[\Sigma F_2])+\lambda^3\left(4F_4+[\Sigma F_3]+\frac{1}{3!}[\Sigma^2F_2]\right)+\cdots\right\}O_\lambda(C)\,. \tag{3.13}$$

Comparing Eqs. (3.8) and (3.13) we get the desired relations for the F_n,

$$2!\,F_2=[(L_4+L_3+L_2)L_1]+[(L_4+L_3)L_2]+[L_4L_3]\,, \tag{3.14a}$$

$$3!\,F_3=-[\Sigma F_2]+[(L_4+L_3+L_2)^2L_1]+[(L_4+L_3)^2L_2]+[L_4^2L_3]\,, \tag{3.14b}$$

$$4!\,F_4=-3![\Sigma F_3]-[\Sigma^2F_2]+[(L_4+L_3+L_2)^3L_1]+[(L_4+L_3)^3L_2]+[L_4^3L_3]\,, \tag{3.14c}$$

$\cdots\,.$

The L_i can be expanded in a power series in a and b. The coefficients are local operators

$$L_1=\sum_{n=1}^{\infty}\frac{1}{n!}a^n\partial_1^{\,n-1}A_1(x_1,x_2)\,, \tag{3.15a}$$

$$L_2=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(n-1)!}\frac{1}{m!}a^{n-1}b^m\partial_1^{\,n-1}\partial_2^{\,m-1}A_2(x_1,x_2)\,, \tag{3.15b}$$

$$L_3=-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n!}\frac{1}{(m-1)!}a^nb^{m-1}\partial_1^{\,n-1}\partial_2^{\,m-1}A_1(x_1,x_2)\,, \tag{3.15c}$$

$$L_4=-\sum_{m=1}\frac{1}{m!}b^m\partial_2^{\,m-1}A_2(x_1,x_2)\,. \tag{3.15d}$$

For an infinitesimal area we only keep terms linear in a and b. We find

$$\Sigma=ab(\partial_1A_2-\partial_2A_1) \tag{3.16}$$

and

$$F_2=-ab[A_1,A_2]\,, \tag{3.17}$$

so that

$$\exp[H(1)]\cong 1+abG_{12} \tag{3.18}$$

as in Eq. (2.3).

Some straightforward, though tedious, calculations lead to the result that, for the particular loop under consideration,

$$W(C) = \mathrm{Tr}\exp[H(1)]$$
$$= \mathrm{Tr}\exp\left(abG_{12} + \tfrac{1}{2}a^2bD_1G_{12} + \tfrac{1}{2}ab^2D_2G_{12} + \frac{1}{2!}\frac{1}{2!}a^2b^2D_1D_2G_{12} + \frac{1}{3!}a^3bD_1^{\,2}G_{12} + \frac{1}{3!}ab^3D_2^{\,2}G_{12} + \frac{1}{3!}\frac{1}{2!}a^3b^2D_1^{\,2}D_2G_{12} + \cdots\right). \tag{3.19}$$

We have only calculated terms to this order. The pattern is clear, except for an ambiguity that arises from the noncommutativity of the covariant derivatives. For example,

$$D_1^{\,2}D_2G_{12} \neq D_1D_2D_1G_{12}.$$

The ambiguity corresponds to that of the parametrization of the surface in the integration of (2.9). A particular choice, such as

$$W(C) = \mathrm{Tr}\exp\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n!}\frac{1}{m!}a^nb^mD_1^{\,n-1}D_2^{\,m-1}G_{12}\right), \tag{3.20}$$

corresponds to a particular surface parametrization. Below we deal further with this equation.

If these were ordinary derivatives, the exponent would be

$$\sum_{n=1}^{\infty}\frac{a^n}{n!}\frac{b^m}{m!}\partial_1^{\,n-1}\partial_2^{\,m-1}G_{12} = \int_0^a dx_1'\int_0^b dx_2' G_{12}(x_1+x_1', x_2+x_2')$$
$$= \int_{x_1}^{x_1+a} dx_1'\int_{x_2}^{x_2+b} dx_2' G_{12}(x_1', x_2'). \tag{3.21}$$

The presence of covariant derivatives changes this result. If we could find a function $\overline{G}_{12}(x_1, x_2)$ defined by

$$\overline{G}_{12}(x_1, x_2) = \phi(x_1, x_2)G_{12}(x_1, x_2)\phi^{-1}(x_1, x_2), \tag{3.22}$$

such that

$$\partial_i\overline{G}_{12} = \phi(x_1, x_2)D_iG_{12}(x_1, x_2)\phi^{-1}(x_1, x_2), \tag{3.23}$$

then (3.20) would yield

$$W(C) = \mathrm{Tr}\exp\left(\int_{x_1}^{x_1+a} dx_1'\int_{x_2}^{x_2+b} dx_2'\phi(x_1', x_2')G_{12}(x_1', x_2')\phi^{-1}(x_1', x_2')\right). \tag{3.24}$$

It is easy to see that

$$\partial_i\overline{G}_{12} = (\partial_i\phi)G_{12}\phi^{-1} + \phi(\partial_iG_{12})\phi^{-1} + \phi G_{12}(\partial_i\phi^{-1})$$
$$= \phi(\partial_iG_{12} + \phi^{-1}(\partial_i\phi)G_{12} - G_{12}\phi^{-1}\partial_i\phi)\phi^{-1}$$
$$= \phi(D_iG_{12})\phi^{-1} \tag{3.25}$$

provided that

$$\phi^{-1}\partial_i\phi = -A_i. \tag{3.26}$$

We see that ϕ can be identified with the string operators U that appear in Sec. I, and we again obtain a form of the non-Abelian Stokes theorem.

The ambiguity expressed in Eq. (3.20) can be seen explicitly in terms of path parametrization by considering a Taylor expansion of Eq. (2.12). Such an expansion involves derivatives such as

$$\frac{\partial}{\partial x_\mu}U_{Ax}G_{\alpha\beta}(x)U_{xA} = U_{Ax}[D_\mu G_{\alpha\beta}(x)]\,U_{xA}.$$

However, it is true that in higher orders, the order of derivatives must be specified, since the argument is path dependent,

$$\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}U_{Ax}G_{\alpha\beta}(x)U_{xA} \neq \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\mu}U_{Ax}G_{\alpha\beta}(x)U_{xA}.$$

The correct Taylor expansion of a path-dependent quantity such as Eq. (2.12) can of course be written down if the path is specified. Such an expansion will, through Eq. (3.25), lead to a unique choice for the expansion (3.20).

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