Quark-quark and quark-antiquark potentials

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The fourth-order quark-quark and quark-antiquark potentials in perturbative quantum chromodynamics are derived with special emphasis on a rigorous investigation of the spin-dependent interaction terms. We consider quarks and antiquarks of equal as well as unequal masses. We also obtain the quark-antiquark annihilation potential to the fourth order.

I. INTRODUCTION

For an understanding of strong interactions, quantum chromodynamics seems to offer the best hope at this time. It is believed that the quark-quark and quark-antiquark potentials can be described by combining a field-theoretical potential resulting from perturbative quantum chromodynamics and a phenomenological confining potential representing nonperturbative effects.[1,2] It is, therefore, of much interest to derive the perturbative quantum-chromodynamical potential as accurately as possible, and higher-order corrections to the well-known second-order potential have been investigated by several authors. These authors, however, have mostly devoted themselves to the calculation of the leading spin-independent interaction terms,[3-5] while the nonleading spin-dependent terms have been treated to a lesser extent.[6,7] The aim of this paper is to derive the fourth-order quark-quark and quark-antiquark potentials with special emphasis on a rigorous investigation of the spin-dependent interaction terms. Our treatment will be applied to quarks and antiquarks of equal as well as unequal masses. We shall also obtain the quark-antiquark annihilation potential, which is of considerable physical interest.[8]

The quark-quark and quark-antiquark potentials will be extracted from the scattering operator by following a straightforward approach,[9] which was also used by us recently for the derivation of the leading as well as the nonleading terms in the fourth-order gravitational[10] and gluonic[11] potentials for scalar particles. For the evaluation of ultraviolet-divergent integrals, we shall use dimensional regularization, while infrared divergences will be handled by introducing the parameter λ and eventually letting $\lambda \to 0$. Renormalization will be performed so that the fourth-order potential, at small momenta, behaves simply as $(1/\vec{k}^2)\times\ln(\vec{k}^2/\mu^2)$, where μ is a renormalization scale. But, we shall also give the result for the potential according to the more convential $\overline{\text{MS}}$ scheme in Sec. VII to facilitate comparison with other investigations.

II. QUARK-QUARK SCATTERING

We take the Lagrangian density for a system of quark and gluon fields as

$$\begin{aligned} L = &-\tfrac{1}{4}(\partial_\mu a_\nu^i - \partial_\nu a_\mu^i)^2 - \tfrac{1}{2}(\partial_\mu a_\mu^i)^2 \\ &-(g/2c\hbar)f^{ijk}(\partial_\mu a_\nu^i - \partial_\nu a_\mu^i)a_\mu^j a_\nu^k \\ &-(g^2/4c^2\hbar^2)f^{ijk}f^{irs}a_\mu^j a_\nu^k a_\mu^r a_\nu^s \\ &-c\hbar(\bar{\psi}\gamma_\mu\partial_\mu\psi + \kappa\bar{\psi}\psi) + ig a_\mu^i \bar{\psi}\gamma_\mu T^i\psi \\ &-\partial_\mu C^{i*}\partial_\mu C^i + (g/c\hbar)f^{ijk}a_\mu^i\partial_\mu C^{j*}C^k , \end{aligned} \qquad (2.1)$$

where ψ, a_μ^i, and C^i are the quark, the gluon, and the gauge-compensating fields. The T^i are $N\times N$ matrices with

$$[T^i, T^j] = if^{ijk}T^k , \qquad (2.2)$$

and the upper indices take the values $1, 2, \ldots, N^2-1$. For the derivation of the scattering matrix elements, we require the contractions in the interaction picture

$$\begin{aligned} \psi_{r,\alpha}(x)^\cdot \bar{\psi}_{s,\beta}(x')^\cdot &= i\delta_{rs}S_{F,\alpha\beta}(x-x') , \\ a_\mu^i(x)^\cdot a_\nu^j(x')^\cdot &= -ic\hbar\delta^{ij}\delta_{\mu\nu}D_F(x-x') , \\ C^i(x)^\cdot C^{j*}(x')^\cdot &= -ic\hbar\delta^{ij}D_F(x-x') , \end{aligned} \qquad (2.3)$$

and the Fourier decomposition

$$\psi(x) = V^{-1/2}\sum_{\vec{p}}[\psi^+(\vec{p})e^{ip\cdot x} + \psi^-(\vec{p})e^{-ip\cdot x}] \qquad (2.4)$$

with

$$p_0 = (\kappa^2 + \vec{p}^2)^{1/2} , \qquad (2.5)$$

where $\vec{p}$, p_0, and κ are related to the momentum $\vec{P}$, energy E, and mass m of the quark as

$$\vec{p} = \vec{P}/\hbar, \quad p_0 = E/c\hbar, \quad \kappa = mc/\hbar . \qquad (2.6)$$

Let us consider the scattering of two quarks of masses m_1 and m_2, whose propagation four-vectors are p and q in the initial state and p' and q'

in the final state, and let

$$k = p' - p = -(q' - q)\,. \tag{2.7}$$

We shall use the center-of-mass system, so that

$$\vec{p} = -\vec{q},\quad \vec{p}\,' = -\vec{q}\,',\quad p_0' = p_0,\quad q_0' = q_0\,,$$
$$\vec{k} = \vec{p}\,' - \vec{p} = -(\vec{q}\,' - \vec{q}),\quad k_0 = 0\,. \tag{2.8}$$

The second-order contribution of the scattering operator for the above process, corresponding to the one-gluon-exchange diagram, is

$$S_2 = -V^{-2}(i/c\hbar)(2\pi)^n\delta(p+q-p'-q') \times \frac{g^2}{k^2+\lambda^2}\bar{\psi}^-(\vec{p}\,')T^i\gamma_\mu\psi^+(\vec{p})\bar{\psi}^-(\vec{q}\,')T^i\gamma_\mu\psi^+(\vec{q})\,, \tag{2.9}$$

which differs from the result in quantum electrodynamics in a trivial way, and yields in the nonrelativistic approximation in the Pauli form

$$S_2 = -V^{-2}(i/c\hbar)(2\pi)^n\delta(p+q-p'-q') \times \phi_1^*(\vec{p}\,')\phi_2^*(\vec{q}\,')\mathcal{V}_2(\vec{k})\phi_2(\vec{q})\phi_1(\vec{p}) \tag{2.10}$$

with

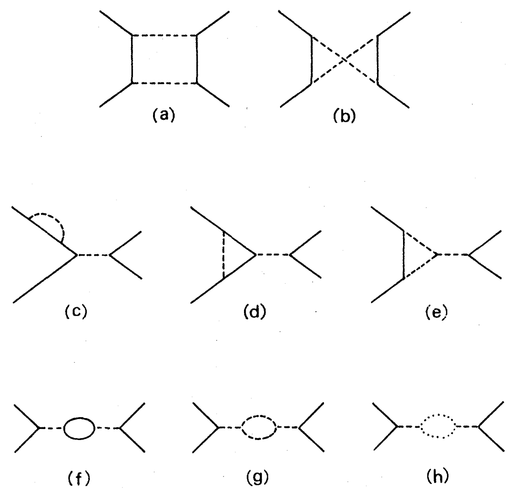

FIG. 1. Fourth-order quark-quark or quark-antiquark scattering. Solid lines represent quarks or antiquarks, while broken and dotted lines represent gluons and gauge-compensating particles, respectively.

$$\mathcal{V}_2(\vec{k}) = T_1^iT_2^ig^2\left(\frac{1}{\vec{k}^2} - \frac{(\kappa_1+\kappa_2)^2}{8\kappa_1^2\kappa_2^2} + \frac{\vec{p}^{\,2}}{\kappa_1\kappa_2\vec{k}^2} - \frac{1}{4\kappa_1\kappa_2\vec{k}^2}(\vec{k}^2\vec{\sigma}_1\cdot\vec{\sigma}_2 - \vec{k}\cdot\vec{\sigma}_1\vec{k}\cdot\vec{\sigma}_2) + \frac{i}{4\kappa_1\kappa_2\vec{k}^2}[(2+\kappa_2/\kappa_1)\vec{\sigma}_1 + (2+\kappa_1/\kappa_2)\vec{\sigma}_2]\cdot(\vec{k}\times\vec{p})\right), \tag{2.11}$$

where the subscripts 1 and 2 refer to the two quarks.

The diagrams for the fourth-order quark-quark scattering are shown in Fig. 1, where tadpole and leaf diagrams with vanishing contributions have been ignored. It is also understood that the external lines in these diagrams are to be labeled with p, p', q, and q' in all possible ways. In order to carry out the nonrelativistic approximation for the fourth-order contributions, we note that in the center-of-mass system

$$\vec{p} = -\vec{q} = -\tfrac{1}{2}(\vec{k}-\vec{s}),\quad \vec{p}\,' = -\vec{q}\,' = \tfrac{1}{2}(\vec{k}+\vec{s})\,,$$
$$p_0 = p_0' = (\kappa_1^2 + \tfrac{1}{4}\vec{k}^2 + \tfrac{1}{4}\vec{s}^{\,2})^{1/2},\quad q_0 = q_0' = (\kappa_2^2 + \tfrac{1}{4}\vec{k}^2 + \tfrac{1}{4}\vec{s}^{\,2})^{1/2}\,, \tag{2.12}$$

where

$$\vec{s} = \vec{p}\,' + \vec{p} = -(\vec{q}\,' + \vec{q}),\quad \vec{s}\cdot\vec{k} = 0\,. \tag{2.13}$$

In the static approximation we would have set $\vec{s} = 0$, but to improve upon the static approximation we shall drop only $O(\vec{s}^{\,2})$ terms. Furthermore, we shall retain spin-independent terms to order $|\vec{k}|^{-1}$, but determine the spin-dependent terms more accurately to order $|\vec{k}|^0$, and for this purpose we shall treat $|\vec{s}|$ as of order $|\vec{k}|$.

A. Diagrams 1(a) and 1(b)

The contribution of the scattering operator for the diagram 1(a) is

$$S_a = V^{-2}\delta(p+q-p'-q')(g^4/c^2\hbar^2)\int dl\,\frac{\bar{\psi}^-(\vec{p}\,')T^i\gamma_\mu[i(p'-l)\cdot\gamma - \kappa_1]T^j\gamma_\nu\psi^+(\vec{p})\bar{\psi}^-(\vec{q}\,')T^i\gamma_\mu[i(q'+l)\cdot\gamma - \kappa_2]T^j\gamma_\nu\psi^+(\vec{q})}{(l^2+\lambda^2)[(p'-p-l)^2+\lambda^2][(p'-l)^2+\kappa_1^2][(q'+l)^2+\kappa_2^2]}\,, \tag{2.14}$$

which can be simplified by using the properties of the γ matrices and the relations

$$\bar{\psi}^-(\vec{p}\,')(ip'\cdot\gamma + \kappa_1) = 0,\quad (ip\cdot\gamma + \kappa_1)\psi^+(\vec{p}) = 0\,,$$
$$\bar{\psi}^-(\vec{q}\,')(iq'\cdot\gamma + \kappa_2) = 0,\quad (iq\cdot\gamma + \kappa_2)\psi^+(\vec{q}) = 0\,. \tag{2.15}$$

Then, by reducing S_a to the Pauli form and carrying out the nonrelativistic approximation as described above, we can express it in the form (2.10) with $\mathcal{V}_2(\vec{k})$ replaced by

$$\mathcal{V}_a(\vec{k})=T_1^iT_1^jT_2^iT_2^j\frac{ig^4}{(2\pi)^n c\hbar}\int dl\frac{a+a_\mu l_\mu+a_{\mu\nu}l_\mu l_\nu}{(l^2+\lambda^2)(l^2-2\vec{l}\cdot\vec{k}+\vec{k}^2+\lambda^2)[l^2-\vec{l}\cdot(\vec{k}+\vec{s})+2l_0p_0][l^2-\vec{l}\cdot(\vec{k}+\vec{s})-2l_0q_0]}, \quad (2.16)$$

where, for $n=4$,

$$\begin{aligned}
a&=4\kappa_1\kappa_2+\vec{k}^2-(\vec{k}^2\vec{\sigma}_1\cdot\vec{\sigma}_2-\vec{k}\cdot\vec{\sigma}_1\vec{k}\cdot\vec{\sigma}_2)+\tfrac{1}{2}i[(2+\kappa_2/\kappa_1)\vec{\sigma}_1+(2+\kappa_1/\kappa_2)\vec{\sigma}_2]\cdot(\vec{k}\times\vec{s})+O(\vec{k}^4),\\
a_i&=-(4+\kappa_1/\kappa_2+\kappa_2/\kappa_1)s_i-2(1-\vec{\sigma}_1\cdot\vec{\sigma}_2)k_i-(\sigma_{1,i}\vec{k}\cdot\vec{\sigma}_2+\vec{k}\cdot\vec{\sigma}_1\sigma_{2,i})\\
&\quad+i\epsilon_{ijk}k_j[(2+\kappa_2/\kappa_1)\sigma_{1,k}+(2+\kappa_1/\kappa_2)\sigma_{2,k}]+O(|\vec{k}|^3),\\
a_4&=-2i(\kappa_1-\kappa_2)+O(\vec{k}^2),\\
a_{ij}+a_{ji}&=4\delta_{ij}(1-\vec{\sigma}_1\cdot\vec{\sigma}_2)+2(\sigma_{1,i}\sigma_{2,j}+\sigma_{1,j}\sigma_{2,i})+O(\vec{k}^2),\\
a_{i4}+a_{4i}&=O(|\vec{k}|),\\
a_{44}&=4-2\vec{\sigma}_1\cdot\vec{\sigma}_2+O(|\vec{k}|).
\end{aligned} \quad (2.17)$$

The integral (2.16) is of the form (A1), and according to (A12) it yields

$$\begin{aligned}
\mathcal{V}_a&=T_1^iT_1^jT_2^iT_2^j\frac{g^2}{4\pi^2c\hbar}\ln\frac{\vec{k}^2}{\lambda^2}\hat{\mathcal{V}}_2(\vec{k})\\
&\quad+T_1^iT_1^jT_2^iT_2^j\frac{g^4}{32c\hbar}\Bigg[\frac{8\kappa_1\kappa_2}{(\kappa_1+\kappa_2)|\vec{k}|^3}-\frac{\kappa_1{}^2+\kappa_2{}^2+\kappa_1\kappa_2}{\kappa_1\kappa_2(\kappa_1+\kappa_2)|\vec{k}|}+\frac{1}{(\kappa_1+\kappa_2)|\vec{k}|^3}(\vec{k}^2\vec{\sigma}_1\cdot\vec{\sigma}_2+\vec{k}\cdot\vec{\sigma}_1\vec{k}\cdot\vec{\sigma}_2)\\
&\qquad+\frac{2}{\pi^2\kappa_1\kappa_2\vec{k}^2}(\vec{k}^2\vec{\sigma}_1\cdot\vec{\sigma}_2-\vec{k}\cdot\vec{\sigma}_1\vec{k}\cdot\vec{\sigma}_2)+\frac{1}{\pi^2\kappa_1\kappa_2}\left(\ln\frac{\vec{k}^2}{\kappa_1\kappa_2}-\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\ln\frac{\kappa_2}{\kappa_1}-4\right)\vec{\sigma}_1\cdot\vec{\sigma}_2\Bigg],
\end{aligned} \quad (2.18)$$

where

$$T_1^iT_2^i\hat{\mathcal{V}}_2(\vec{k})=\mathcal{V}_2(\vec{k}). \quad (2.19)$$

Similarly, for diagram 1(b),

$$S_b=V^{-2}\delta(p+q-p'-q')(g^4/c^2\hbar^2)\int dl\frac{\bar{\psi}^-(\vec{p}')T^i\gamma_\mu[i(p'-l)\cdot\gamma-\kappa_1]T^j\gamma_\nu\psi^+(\vec{p})\bar{\psi}^-(\vec{q}')T^j\gamma_\nu[i(q-l)\cdot\gamma-\kappa_2]T^i\gamma_\mu\psi^+(\vec{q})}{(l^2+\lambda^2)[(p'-p-l)^2+\lambda^2][(p'-l)^2+\kappa_1{}^2][(q-l)^2+\kappa_2{}^2]}, \quad (2.20)$$

which can be expressed in the form (2.10) with

$$\mathcal{V}_b=T_1^iT_1^jT_2^jT_2^i\frac{ig^4}{(2\pi)^n c\hbar}\int dl\frac{b+b_\mu l_\mu+b_{\mu\nu}l_\mu l_\nu}{(l^2+\lambda^2)(l^2-2\vec{l}\cdot\vec{k}+\vec{k}^2+\lambda^2)[l^2-\vec{l}\cdot(\vec{k}+\vec{s})+2l_0p_0][l^2-\vec{l}\cdot(\vec{k}-\vec{s})+2l_0q_0]}, \quad (2.21)$$

where, for $n=4$,

$$\begin{aligned}
b&=4\kappa_1\kappa_2-\vec{k}^2-(\vec{k}^2\vec{\sigma}_1\cdot\vec{\sigma}_2-\vec{k}\cdot\vec{\sigma}_1\vec{k}\cdot\vec{\sigma}_2)+\tfrac{1}{2}i[(2+\kappa_2/\kappa_1)\vec{\sigma}_1+(2+\kappa_1/\kappa_2)\vec{\sigma}_2]\cdot(\vec{k}\times\vec{s})+O(\vec{k}^4),\\
b_i&=(\kappa_1/\kappa_2-\kappa_2/\kappa_1)s_i+2(1+\vec{\sigma}_1\cdot\vec{\sigma}_2)k_i-(\sigma_{1,i}\vec{k}\cdot\vec{\sigma}_2+\vec{k}\cdot\vec{\sigma}_1\sigma_{2,i})\\
&\quad-i\epsilon_{ijk}k_j[(2-\kappa_2/\kappa_1)\sigma_{1,k}-(2-\kappa_1/\kappa_2)\sigma_{2,k}]+O(|\vec{k}|^3),\\
b_4&=2i(\kappa_1+\kappa_2)+O(\vec{k}^2),\\
b_{ij}+b_{ji}&=4\delta_{ij}(1-\vec{\sigma}_1\cdot\vec{\sigma}_2)+2(\sigma_{1,i}\sigma_{2,j}+\sigma_{1,j}\sigma_{2,i})+O(\vec{k}^2),\\
b_{i4}+b_{4i}&=O(|\vec{k}|),\\
b_{44}&=-4-2\vec{\sigma}_1\cdot\vec{\sigma}_2+O(|\vec{k}|).
\end{aligned} \quad (2.22)$$

Then, according to (A14),

$$\begin{aligned}
\mathcal{V}_b&=-T_1^iT_1^jT_2^jT_2^i\frac{g^2}{4\pi^2c\hbar}\ln\frac{\vec{k}^2}{\lambda^2}\hat{\mathcal{V}}_2(\vec{k})-T_1^iT_1^jT_2^jT_2^i\frac{g^4}{32c\hbar}\Bigg[-\frac{\kappa_1+\kappa_2}{\kappa_1\kappa_2|\vec{k}|}+\frac{2}{\pi^2\kappa_1\kappa_2\vec{k}^2}(\vec{k}^2\vec{\sigma}_1\cdot\vec{\sigma}_2-\vec{k}\cdot\vec{\sigma}_1\vec{k}\cdot\vec{\sigma}_2)\\
&\qquad+\frac{1}{\pi^2\kappa_1\kappa_2}\left(\ln\frac{\vec{k}^2}{\kappa_1\kappa_2}-\frac{\kappa_1+\kappa_2}{\kappa_1-\kappa_2}\ln\frac{\kappa_2}{\kappa_1}-4\right)\vec{\sigma}_1\cdot\vec{\sigma}_2\Bigg].
\end{aligned} \quad (2.23)$$

B. Diagrams 1(c) and 1(d)

For the diagrams 1(c) and 1(d), we have

$$S_c=\sum_{r=1}^{4} S_c^{(r)}, \tag{2.24}$$

where

$$S_c^{(1)}=-V^{-2}(i/c\hbar)(2\pi)^n\delta(p+q-p'-q')\frac{g^2}{k^2+\lambda^2}\bar\psi^-(\vec p')T^i\gamma_\mu\frac{ip\cdot\gamma-\kappa_1}{p^2+\kappa_1^2}\Sigma(p)\psi^+(\vec p)\bar\psi^-(\vec q')T^i\gamma_\mu\psi^+(\vec q) \tag{2.25}$$

with

$$\Sigma(p)=\frac{ig^2}{(2\pi)^n c\hbar}\int dl\,\frac{T^j\gamma_\mu[i(p-l)\cdot\gamma-\kappa_1]T^j\gamma_\mu}{(l^2+\lambda^2)[(p-l)^2+\kappa_1^2]}, \tag{2.26}$$

and

$$S_d=\sum_{r=1}^{2} S_d^{(r)}, \tag{2.27}$$

where

$$S_d^{(1)}=-V^{-2}(i/c\hbar)(2\pi)^n\delta(p+q-p'-q')\frac{g^2}{k^2+\lambda^2}\bar\psi^-(\vec p')\Lambda_\mu^i(p',p)\psi^+(\vec p)\bar\psi^-(\vec q')T^i\gamma_\mu\psi^+(\vec q) \tag{2.28}$$

with

$$\Lambda_\mu^i(p',p)=\frac{ig^2}{(2\pi)^n c\hbar}\int dl\,\frac{T^j\gamma_\nu[i(p'-l)\cdot\gamma-\kappa_1]T^i\gamma_\mu[i(p-l)\cdot\gamma-\kappa_1]T^j\gamma_\nu}{(l^2+\lambda^2)[(p'-l)^2+\kappa_1^2][(p-l)^2+\kappa_1^2]}, \tag{2.29}$$

while other terms in (2.24) and (2.27) have similar forms.

These diagrams also appear in quantum electrodynamics, and their treatment is well known. After renormalization and neglect of the $O(|\vec k|^2)$ terms in $\Lambda_\mu^i(p',p)$,

$$S_c=-V^{-2}(i/c\hbar)(2\pi)^n\delta(p+q-p'-q')(g^2/\vec k^2)(-g^2/16\pi^2c\hbar)\ln(\lambda^2/\mu^2)$$
$$\times[\bar\psi^-(\vec p')(T^jT^jT^i+T^iT^jT^j)\gamma_\mu\psi^+(\vec p)\bar\psi^-(\vec q')T^i\gamma_\mu\psi^+(\vec q)+\bar\psi^-(\vec p')T^i\gamma_\mu\psi^+(\vec p)\bar\psi^-(\vec q')(T^jT^jT^i+T^iT^jT^j)\gamma_\mu\psi^+(\vec q)],$$

$$S_d=-V^{-2}(i/c\hbar)(2\pi)^n\delta(p+q-p'-q')(g^2/\vec k^2)(g^2/8\pi^2c\hbar)$$
$$\times\left[\bar\psi^-(\vec p')T^jT^iT^j\left(\ln\frac{\lambda^2}{\mu^2}\gamma_\mu-\frac{1}{2\kappa_1}\sigma_{\mu\nu}k_\nu\right)\psi^+(\vec p)\bar\psi^-(\vec q')T^i\gamma_\mu\psi^+(\vec q)+\bar\psi^-(\vec p')T^i\gamma_\mu\psi^+(\vec p)\bar\psi^-(\vec q')T^jT^iT^j\left(\ln\frac{\lambda^2}{\mu^2}\gamma_\mu+\frac{1}{2\kappa_2}\sigma_{\mu\nu}k_\nu\right)\psi^+(\vec q)\right],$$

and then reduction to the Pauli form leads to

$$\mathcal{V}_c=-T_1^iT_2^i(T_1^jT_1^j+T_2^jT_2^j)\frac{g^2}{8\pi^2c\hbar}\ln\frac{\lambda^2}{\mu^2}\hat{\mathcal{V}}_2(\vec k), \tag{2.30}$$

$$\mathcal{V}_d=T_1^iT_2^j(T_1^jT_1^i+T_2^iT_2^j)\frac{g^2}{8\pi^2c\hbar}\ln\frac{\lambda^2}{\mu^2}\hat{\mathcal{V}}_2(\vec k)+T_1^iT_2^j\frac{g^4}{32c\hbar}\left[-(T_1^jT_1^i+T_2^iT_2^j)\frac{1}{\pi^2\kappa_1\kappa_2\vec k^2}(\vec k^2\vec\sigma_1\cdot\vec\sigma_2-\vec k\cdot\vec\sigma_1\vec k\cdot\vec\sigma_2)\right.$$
$$\left.+\frac{2i(\kappa_1+\kappa_2)}{\pi^2\kappa_1^2\kappa_2^2\vec k^2}(T_1^jT_1^i\kappa_2\vec\sigma_1+T_2^iT_2^j\kappa_1\vec\sigma_2)\cdot(\vec k\times\vec p)\right]. \tag{2.31}$$

C. Diagram 1(e)

In the case of the diagram 1(e),

$$S_e=\sum_{r=1}^{2} S_e^{(r)}, \tag{2.32}$$

where

$$S_e^{(1)}=-V^{-2}(i/c\hbar)(2\pi)^n\delta(p+q-p'-q')\frac{g^2}{k^2+\lambda^2}\bar\psi^-(\vec p')\Lambda_\mu'^{\,i}(p',p)\psi^+(\vec p)\bar\psi^-(\vec q')T^i\gamma_\mu\psi^+(\vec q) \tag{2.33}$$

with

$$\Lambda_{\mu}^{\prime i}(p',p)=if^{ijk}\frac{g^2}{(2\pi)^n c\hbar}\int dl\frac{[\delta_{\nu\lambda}(2l-k)_\mu+\delta_{\lambda\mu}(2k-l)_\nu-\delta_{\mu\nu}(l+k)_\lambda]T^j\gamma_\nu[i(p'-l)\cdot\gamma-\kappa_1]T^k\gamma_\lambda}{(l^2+\lambda^2)[(l-k)^2+\lambda^2][(p'-l)^2+\kappa_1^{\,2}]}, \tag{2.34}$$

while the other term in (2.32) has a similar form.

After integration over l, simplification with the use of the properties of the γ matrices and the relations (2.15) and (2.12), and renormalization, we obtain

$$\Lambda_{\mu}^{\prime i}(p',p)=-if^{ijk}T^jT^k(g^2/8\pi^2c\hbar)\int_0^1 du_1\int_0^{u_1}du_2\Bigg[3\gamma_\mu-3\ln\left(\frac{\kappa_1^{\,2}u_2^{\,2}+\vec{k}^2(u_1-u_2)(1-u_1)}{\kappa_1^{\,2}u_2^{\,2}}\right)\gamma_\mu$$
$$+\frac{-3\kappa_1^{\,2}u_2^{\,2}\gamma_\mu+\vec{k}^2(1-u_1+u_1^{\,2}-u_1u_2)\gamma_\mu+\kappa_1(u_2-u_2^{\,2})\sigma_{\mu\nu}k_\nu}{\kappa_1^{\,2}u_2^{\,2}+\vec{k}^2(u_1-u_2)(1-u_1)}\Bigg],$$

where the k_μ terms have been dropped in view of

$$k_\mu\bar{\psi}^-(\vec{q}')\gamma_\mu\psi^+(\vec{q})=0\,.$$

The integrals over u_2 and u_1 are similar to those in Appendix A, and upon evaluation

$$\Lambda_{\mu}^{\prime i}(p',p)=-if^{ijk}T^jT^k\frac{g^2}{16c\hbar}\left(\frac{|\vec{k}|}{2\kappa_1}\gamma_\mu-\frac{\sigma_{\mu\nu}k_\nu}{\pi^2\kappa_1}\ln\frac{\vec{k}^2}{\kappa_1^{\,2}}\right)+O(\vec{k}^2)\,. \tag{2.35}$$

It follows that

$$S_e=-V^{-2}(i/c\hbar)(2\pi)^n\delta(p+q-p'-q')(-if^{ijk}g^4/16c\hbar\vec{k}^2)\Bigg[\bar{\psi}^-(\vec{p}')T^jT^k\left(\frac{|\vec{k}|}{2\kappa_1}\gamma_\mu-\frac{\sigma_{\mu\nu}k_\nu}{\pi^2\kappa_1}\ln\frac{\vec{k}^2}{\kappa_1^{\,2}}\right)\psi^+(\vec{p})\bar{\psi}^-(\vec{q}')T^i\gamma_\mu\psi^+(\vec{q})$$
$$+\bar{\psi}^-(\vec{p}')T^i\gamma_\mu\psi^+(\vec{p})\bar{\psi}^-(\vec{q}')T^jT^k\left(\frac{|\vec{k}|}{2\kappa_2}\gamma_\mu+\frac{\sigma_{\mu\nu}k_\nu}{\pi^2\kappa_2}\ln\frac{\vec{k}^2}{\kappa_2^{\,2}}\right)\psi^+(\vec{q})\Bigg],$$

and, with reduction to the Pauli form,

$$\mathcal{V}_e=if^{ijk}T_1^iT_2^j\frac{g^4}{32c\hbar}\Bigg[\frac{1}{\kappa_1\kappa_2|\vec{k}|}(T_1^k\kappa_2-T_2^k\kappa_1)-\frac{1}{\pi^2\kappa_1\kappa_2\vec{k}^2}\left(T_1^k\ln\frac{\vec{k}^2}{\kappa_1^{\,2}}-T_2^k\ln\frac{\vec{k}^2}{\kappa_2^{\,2}}\right)(\vec{k}^2\vec{\sigma}_1\cdot\vec{\sigma}_2-\vec{k}\cdot\vec{\sigma}_1\vec{k}\cdot\vec{\sigma}_2)$$
$$+\frac{2i(\kappa_1+\kappa_2)}{\pi^2\kappa_1^{\,2}\kappa_2^{\,2}\vec{k}^2}\left(T_1^k\kappa_2\ln\frac{\vec{k}^2}{\kappa_1^{\,2}}\vec{\sigma}_1-T_2^k\kappa_1\ln\frac{\vec{k}^2}{\kappa_2^{\,2}}\vec{\sigma}_2\right)\cdot(\vec{k}\times\vec{p})\Bigg]\,. \tag{2.36}$$

D. Diagrams 1(f), 1(g), and 1(h)

The treatment for diagram 1(f), which also appears in quantum electrodynamics, is well known, and it is easy to show that within the approximation under consideration

$$\mathcal{V}_f=0\,. \tag{2.37}$$

The contribution for diagrams 1(g) and 1(h) is expressible as

$$S_g+S_h=-V^{-2}(i/c\hbar)(2\pi)^n\delta(p+q-p'-q')\frac{g^2}{(k^2+\lambda^2)^2}\bar{\psi}^-(\vec{p}')T^i\gamma_\mu\psi^+(\vec{p})\Pi_{\mu\nu}^{\prime ij}(k)\bar{\psi}^-(\vec{q}')T^j\gamma_\nu\psi^+(\vec{q})\,, \tag{2.38}$$

where

$$\Pi_{\mu\nu}^{\prime ij}(k)=-f^{ikl}f^{jkl}\frac{ig^2}{(2\pi)^n 2c\hbar}\int\frac{dl}{(l^2+\lambda^2)[(k-l)^2+\lambda^2]}[\delta_{\alpha\beta}(2l-k)_\mu+\delta_{\beta\mu}(2k-l)_\alpha-\delta_{\mu\alpha}(l+k)_\beta]$$
$$\times[\delta_{\alpha\beta}(2l-k)_\nu+\delta_{\beta\nu}(2k-l)_\alpha-\delta_{\nu\alpha}(l+k)_\beta]$$
$$-f^{ikl}f^{jkl}\frac{ig^2}{(2\pi)^n c\hbar}\int dl\frac{l_\mu(k-l)_\nu}{(l^2+\lambda^2)[(k-l)^2+\lambda^2]}\,. \tag{2.39}$$

After integration and renormalization,

$$\Pi_{\mu\nu}^{\prime ij}(k)=-f^{ikl}f^{jkl}\frac{5g^2}{48\pi^2c\hbar}(\delta_{\mu\nu}k^2-k_\mu k_\nu)\ln\frac{k^2}{\mu^2}\,, \tag{2.40}$$

and by substituting (2.40) into (2.38) and carrying out reduction to the Pauli form, it is found that

$$\mathcal{V}_g+\mathcal{V}_h=-f^{ikl}f^{jkl}T_1^iT_2^j\frac{5g^2}{48\pi^2c\hbar}\ln\frac{\vec{k}^2}{\mu^2}\hat{\mathcal{V}}_2(\vec{k})\,. \tag{2.41}$$

III. QUARK-QUARK POTENTIAL

The second-order quark-quark potential immediately follows from (2.11), and takes a form familiar in quantum electrodynamics. However, for the fourth-order potential, in addition to the contributions derived in Sec. II, we require

$$\delta\mathcal{V}_4 = -\frac{1}{(2\pi)^3}\int d\vec{p}\,''\frac{\mathcal{V}_2(\vec{p}\,',\vec{p}\,'')\mathcal{V}_2(\vec{p}\,'',\vec{p})}{(p_0+q_0)c\hbar-(p_0''+q_0'')c\hbar}, \tag{3.1}$$

because the effect of the iteration of the second-order potential must be subtracted from the fourth-order contributions.

For any $\vec{p}$ and $\vec{p}\,'$, (2.11) can be written as

$$\mathcal{V}_2(\vec{p}\,',\vec{p}) = T_1^i T_2^i \frac{g^2}{(\vec{p}\,'-\vec{p})^2+\lambda^2}\left(1+\frac{1}{4\kappa_1\kappa_2}[\alpha_{ij}(p'-p)_i(p'-p)_j+\beta_{ij}(p'-p)_i p_j+4\vec{p}^2]\right) \tag{3.2}$$

with

$$\alpha_{ij} = -\frac{(\kappa_1+\kappa_2)^2}{2\kappa_1\kappa_2}\delta_{ij}-\vec{\sigma}_1\cdot\vec{\sigma}_2\delta_{ij}+\sigma_{1,i}\sigma_{2,j},$$

$$\beta_{ij} = \frac{i}{\kappa_1\kappa_2}\epsilon_{ijk}[(2\kappa_1+\kappa_2)\kappa_2\sigma_{1,k}+(\kappa_1+2\kappa_2)\kappa_1\sigma_{2,k}], \tag{3.3}$$

and in the nonrelativistic approximation in the center-of-mass system

$$\frac{1}{(p_0+q_0)-(p_0''+q_0'')} = \frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\frac{1}{\vec{p}^2-\vec{p}\,''^2}\left(1+\frac{\kappa_1^2+\kappa_2^2-\kappa_1\kappa_2}{4\kappa_1^2\kappa_2^2}(\vec{p}^2+\vec{p}\,''^2)+\cdots\right). \tag{3.4}$$

Substituting (3.2) and (3.4) into (3.1), and retaining terms up to second order in $\vec{p}$, $\vec{p}\,'$, and $\vec{p}\,''$ in the numerator, we find

$$\delta\mathcal{V}_4 = -T_1^i T_1^j T_2^i T_2^j \frac{g^4}{(2\pi)^3 c\hbar}\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\int\frac{d\vec{p}\,''}{[(\vec{p}\,'-\vec{p}\,'')^2+\lambda^2][(\vec{p}\,''-\vec{p})^2+\lambda^2](\vec{p}^2-\vec{p}\,''^2)}$$
$$\times\left[1+\frac{1}{4\kappa_1\kappa_2}\left(\alpha_{ij}[(p'-p'')_i(p'-p'')_j+(p''-p)_i(p''-p)_j]\right.\right.$$
$$\left.\left.+\beta_{ij}[(p'-p'')_i p_j''+(p''-p)_i p_j]+\frac{\kappa_1^2+\kappa_2^2+3\kappa_1\kappa_2}{\kappa_1\kappa_2}(\vec{p}^2+\vec{p}\,''^2)\right)\right],$$

and then upon setting

$$\vec{p} = -\tfrac{1}{2}\vec{k}+\tfrac{1}{2}\vec{s},\quad \vec{p}\,' = \tfrac{1}{2}\vec{k}+\tfrac{1}{2}\vec{s},\quad \vec{p}\,''-\vec{p} = \vec{l},\quad \vec{p}\,'-\vec{p}\,'' = \vec{k}-\vec{l}, \tag{3.5}$$

integrating over $\vec{l}$, and dropping terms of second order in $\vec{s}$,

$$\delta\mathcal{V}_4 = T_1^i T_1^j T_2^i T_2^j \frac{g^4}{32\pi c\hbar(\kappa_1+\kappa_2)}$$
$$\times\int_0^1 du_1\int_0^{u_1}du_2\left\{\frac{1}{|\vec{k}|\Delta}\left(2\alpha_{ii}+\frac{3(\kappa_1^2+\kappa_2^2+3\kappa_1\kappa_2)}{\kappa_1\kappa_2}\right)\right.$$
$$+\frac{1}{|\vec{k}|^3\Delta^3}\left[4\kappa_1\kappa_2+k_i k_j\left(2\alpha_{ij}+\frac{\kappa_1^2+\kappa_2^2+3\kappa_1\kappa_2}{\kappa_1\kappa_2}\delta_{ij}\right)[\tfrac{1}{2}-u_1+\tfrac{1}{2}u_2+(u_1-\tfrac{1}{2}u_2)^2]\right.$$
$$\left.\left.+k_i s_j[\tfrac{1}{2}\beta_{ij}(1-u_2)+(\alpha_{ij}+\alpha_{ji})(\tfrac{1}{2}-u_1+\tfrac{1}{2}u_2)u_2]\right]\right\},$$

where Δ is defined by (A9). Evaluation of parameter integrals, which are of the same form as those in Appendix A, gives

$$\delta\mathcal{V}_4 = T_1^i T_1^j T_2^i T_2^j \frac{g^4}{32c\hbar(\kappa_1+\kappa_2)|\vec{k}|^3}\left[-8\kappa_1\kappa_2-\alpha_{ij}k_i k_j+\left(\alpha_{ii}+\frac{\kappa_1^2+\kappa_2^2+3\kappa_1\kappa_2}{\kappa_1\kappa_2}\right)\vec{k}^2\right],$$

which becomes, on using (3.3),

$$\delta\mathcal{V}_4 = T_1^i T_1^j T_2^i T_2^j \frac{g^4}{32c\hbar(\kappa_1+\kappa_2)|\vec{k}|^3}(-8\kappa_1\kappa_2+\vec{k}^2-\vec{k}^2\vec{\sigma}_1\cdot\vec{\sigma}_2-\vec{k}\cdot\vec{\sigma}_1\vec{k}\cdot\vec{\sigma}_2). \tag{3.6}$$

By adding (2.18), (2.23), (2.30), (2.31), (2.36), (2.41), and (3.6), and simplifying the products of T matrices with the use of the SU(N) algebra, we obtain for the Fourier transform of the fourth-order quark-quark potential

$$\mathcal{V}_4(\vec{k}) = -\frac{g^2}{4\pi^2 c\hbar}\frac{11N}{12}\ln\frac{\vec{k}^2}{\mu^2}\mathcal{V}_2(\vec{k}) - \frac{g^4}{32\pi^2 c\hbar\kappa_1\kappa_2}\left[A + B\frac{\vec{k}^2\vec{\sigma}_1\cdot\vec{\sigma}_2 - \vec{k}\cdot\vec{\sigma}_1\vec{k}\cdot\vec{\sigma}_2}{\vec{k}^2} + C\vec{\sigma}_1\cdot\vec{\sigma}_2 + D\frac{i(\kappa_2\vec{\sigma}_1 + \kappa_1\vec{\sigma}_2)\cdot\vec{k}\times\vec{p}}{\vec{k}^2} + E\frac{i(\kappa_2\vec{\sigma}_1 - \kappa_1\vec{\sigma}_2)\cdot\vec{k}\times\vec{p}}{\vec{k}^2}\right] \quad (3.7)$$

with

$$A = -NT_1^iT_2^i\frac{\pi^2(\kappa_1+\kappa_2)}{|\vec{k}|} + \left(\frac{1}{N}T_1^iT_2^i - \frac{N^2-1}{4N^2}\right)\frac{2\pi^2\kappa_1\kappa_2}{(\kappa_1+\kappa_2)|\vec{k}|},$$

$$B = NT_1^iT_2^i\ln\frac{\vec{k}^2}{\kappa_1\kappa_2} + \frac{N^2-1}{N}T_1^iT_2^i,$$

$$C = NT_1^iT_2^i\frac{1}{2}\left(\ln\frac{\vec{k}^2}{\kappa_1\kappa_2} - 4\right) - \left[\left(\frac{1}{N}T_1^iT_2^i - \frac{N^2-1}{4N^2}\right)\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2} + \left(\frac{N^2-2}{2N}T_1^iT_2^i + \frac{N^2-1}{4N^2}\right)\frac{\kappa_1+\kappa_2}{\kappa_1-\kappa_2}\right]\ln\frac{\kappa_2}{\kappa_1}, \quad (3.8)$$

$$D = -NT_1^iT_2^i\frac{\kappa_1+\kappa_2}{\kappa_1\kappa_2}\ln\frac{\vec{k}^2}{\kappa_1\kappa_2} + \frac{1}{N}T_1^iT_2^i\frac{\kappa_1+\kappa_2}{\kappa_1\kappa_2},$$

$$E = -NT_1^iT_2^i\frac{\kappa_1+\kappa_2}{\kappa_1\kappa_2}\ln\frac{\kappa_2}{\kappa_1},$$

and we observe that the spin-independent part of (3.7) agrees with the result for the scalar-scalar case.[11]

For quarks of equal mass κ, (2.11) and (3.7) reduce to

$$\mathcal{V}_2(\vec{k}) = T_1^iT_2^ig^2\left(\frac{1}{\vec{k}^2} - \frac{1}{2\kappa^2} + \frac{\vec{p}^2}{\kappa^2\vec{k}^2} - \frac{1}{4\kappa^2}\frac{\vec{k}^2\vec{\sigma}_1\cdot\vec{\sigma}_2 - \vec{k}\cdot\vec{\sigma}_1\vec{k}\cdot\vec{\sigma}_2}{\vec{k}^2} + \frac{3}{4\kappa^2}\frac{i(\vec{\sigma}_1+\vec{\sigma}_2)\cdot\vec{k}\times\vec{p}}{\vec{k}^2}\right), \quad (3.9)$$

and

$$\mathcal{V}_4(\vec{k}) = -\frac{g^2}{4\pi^2 c\hbar}\frac{11N}{12}\ln\frac{\vec{k}^2}{\mu^2}\mathcal{V}_2(\vec{k}) - \frac{g^4}{32\pi^2 c\hbar\kappa^2}\left[\bar{A} + \bar{B}\frac{\vec{k}^2\vec{\sigma}_1\cdot\vec{\sigma}_2 - \vec{k}\cdot\vec{\sigma}_1\vec{k}\cdot\vec{\sigma}_2}{\vec{k}^2} + \bar{C}\vec{\sigma}_1\cdot\vec{\sigma}_2 + \bar{D}\frac{i(\vec{\sigma}_1+\vec{\sigma}_2)\cdot\vec{k}\times\vec{p}}{\vec{k}^2}\right] \quad (3.10)$$

with

$$\bar{A} = -\left(\frac{2N^2-1}{N}T_1^iT_2^i + \frac{N^2-1}{4N^2}\right)\frac{\pi^2\kappa}{|\vec{k}|},$$

$$\bar{B} = NT_1^iT_2^i\ln\frac{\vec{k}^2}{\kappa^2} + \frac{N^2-1}{N}T_1^iT_2^i,$$

$$\bar{C} = NT_1^iT_2^i\tfrac{1}{2}\ln\frac{\vec{k}^2}{\kappa^2} - \frac{N^2+2}{N}T_1^iT_2^i + \frac{N^2-1}{2N^2}, \quad (3.11)$$

$$\bar{D} = -2NT_1^iT_2^i\ln\frac{\vec{k}^2}{\kappa^2} + \frac{2}{N}T_1^iT_2^i,$$

where we have made use of the fact that

$$\frac{1}{\kappa_1-\kappa_2}\ln\frac{\kappa_2}{\kappa_1} = -\frac{1}{\kappa} \quad \text{for } \kappa_2 = \kappa_1 = \kappa.$$

IV. QUARK-ANTIQUARK DIRECT POTENTIAL

The quark-antiquark scattering can be described by the same diagrams as those for the quark-quark scattering provided that the external lines are labeled appropriately with the propagation four-vectors p and p' of the quark and q and q' of the antiquark. If the quark and the antiquark belong to different quark fields, only the direct quark-antiquark scattering is possible, and the contributions of various diagrams can be obtained from those in Sec. II by the replacements

$$\bar{\psi}^-(\vec{q}') \to \bar{\psi}^+(\vec{q}), \quad \psi^+(\vec{q}) \to \psi^-(\vec{q}'), \quad q' \to -q, \quad q \to -q'. \quad (4.1)$$

As a result of these replacements, the scattering contributions involve antiquark factors of the form

$$\bar{\psi}^+(\vec{q})\gamma_\mu T^i\gamma_\nu T^j\cdots\gamma_\lambda T^k\psi^-(\vec{q}'), \quad (4.2)$$

which can be expressed by projecting out the color factor as[12]

$$\bar{\psi}^+(\vec{q})\gamma_\mu\gamma_\nu\cdots\gamma_\lambda\psi^-(\vec{q}')[\eta^*(\vec{q})T^iT^j\cdots T^k\eta(\vec{q}')]. \quad (4.3)$$

The charge-conjugation relations for the two factors in (4.3) are

$$\bar{\psi}^+(\vec{q})\gamma_\mu\gamma_\nu\cdots\gamma_\lambda\psi^-(\vec{q}') = \pm\bar{\psi}_C^-(\vec{q}')\gamma_\lambda\cdots\gamma_\nu\gamma_\mu\psi_C^+(\vec{q}), \quad (4.4)$$

$$\eta^*(\vec{q})T^iT^j\cdots T^k\eta(\vec{q}') = \pm\eta_C^*(\vec{q}')T^k\cdots T^jT^i\eta_C(\vec{q}), \quad (4.5)$$

where the upper sign in (4.4) or (4.5) corresponds to an even number of the γ's or the T's, while the lower sign corresponds to an odd number. It follows that

$$\bar{\psi}^{+}(\vec{q})\gamma_\mu T^i \gamma_\nu T^j \cdots \gamma_\lambda T^k \psi^{-}(\vec{q}\,') = \bar{\psi}_C^{-}(\vec{q}\,')\gamma_\lambda T^k \cdots \gamma_\nu T^j \gamma_\mu T^i \psi_C^{+}(\vec{q}) . \quad (4.6)$$

With the replacements (4.1) in (2.9), we get for the second-order quark-antiquark scattering

$$S_2' = -V^{-2}(i/c\hbar)(2\pi)^n \delta(p+q-p'-q') \times \frac{g^2}{k^2+\lambda^2}\bar{\psi}^{-}(\vec{p}\,')T^i\gamma_\mu \psi^{+}(\vec{p})\bar{\psi}^{+}(\vec{q})T^i\gamma_\mu\psi^{-}(\vec{q}\,') ,$$

and then, with the application of (4.6),

$$S_2' = -V^{-2}(i/c\hbar)(2\pi)^n \delta(p+q-p'-q') \times \frac{g^2}{k^2+\lambda^2}\bar{\psi}^{-}(\vec{p}\,')T^i\gamma_\mu \psi^{+}(\vec{p})\bar{\psi}_C^{-}(\vec{q}\,')T^i\gamma_\mu\psi_C^{+}(\vec{q}) ,$$

which shows that

$$\mathcal{V}_2'(\vec{k}) = \mathcal{V}_2(\vec{k}) , \quad (4.7)$$

where $\mathcal{V}_2(\vec{k})$ is given by (2.11) or (3.9). Similarly, carrying out the replacements (4.1) and applying (4.6), we find that all the fourth-order diagrams for the quark-antiquark scattering yield the same potential contributions as in Sec. II except that the contributions of the box and crossed box diagrams are interchanged. Moreover, in view of (4.7), $\delta\mathcal{V}_4' = \delta\mathcal{V}_4$. Therefore,

$$\mathcal{V}_4'(\vec{k}) = \mathcal{V}_4(\vec{k}) , \quad (4.8)$$

where $\mathcal{V}_4(\vec{k})$ is given by (3.7) or (3.10).

V. QUARK-ANTIQUARK ANNIHILATION SCATTERING

If a quark and an antiquark belong to the same quark field, we have to deal with the direct scattering as well as the annihilation scattering, keeping of course in mind that the quark and the antiquark have the same mass. The direct scattering has already been discussed in Sec. IV, while the contributions of various diagrams for the annihilation scattering can be obtained from those in Sec. II by carrying out the replacements

$$\bar{\psi}^{-}(\vec{q}\,') \to \bar{\psi}^{-}(\vec{p}\,') , \quad \bar{\psi}^{-}(\vec{p}\,') \to \bar{\psi}^{+}(\vec{q}) , \quad \psi^{+}(\vec{q}) \to \psi^{-}(\vec{q}\,') ,$$
$$q' \to p' , \quad p' \to -q , \quad q \to -q' , \quad (5.1)$$

and setting $\kappa_1 = \kappa_2 = \kappa$.

The treatment of the annihilation scattering contributions requires an elaboration of the procedure in quantum electrodynamics.[13] We shall reduce the scattering contributions by expressing them in terms of the Pauli spinors ϕ and χ for quarks and antiquarks and the color-space spinor η, and then apply the exchange relations for the Pauli matrices

$$1_{s'r}1_{r's} = \tfrac{1}{2}1_{r'r}1_{s's} + \tfrac{1}{2}\vec{\sigma}_{r'r}\cdot\vec{\sigma}_{s's} ,$$
$$\vec{\sigma}_{s'r}\cdot\vec{\sigma}_{r's} = \tfrac{3}{2}1_{r'r}1_{s's} - \tfrac{1}{2}\vec{\sigma}_{r'r}\cdot\vec{\sigma}_{s's} , \quad (5.2)$$

and the color matrices

$$1_{s'r}1_{r's} = \frac{1}{N}1_{s's}1_{r'r} + 2T^i_{s's}T^i_{r'r} ,$$
$$T^i_{s'r}T^i_{r's} = \frac{N^2-1}{2N^2}1_{s's}1_{r'r} - \frac{1}{N}T^i_{s's}T^i_{r'r} . \quad (5.3)$$

This will be followed by charge conjugation with the use of

$$\chi^*(\vec{q})\chi(\vec{q}\,') = -\phi_C^*(\vec{q}\,')\phi_C(\vec{q}) ,$$
$$\chi^*(\vec{q})\sigma_i\chi(\vec{q}\,') = \phi_C^*(\vec{q}\,')\sigma_i\phi_C(\vec{q}) ,$$
$$\eta^*(\vec{q})\eta(\vec{q}\,') = \eta_C^*(\vec{q}\,')\eta_C(\vec{q}) ,$$
$$\eta^*(\vec{q})T^i\eta(\vec{q}\,') = -\eta_C^*(\vec{q}\,')T^i\eta_C(\vec{q}) . \quad (5.4)$$

Let us consider the second-order annihilation scattering contribution, given by

$$S_2'' = -V^{-2}(i/c\hbar)(2\pi)^n\delta(p+q-p'-q') \times \frac{g^2}{\bar{k}^2}\bar{\psi}^{-}(\vec{p}\,')T^i\gamma_\mu\psi^{-}(\vec{q}\,')\bar{\psi}^{+}(\vec{q})T^i\gamma_\mu\psi^{+}(\vec{p}) , \quad (5.5)$$

where

$$\bar{k} = -(p+q) . \quad (5.6)$$

After projecting out the color factor, carrying out reduction to the Pauli form, and taking into account the anticommutation property of the Pauli spinors, it is possible to express (5.5) in the nonrelativistic approximation as

$$S_2'' = -V^{-2}(i/c\hbar)(2\pi)^n\delta(p+q-p'-q') \times(-g^2/4\kappa^2)\phi_2^*(\vec{p}\,')\chi_1^*(\vec{q})\vec{\sigma}_1\cdot\vec{\sigma}_2\chi_2(\vec{q}\,')\phi_1(\vec{p}) \times[\eta_2^*(\vec{p}\,')\eta_1^*(\vec{q})T_1^iT_2^i\eta_2(\vec{q}\,')\eta_1(\vec{p})] ,$$

which becomes, in view of the exchange relations (5.2) and (5.3),

$$S_2'' = -V^{-2}(i/c\hbar)(2\pi)^n\delta(p+q-p'-q') \times(-g^2/8\kappa^2)\phi_1^*(\vec{p}\,')\chi_2^*(\vec{q})(3-\vec{\sigma}_1\cdot\vec{\sigma}_2)\chi_2(\vec{q}\,')\phi_1(\vec{p}) \times\left[\eta_1^*(\vec{p}\,')\eta_2^*(\vec{q})\left(\frac{N^2-1}{2N^2}-\frac{1}{N}T_1^iT_2^i\right)\eta_2(\vec{q}\,')\eta_1(\vec{p})\right].$$

Then, the use of the charge-conjugation relations (5.4) gives

$$S_2'' = -V^{-2}(i/c\hbar)(2\pi)^n\delta(p+q-p'-q') \times(g^2/8\kappa^2)\phi_1^*(\vec{p}\,')\phi_{2C}^*(\vec{q}\,')(3+\vec{\sigma}_1\cdot\vec{\sigma}_2)\phi_{2C}(\vec{q})\phi_1(\vec{p}) \times\left[\eta_1^*(\vec{p}\,')\eta_{2C}^*(\vec{q}\,')\left(\frac{N^2-1}{2N^2}+\frac{1}{N}T_1^iT_2^i\right)\eta_{2C}(\vec{q})\eta_1(\vec{p})\right] , \quad (5.7)$$

whence it follows that

$$\mathcal{V}_2''(\vec{k})=\left(\frac{N^2-1}{2N^2}+\frac{1}{N}T_1^iT_2^i\right)\frac{g^2}{8\kappa^2}(3+\vec{\sigma}_1\cdot\vec{\sigma}_2). \qquad (5.8)$$

A similar treatment will now be applied to the fourth-order annihilation scattering contributions corresponding to the various diagrams in Fig. 1. We shall again carry out approximations as explained in Sec. II, and simplify by using the properties of the γ matrices and the relations

$$\bar{\psi}^-(\vec{p}\,')(ip'\cdot\gamma+\kappa)=0\,,\quad (ip\cdot\gamma+\kappa)\psi^+(\vec{p})=0\,,$$
$$\bar{\psi}^+(\vec{q})(iq\cdot\gamma-\kappa)=0\,,\quad (iq'\cdot\gamma-\kappa)\psi^-(\vec{q}\,')=0\,. \qquad (5.9)$$

A. Annihilation diagrams 1(a) and 1(b)

The annihilation scattering contribution corresponding to diagram 1(a) is

$$S_a''=-V^{-2}\delta(p+q-p'-q')(g^4/c^2\hbar^2)\times\int dl\,\frac{\bar{\psi}^-(\vec{p}\,')T^i\gamma_\mu[i(p'+l)\cdot\gamma-\kappa]T^j\gamma_\nu\psi^-(\vec{q}\,')\bar{\psi}^+(\vec{q})T^i\gamma_\mu[i(q+l)\cdot\gamma+\kappa]T^j\gamma_\nu\psi^+(\vec{p})}{(l^2+\lambda^2)[(p+q+l)^2+\lambda^2][(p'+l)^2+\kappa^2][(q+l)^2+\kappa^2]}\,. \qquad (5.10)$$

After projecting out the color factor, and simplifying and reducing to the Pauli form, (5.10) can be expressed in the nonrelativistic approximation as

$$S_a''=V^{-2}\delta(p+q-p'-q')(g^4/c^2\hbar^2)\times\int dl\,\frac{\phi_2^*(\vec{p}\,')\chi_1^*(\vec{q})(a'+a_\mu'l_\mu+a_{\mu\nu}'l_\mu l_\nu)\chi_2(\vec{q}\,')\phi_1(\vec{p})}{(l^2+\lambda^2)(l^2-4l_0p_0-4p_0^{\,2}+\lambda^2)[l^2+\vec{l}\cdot(\vec{k}+\vec{s})-2l_0p_0][l^2+\vec{l}\cdot(\vec{k}-\vec{s})-2l_0p_0]}$$
$$\times[\eta_2^*(\vec{p}\,')\eta_1^*(\vec{q})T_1^iT_1^jT_2^iT_2^j\eta_2(\vec{q}\,')\eta_1(\vec{p})]\,, \qquad (5.11)$$

where, for $n=4$,

$$a'=-4\kappa^2\vec{\sigma}_1\cdot\vec{\sigma}_2+O(|\vec{k}|),\quad a_4'=4i\kappa\vec{\sigma}_1\cdot\vec{\sigma}_2+O(|\vec{k}|),$$
$$a_{ii}'=8\vec{\sigma}_1\cdot\vec{\sigma}_2-6+O(|\vec{k}|),\quad a_{44}'=2\vec{\sigma}_1\cdot\vec{\sigma}_2+O(|\vec{k}|). \qquad (5.12)$$

Upon integration with the help of (B4) and simplification of the product of T matrices, (5.11) becomes

$$S_a''=-V^{-2}(i/c\hbar)(2\pi)^n\delta(p+q-p'-q')(g^4/16\pi^2c\hbar\kappa^2)$$
$$\times\phi_2^*(\vec{p}\,')\chi_1^*(\vec{q})[(1-\ln 2)(1-\tfrac{1}{3}\vec{\sigma}_1\cdot\vec{\sigma}_2)-\ln(\kappa^2/\lambda^2)\vec{\sigma}_1\cdot\vec{\sigma}_2]\chi_2(\vec{q}\,')\phi_1(\vec{p})\left[\eta_2^*(\vec{p}\,')\eta_1^*(\vec{q})\left(\frac{N^2-1}{4N^2}-\frac{1}{N}T_1^iT_2^i\right)\eta_2(\vec{q}\,')\eta_1(\vec{p})\right].$$

Then, with the use of the exchange and charge-conjugation relations, S_a'' can be converted into a form similar to (5.7), and thus

$$\mathcal{V}_a''=-\left(\frac{N^2-1}{4N^3}+\frac{N^2+1}{2N^2}T_1^iT_2^i\right)\frac{g^4}{16\pi^2c\hbar\kappa^2}[\tfrac{2}{3}(1-\ln 2)\vec{\sigma}_1\cdot\vec{\sigma}_2+\tfrac{1}{2}(3+\vec{\sigma}_1\cdot\vec{\sigma}_2)\ln(\kappa^2/\lambda^2)]\,. \qquad (5.13)$$

Similarly,

$$S_b''=V^{-2}\delta(p+q-p'-q')(g^4/c^2\hbar^2)\int dl\,\frac{\bar{\psi}^-(\vec{p}\,')T^i\gamma_\mu[i(q'+l)\cdot\gamma+\kappa]T^j\gamma_\nu\psi^-(\vec{q}\,')\bar{\psi}^+(\vec{q})T^j\gamma_\nu[i(q+l)\cdot\gamma+\kappa]T^i\gamma_\mu\psi^+(\vec{p})}{(l^2+\lambda^2)[(p+q+l)^2+\lambda^2][(q'+l)^2+\kappa^2][(q+l)^2+\kappa^2]}\,, \qquad (5.14)$$

which becomes, in the nonrelativistic approximation,

$$S_b''=-V^{-2}\delta(p+q-p'-q')(g^4/c^2\hbar^2)\times\int dl\,\frac{\phi_2^*(\vec{p}\,')\chi_1^*(\vec{q})(b'+b_\mu'l_\mu+b_{\mu\nu}'l_\mu l_\nu)\chi_2(\vec{q}\,')\phi_1(\vec{p})}{(l^2+\lambda^2)(l^2-4l_0p_0-4p_0^2+\lambda^2)[l^2-\vec{l}\cdot(\vec{k}+\vec{s})-2l_0p_0][l^2+\vec{l}\cdot(\vec{k}-\vec{s})-2l_0p_0]}$$
$$\times[\eta_2^*(\vec{p}\,')\eta_1^*(\vec{q})T_1^iT_1^jT_2^jT_2^i\eta_2(\vec{q}\,')\eta_1(\vec{p})]\,, \qquad (5.15)$$

where, for $n=4$,

$$b'=-4\kappa^2\vec{\sigma}_1\cdot\vec{\sigma}_2+O(|\vec{k}|),\quad b_4'=4i\kappa\vec{\sigma}_1\cdot\vec{\sigma}_2+O(|\vec{k}|),$$
$$b_{ii}'=8\vec{\sigma}_1\cdot\vec{\sigma}_2+6+O(|\vec{k}|),\quad b_{44}'=2\vec{\sigma}_1\cdot\vec{\sigma}_2+O(|\vec{k}|)\,. \qquad (5.16)$$

After integration with the help of (B7), (5.15) is expressible as

$$S_b'' = -V^{-2}(i/c\hbar)(2\pi)^n\delta(p+q-p'-q')(g^4/16\pi^2c\hbar\kappa^2)\ \phi_2^*(\vec{p}')\chi_1^*(\vec{q})[(1-\ln 2)(1+\tfrac{1}{3}\vec{\sigma}_1\cdot\vec{\sigma}_2)+\ln(\kappa^2/\lambda^2)\vec{\sigma}_1\cdot\vec{\sigma}_2]\chi_2(\vec{q}')\phi_1(\vec{p})$$
$$\times\left[\eta_2^*(\vec{p}')\eta_1^*(\vec{q})\left(\frac{N^2-1}{4N^2}+\frac{N^2-2}{2N}T_1^iT_2^i\right)\eta_2(\vec{q}')\eta_1(\vec{p})\right],$$

which leads, with the use of the exchange and charge-conjugation relations, to

$$\mathcal{V}_b'' = -\left(\frac{(N^2-1)^2}{4N^3}-\frac{1}{2N^2}T_1^iT_2^i\right)\frac{g^4}{16\pi^2c\hbar\kappa^2}[(1-\ln 2)(1-\tfrac{1}{3}\vec{\sigma}_1\cdot\vec{\sigma}_2)+\tfrac{1}{2}(3+\vec{\sigma}_1\cdot\vec{\sigma}_2)\ln(\kappa^2/\lambda^2)]\,. \quad (5.17)$$

B. Annihilation diagrams 1(c) and 1(d)

By proceeding in the same manner as in Sec. II, the contribution for the annihilation diagram 1(c) is found to be

$$S_c'' = -V^{-2}(i/c\hbar)(2\pi)^n\delta(p+q-p'-q')(g^2/\bar{k}^2)(-g^2/16\pi^2c\hbar)\ln(\lambda^2/\mu^2)$$
$$\times[\bar{\psi}^-(\vec{p}')(T^jT^jT^i+T^iT^jT^j)\gamma_\mu\psi^-(\vec{q}')\bar{\psi}^+(\vec{q})T^i\gamma_\mu\psi^+(\vec{p})+\bar{\psi}^-(\vec{p}')T^i\gamma_\mu\psi^-(\vec{q}')\bar{\psi}^+(\vec{q})(T^jT^jT^i+T^iT^jT^j)\gamma_\mu\psi^+(\vec{p})],$$

which can be expressed, after simplification of the color factors, as

$$S_c'' = -\frac{N^2-1}{N}\frac{g^2}{8\pi^2c\hbar}\ln\frac{\lambda^2}{\mu^2}S_2'',$$

and therefore

$$\mathcal{V}_c'' = -\frac{N^2-1}{N}\frac{g^2}{8\pi^2c\hbar}\ln\frac{\lambda^2}{\mu^2}\mathcal{V}_2''(\vec{k})\,. \quad (5.18)$$

For the annihilation diagram 1(d),

$$S_d'' = -V^{-2}(i/c\hbar)(2\pi)^n\delta(p+q-p'-q')\frac{g^2}{\bar{k}^2}[\bar{\psi}^-(\vec{p}')T^i\gamma_\mu\psi^-(\vec{q}')\bar{\psi}^+(\vec{q})\Lambda_\mu^i(-q,p)\psi^+(\vec{p})$$
$$+\bar{\psi}^-(\vec{p}')\Lambda_\mu^i(p',-q')\psi^-(\vec{q}')\bar{\psi}^+(\vec{q})T^i\gamma_\mu\psi^+(\vec{p})] \quad (5.19)$$

with

$$\Lambda_\mu^i(-q,p) = \frac{-ig^2}{(2\pi)^nc\hbar}\int dl\,\frac{T^j\gamma_\nu[i(q+l)\cdot\gamma+\kappa]T^i\gamma_\mu[i(p-l)\cdot\gamma-\kappa]T^j\gamma_\nu}{(l^2+\lambda^2)[(q+l)^2+\kappa^2][(p-l)^2+\kappa^2]}\,. \quad (5.20)$$

Simplification and evaluation yield, after renormalization,

$$\Lambda_\mu^i(-q,p) = \Lambda_\mu^i(p',-q')$$
$$= -T^jT^iT^j\frac{g^2}{8\pi^2c\hbar}\left(\frac{\pi^2\kappa}{|\vec{k}|}+\ln\frac{\vec{k}^2}{\kappa^2}-\ln\frac{\lambda^2}{\mu^2}+2\right)\gamma_\mu+O(|\vec{k}|)\,, \quad (5.21)$$

and, upon substitution of (5.21) into (5.19) and simplification of color factors,

$$S_d'' = \frac{1}{N}\frac{g^2}{8\pi^2c\hbar}\left(\frac{\pi^2\kappa}{|\vec{k}|}+\ln\frac{\vec{k}^2}{\kappa^2}-\ln\frac{\lambda^2}{\mu^2}+2\right)S_2'',$$

which shows that

$$\mathcal{V}_d'' = \frac{1}{N}\frac{g^2}{8\pi^2c\hbar}\left(\frac{\pi^2\kappa}{|\vec{k}|}+\ln\frac{\vec{k}^2}{\kappa^2}-\ln\frac{\lambda^2}{\mu^2}+2\right)\mathcal{V}_2''(\vec{k})\,. \quad (5.22)$$

C. Annihilation diagram 1(e)

In the case of the annihilation diagram 1(e),

$$S_e'' = -V^{-2}(i/c\hbar)(2\pi)^n\delta(p+q-p'-q')\frac{g^2}{\bar{k}^2}[\bar{\psi}^-(\vec{p}')T^i\gamma_\mu\psi^-(\vec{q}')\bar{\psi}^+(\vec{q})\Lambda_\mu'^i(-q,p)\psi^+(\vec{p})$$
$$+\bar{\psi}^-(\vec{p}')\Lambda_\mu'^i(p',-q')\psi^-(\vec{q}')\bar{\psi}^+(\vec{q})T^i\gamma_\mu\psi^+(\vec{p})] \quad (5.23)$$

with

$$\Lambda_\mu'^i(-q,p) = -if^{ijk}\frac{g^2}{(2\pi)^nc\hbar}\int dl\,\frac{[\delta_{\nu\lambda}(2l-\bar{k})_\mu+\delta_{\lambda\mu}(2\bar{k}-l)_\nu-\delta_{\mu\nu}(l+\bar{k})_\lambda]T^j\gamma_\nu[i(q+l)\cdot\gamma+\kappa]T^k\gamma_\lambda}{(l^2+\lambda^2)[(l-\bar{k})^2+\lambda^2][(q+l)^2+\kappa^2]}\,. \quad (5.24)$$

After simplification, integration over l, and renormalization, we obtain

$$\Lambda_\mu'^{i}(-q,p)=\Lambda_\mu'^{i}(p',-q')$$
$$=-if^{ijk}T^jT^k(g^2/8\pi^2 c\hbar)\int_0^1 du_1\int_0^{u_1}du_2\left(3-3\ln\frac{u_2{}^2-4(u_1-u_2)(1-u_1)}{u_2{}^2}-\frac{1-u_1+\frac{1}{2}u_2+\frac{1}{4}(2u_1-u_2)^2}{u_2{}^2-4(u_1-u_2)(1-u_1)}\right)\gamma_\mu+O(|\vec{k}|)\,,$$

where the integrals over u_2 and u_1 are similar to those in Appendix B, and upon evaluation

$$\Lambda_\mu'^{i}(-q,p)=\Lambda_\mu'^{i}(p',-q')=-if^{ijk}T^jT^k\frac{g^2}{12\pi^2 c\hbar}(1+2\ln 2)\gamma_\mu+O(|\vec{k}|)\,. \tag{5.25}$$

Substitution of (5.25) and simplification of color factors enable us to put (5.23) in the form

$$S_e''=N\frac{g^2}{12\pi^2 c\hbar}(1+2\ln 2)S_2''\,,$$

and thus

$$\mathcal{V}_e''=N\frac{g^2}{12\pi^2 c\hbar}(1+2\ln 2)\mathcal{V}_2''(\vec{k})\,. \tag{5.26}$$

D. Annihilation diagrams 1(f), 1(g), and 1(h)

The treatment for diagram 1(f) is similar to that in quantum electrodynamics, and

$$S_f''=-V^{-2}(i/c\hbar)(2\pi)^n\delta(p+q-p'-q')\frac{g^2}{\bar{k}^4}\bar{\psi}^-(\vec{p}\,')T^i\gamma_\mu\psi^-(\vec{q}\,')\Pi_{\mu\nu}^{ij}(\bar{k})\bar{\psi}^+(\vec{q})T^j\gamma_\nu\psi^+(\vec{p})\,, \tag{5.27}$$

where

$$\Pi_{\mu\nu}^{ij}(\bar{k})=-\frac{ig^2}{(2\pi)^n c\hbar}\int dl\frac{\mathrm{Tr}\{(il\cdot\gamma-\kappa)T^i\gamma_\mu[i(l+\bar{k})\cdot\gamma-\kappa]T^j\gamma_\nu\}}{(l^2+\kappa^2)[(l+\bar{k})^2+\kappa^2]}\,. \tag{5.28}$$

Evaluation and renormalization of (5.28) give

$$\Pi_{\mu\nu}^{ij}(\bar{k})=\delta^{ij}\frac{g^2}{4\pi^2 c\hbar}(\bar{k}^2\delta_{\mu\nu}-\bar{k}_\mu\bar{k}_\nu)\int_0^1 du\,u(1-u)\ln\left(1+\frac{u(1-u)\bar{k}^2}{\kappa^2}\right)$$
$$=\delta^{ij}\frac{g^2}{4\pi^2 c\hbar}(\bar{k}^2\delta_{\mu\nu}-\bar{k}_\mu\bar{k}_\nu)[-\tfrac{4}{9}+O(\vec{k}^2)]\,, \tag{5.29}$$

so that (5.27) becomes

$$S_f''=-\frac{g^2}{9\pi^2 c\hbar}S_2''\,,$$

and

$$\mathcal{V}_f''=-\frac{g^2}{9\pi^2 c\hbar}\mathcal{V}_2''(\vec{k})\,. \tag{5.30}$$

For the annihilation diagrams 1(g) and 1(h),

$$S_g''+S_h''=-V^{-2}(i/c\hbar)(2\pi)^n\delta(p+q-p'-q')\frac{g^2}{\bar{k}^4}\bar{\psi}^-(\vec{p}\,')T^i\gamma_\mu\psi^-(\vec{q}\,')\Pi_{\mu\nu}'^{ij}(\bar{k})\bar{\psi}^+(\vec{q})T^j\gamma_\nu\psi^+(\vec{p})\,, \tag{5.31}$$

where, after renormalization,

$$\Pi_{\mu\nu}'^{ij}(\bar{k})=-f^{ikl}f^{jkl}\frac{5g^2}{48\pi^2 c\hbar}(\delta_{\mu\nu}\bar{k}^2-\bar{k}_\mu\bar{k}_\nu)\ln\frac{\bar{k}^2}{\mu^2}\,, \tag{5.32}$$

which can be obtained from (2.40) by replacing k by $\bar{k}$. Dropping the imaginary part in (5.32), we have

$$\Pi_{\mu\nu}'^{ij}(\bar{k})=-f^{ikl}f^{jkl}\frac{5g^2}{48\pi^2 c\hbar}(\delta_{\mu\nu}\bar{k}^2-\bar{k}_\mu\bar{k}_\nu)\left(\ln\frac{4\kappa^2}{\mu^2}+O(\vec{k}^2)\right)\,, \tag{5.33}$$

which, when substituted into (5.31), gives

$$S_g'' + S_h'' = -N\frac{5g^2}{48\pi^2 c\hbar}\left(2\ln 2 + \ln\frac{\kappa^2}{\mu^2}\right)S_2''$$

or

$$\mathcal{V}_g'' + \mathcal{V}_h'' = -N\frac{5g^2}{48\pi^2 c\hbar}\left(2\ln 2 + \ln\frac{\kappa^2}{\mu^2}\right)\mathcal{V}_2''(\vec{k})\,. \tag{5.34}$$

VI. QUARK-ANTIQUARK ANNIHILATION POTENTIAL

The second-order quark-antiquark annihilation potential follows from $\mathcal{V}_2''(\vec{k})$ given by (5.8). However, for the fourth-order potential, it should be kept in mind that the addition of $\mathcal{V}_2''(\vec{k})$ to $\mathcal{V}_2'(\vec{k})$ generates extra terms in the iteration of the second-order potential, which must be subtracted from the fourth-order contributions derived in Sec. V.

For this purpose, it is sufficient within our approximation to retain the cross terms generated by $\mathcal{V}_2''(\vec{k})$ and the Coulomb term in $\mathcal{V}_2'(\vec{k})$, and thus

$$\delta\mathcal{V}_4'' = -\frac{1}{(2\pi)^3}\int d\vec{p}\,''\frac{\mathcal{V}_2''(\vec{p}\,',\vec{p}\,'')\mathcal{V}_2'(\vec{p}\,'',\vec{p}) + \mathcal{V}_2'(\vec{p}\,',\vec{p}\,'')\mathcal{V}_2''(\vec{p}\,'',\vec{p})}{(p_0+q_0)c\hbar - (p_0''-q_0'')c\hbar}, \tag{6.1}$$

where, for any $\vec{p}$ and $\vec{p}\,'$,

$$\mathcal{V}_2'(\vec{p}\,',\vec{p}) = T_1^i T_2^i\frac{g^2}{(\vec{p}\,'-\vec{p})^2+\lambda^2},$$

$$\mathcal{V}_2''(\vec{p}\,',\vec{p}) = \left(\frac{N^2-1}{2N^2}+\frac{1}{N}T_1^i T_2^i\right)\frac{g^2}{8\kappa^2}(3+\vec{\sigma}_1\cdot\vec{\sigma}_2)\,. \tag{6.2}$$

Upon substitution of (6.2) and simplification of the product of T matrices, (6.1) becomes within our approximation

$$\delta\mathcal{V}_4'' = -\left(\frac{N^2-1}{2N^3}+\frac{N^2-3}{N^2}T_1^i T_2^i\right)\frac{g^4}{16c\hbar\kappa}(3+\vec{\sigma}_1\cdot\vec{\sigma}_2)\frac{1}{(2\pi)^3}\int d\vec{p}\,''\left(\frac{1}{(\vec{p}\,''-\vec{p})^2+\lambda^2}+\frac{1}{(\vec{p}\,'-\vec{p}\,'')^2+\lambda^2}\right)\frac{1}{\vec{p}^{\,2}-\vec{p}\,''^2}\,.$$

Then, using (3.5), combining the denominators, and performing integrations, we obtain

$$\begin{aligned}\delta\mathcal{V}_4'' &= -\left(\frac{N^2-1}{2N^3}+\frac{N^2-3}{N^2}T_1^i T_2^i\right)\frac{g^4}{16c\hbar\kappa}(3+\vec{\sigma}_1\cdot\vec{\sigma}_2)\frac{1}{(2\pi)^3}\int_0^1 du\int d\vec{l}\,\frac{-2}{[\vec{l}^{\,2}-\frac{1}{4}\vec{k}^2u^2+(1-u)\lambda^2]^2}\\ &= -\left(\frac{N^2-1}{2N^3}+\frac{N^2-3}{N^2}T_1^i T_2^i\right)\frac{g^4}{64c\hbar\kappa|\vec{k}|}(3+\vec{\sigma}_1\cdot\vec{\sigma}_2)\,.\end{aligned} \tag{6.3}$$

According to (5.13), (5.17), (5.18), (5.22), (5.26), (5.30), (5.34), and (6.3), the Fourier transform of the fourth-order quark-antiquark annihilation potential is given by

$$\begin{aligned}\mathcal{V}_4''(\vec{k}) = &-\frac{g^2}{4\pi^2 c\hbar}\left(\frac{11N}{12}\ln\frac{\kappa^2}{\mu^2}-\frac{1}{2N}\ln\frac{\vec{k}^2}{\kappa^2}-\frac{(N-1)(5N+3)}{3N}+\frac{N^2-2}{2}\ln 2\right)\mathcal{V}_2''(\vec{k})\\ &-\frac{g^4}{32\pi^2 c\hbar\kappa^2}\left(\frac{N^2-4}{N^2}T_1^i T_2^i\frac{\pi^2\kappa}{2|\vec{k}|}(3+\vec{\sigma}_1\cdot\vec{\sigma}_2)+T_1^i T_2^i(1-\ln 2)\vec{\sigma}_1\cdot\vec{\sigma}_2\right).\end{aligned} \tag{6.4}$$

It is interesting that for color-singlet states, (5.8) and (6.4) reduce to

$$\mathcal{V}_2''(\vec{k}) = 0\,, \tag{6.5}$$

$$\mathcal{V}_4''(\vec{k}) = \frac{g^4}{64\pi^2 c\hbar\kappa^2}\left(\frac{(N^2-1)(N^2-4)}{N^3}\frac{\pi^2\kappa}{2|\vec{k}|}(3+\vec{\sigma}_1\cdot\vec{\sigma}_2)+\frac{N^2-1}{N}(1-\ln 2)\vec{\sigma}_1\cdot\vec{\sigma}_2\right). \tag{6.6}$$

VII. CONCLUSION

In view of the complicated nature of our results, we have given the Fourier transforms of various potentials in the preceding sections, whence the potentials can be obtained through the relation

$$\mathcal{V}(\vec{x}) = (2\pi)^{-3}\int d\vec{k}\,e^{i\vec{k}\cdot\vec{x}}\mathcal{V}(\vec{k})\,. \tag{7.1}$$

We have found that the Fourier transform of the fourth-order quark-quark potential is given by (3.7) for unequal masses and by (3.10) for equal masses. We have also shown in Sec. IV that if a quark and an antiquark belong to different quark fields, the quark-antiquark potential is equal to the quark-quark potential. But, if a quark and an antiquark belong to the same quark field, the Fourier transform of the fourth-order quark-antiquark potential is

$$\mathcal{V}_4(\vec{k}) = \mathcal{V}_4'(\vec{k}) + \mathcal{V}_4''(\vec{k})\,, \tag{7.2}$$

where $\mathcal{V}_4'(\vec{k}) = \mathcal{V}_4(\vec{k})$ is given by (3.10), while $\mathcal{V}_4''(\vec{k})$ is given by (6.4) or (6.6). These results represent significant improvement over the earlier perturbative calculations.

It should be mentioned that several renormalization prescriptions are currently being used for the choice of the finite parts of renormalization constants, which leads to prescription dependence in physical results. Of course, whatever renormalization prescription is used, it must be ensured that all renormalization constants for the annihilation diagrams are equal to those for the corresponding direct diagrams. In Sec. II renormalization was performed by choosing the finite parts such that the fourth-order quark-quark potential takes the simplest form. For simplicity, the contributions from light-quark loops were also ignored. When renormalization is performed in a more conventional manner by using the $\overline{\text{MS}}$ scheme,[14] and the contributions of n_f light quarks are included, we find that the quark-quark potential (3.7) acquires the additional terms

$$\Delta\mathcal{V}_4(\vec{k}) = \frac{g^2}{24\pi^2 c\hbar}\left(\frac{31N}{6} + \ln\frac{\kappa_1^2}{\mu^2} + \ln\frac{\kappa_2^2}{\mu^2} + n_f \ln\frac{\vec{k}^2}{\mu^2} - \tfrac{5}{3}n_f\right)\mathcal{V}_2(\vec{k})\,. \tag{7.3}$$

The same result holds for the quark-antiquark direct potential, while the quark-antiquark annihilation potential remains unchanged for color-singlet states because of (6.5).

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APPENDIX A: EVALUATION OF INTEGRALS FOR DIRECT DIAGRAMS

We shall evaluate the multiple integrals encountered in our treatment of the direct box and crossed box diagrams. Since the integration techniques have already been fully explained,[15] we shall give only a brief account of evaluation in accordance with the approximations described in Sec. II.

Let us evaluate the integral for the direct box diagram

$$I_a = \int dl \frac{a + a_\mu l_\mu + a_{\mu\nu} l_\mu l_\nu}{(l^2+\lambda^2)(l^2 - 2\vec{l}\cdot\vec{k} + \vec{k}^2 + \lambda^2)[l^2 - \vec{l}\cdot(\vec{k}+\vec{s}) + 2l_0 p_0][l^2 - \vec{l}\cdot(\vec{k}+\vec{s}) - 2l_0 q_0]} \tag{A1}$$

by retaining the spin-independent terms to order $|\vec{k}|^{-1}$ and the spin-dependent terms to order $|\vec{k}|^0$. Since we shall retain only the relevant terms at various stages of calculation, it is important to note that, according to (2.17), a_i and $a_{i4} + a_{4i}$ are of order $|\vec{k}|$, and the spin-dependent part of a is of order $\vec{k}^2$. After integrating over l, and dropping the $O(\vec{s}^2)$ terms, we get

$$I_a = \frac{i\pi^2}{2}\int_0^1 du_1 \int_0^{u_1} du_2 \int_0^{u_2} du_3 \left(\frac{a_{\mu\mu}}{D} + \frac{2N}{D^2}\right) \tag{A2}$$

with

$$N = a + \vec{a}\cdot\vec{k}u_1 - \tfrac{1}{2}\vec{a}\cdot(\vec{k}-\vec{s})u_2 + a_{ij}k_i k_j(u_1 - \tfrac{1}{2}u_2)^2 + \tfrac{1}{2}(a_{ij}+a_{ji})k_i s_j(u_1 - \tfrac{1}{2}u_2)u_2 - a_{44}(p_0u_3 + q_0u_3 - p_0u_2)^2$$
$$- i[a_4 + (a_{i4}+a_{4i})(k_iu_1 - \tfrac{1}{2}k_iu_2 + \tfrac{1}{2}s_iu_2)](p_0u_3 + q_0u_3 - p_0u_2)\,, \tag{A3}$$

$$D = (p_0u_3 + q_0u_3 - p_0u_2)^2 + \vec{k}^2[(u_1 - u_2)(1-u_1) - \tfrac{1}{4}u_2^2] + \lambda^2(1-u_2)\,. \tag{A4}$$

Then, integration over u_3 leads to

$$I_a = I_0 + I_1 + I_2\,, \tag{A5}$$

where

$$I_0 = \frac{i\pi^2}{2(p_0+q_0)}\int_0^1 du_1 \int_0^{u_1} du_2 \pi\left(\frac{a_{ii}}{|\vec{k}|\Delta} + \frac{\overline{N}}{|\vec{k}|^3\Delta^3}\right), \tag{A6}$$

$$I_1 = \frac{i\pi^2}{2(p_0+q_0)}\int_0^1 du_1 \int_0^{u_1} du_2 \left[-\left(\frac{\overline{N}}{|\vec{k}|^3\Delta^3} + \frac{a_{ii}}{|\vec{k}|\Delta}\right)\tan^{-1}\left(\frac{|\vec{k}|\Delta}{q_0u_2}\right) + \left(ia_4 + i(a_{i4}+a_{4i})(k_iu_1 - \tfrac{1}{2}k_iu_2 + \tfrac{1}{2}s_iu_2) + a_{44}q_0u_2 + \frac{\overline{N}}{\vec{k}^2\Delta^2}q_0u_2\right)\frac{1}{q_0^2u_2^2 + \vec{k}^2\Delta^2}\right], \tag{A7}$$

with

$$\overline{N}=a+\vec{a}\cdot\vec{k}u_1-\tfrac{1}{2}\vec{a}\cdot(\vec{k}-\vec{s})u_2+a_{ij}k_ik_j(u_1-\tfrac{1}{2}u_2)^2+\tfrac{1}{2}(a_{ij}+a_{ji})k_is_j(u_1-\tfrac{1}{2}u_2)u_2\,, \tag{A8}$$

$$\Delta^2=(u_1-u_2)(1-u_1)-\tfrac{1}{4}u_2{}^2+(\lambda^2/\vec{k}^2)(1-u_2)\,, \tag{A9}$$

and I_2 can be obtained from I_1 by the replacements $p_0\to q_0$, $q_0\to p_0$, $a_4\to -a_4$, and $a_{i4}+a_{4i}\to-(a_{i4}+a_{4i})$. Final integrations over u_2 and u_1 yield

$$I_0=-\frac{i\pi^4}{(p_0+q_0)|\vec{k}|^3}[a+\tfrac{1}{2}\vec{a}\cdot\vec{s}-\tfrac{1}{4}a_{ii}\vec{k}^2-\tfrac{1}{4}a_{ij}k_ik_j+\tfrac{1}{4}(a_{ij}+a_{ji})k_is_j]\,, \tag{A10}$$

$$I_1=\frac{i\pi^2}{2q_0(p_0+q_0)}\left[-a_{ii}+\frac{a_{ii}-a_{44}}{2}\ln\frac{\vec{k}^2}{q_0{}^2}-\frac{2a+\vec{a}\cdot\vec{k}}{\vec{k}^2}\ln\frac{\vec{k}^2}{\lambda^2}+\frac{ia_4}{2}\left(\frac{\pi^2}{|\vec{k}|}-\frac{1}{q_0}\ln\frac{\vec{k}^2}{\lambda^2}\right)+\frac{a_{ij}k_ik_j}{\vec{k}^2}\left(1-\ln\frac{\vec{k}^2}{\lambda^2}\right)\right], \tag{A11}$$

while I_2 is obtainable from I_1.

From the above results for I_0, I_1, and I_2, it is found, by using

$$p_0=\kappa_1+\vec{k}^2/8\kappa_1+\cdots\,,\quad q_0=\kappa_2+\vec{k}^2/8\kappa_2+\cdots\,,$$

that

$$\begin{aligned}I_a=&-\frac{i\pi^4}{(\kappa_1+\kappa_2)|\vec{k}|^3}\left[a+\tfrac{1}{2}\vec{a}\cdot\vec{s}-\tfrac{1}{4}\left(a_{ii}+\frac{a}{2\kappa_1\kappa_2}\right)\vec{k}^2-\tfrac{1}{4}a_{ij}k_ik_j+\tfrac{1}{4}(a_{ij}+a_{ji})k_is_j\right]\\&+\frac{i\pi^2}{2\kappa_1\kappa_2}\left[-a_{ii}+\frac{a_{ii}-a_{44}}{2}\left(\ln\frac{\vec{k}^2}{\kappa_1\kappa_2}-\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\ln\frac{\kappa_2}{\kappa_1}\right)-\frac{2a+\vec{a}\cdot\vec{k}}{\vec{k}^2}\ln\frac{\vec{k}^2}{\lambda^2}\right.\\&\left.+\frac{ia_4}{2}\left(\frac{\pi^2(\kappa_1-\kappa_2)}{(\kappa_1+\kappa_2)|\vec{k}|}-\frac{\kappa_1-\kappa_2}{\kappa_1\kappa_2}\ln\frac{\vec{k}^2}{\lambda^2}\right)+\frac{a_{ij}k_ik_j}{\vec{k}^2}\left(1-\ln\frac{\vec{k}^2}{\lambda^2}\right)\right].\end{aligned} \tag{A12}$$

The integral for the direct crossed box diagram

$$I_b=\int dl\frac{b+b_\mu l_\mu+b_{\mu\nu}l_\mu l_\nu}{(l^2+\lambda^2)(l^2-2\vec{l}\cdot\vec{k}+\vec{k}^2+\lambda^2)[l^2-\vec{l}\cdot(\vec{k}+\vec{s})+2l_0p_0][l^2-\vec{l}\cdot(\vec{k}-\vec{s})+2l_0q_0]} \tag{A13}$$

can be evaluated in a similar manner. However, in this case we do not get any terms corresponding to I_0, while the terms corresponding to I_1 and I_2 yield

$$\begin{aligned}I_b=&\frac{i\pi^2}{2\kappa_1\kappa_2}\left[b_{ii}-\frac{b_{ii}-b_{44}}{2}\left(\ln\frac{\vec{k}^2}{\kappa_1\kappa_2}-\frac{\kappa_1+\kappa_2}{\kappa_1-\kappa_2}\ln\frac{\kappa_2}{\kappa_1}\right)+\frac{2b+\vec{b}\cdot\vec{k}}{\vec{k}^2}\ln\frac{\vec{k}^2}{\lambda^2}\right.\\&\left.+\frac{ib_4}{2}\left(\frac{\pi^2}{|\vec{k}|}-\frac{\kappa_1+\kappa_2}{\kappa_1\kappa_2}\ln\frac{\vec{k}^2}{\lambda^2}\right)-\frac{b_{ij}k_ik_j}{\vec{k}^2}\left(1-\ln\frac{\vec{k}^2}{\lambda^2}\right)\right].\end{aligned} \tag{A14}$$

APPENDIX B: EVALUATION OF INTEGRALS FOR ANNIHILATION DIAGRAMS

The treatment of integrals for the annihilation box and crossed box diagrams, which we shall now describe, is much simpler than that for the direct diagrams.

The integral for the annihilation box diagram

$$I_a'=\int dl\frac{a'+a_\mu'l_\mu+a_{\mu\nu}'l_\mu l_\nu}{(l^2+\lambda^2)(l^2-4l_0p_0-4p_0{}^2+\lambda^2)[l^2+\vec{l}\cdot(\vec{k}+\vec{s})-2l_0p_0][l^2+\vec{l}\cdot(\vec{k}-\vec{s})-2l_0p_0]} \tag{B1}$$

gives, upon integration over l,

$$I_a'=-\frac{i\pi^2}{\kappa^2}\int_0^1du_1\int_0^{u_1}du_2\int_0^{u_2}du_3\left(\frac{a_{\mu\mu}'}{2D'}+\frac{a'-2ia_4'\kappa(u_1-\frac{1}{2}u_2)-4a_{44}'\kappa^2(u_1-\frac{1}{2}u_2)^2}{\kappa^2D'^2}\right)+O(|\vec{k}|)\,, \tag{B2}$$

where

$$D'=u_2{}^2-4(1-u_1)(u_1-u_2)+(1-u_2)\lambda^2/\kappa^2\,. \tag{B3}$$

After substituting (5.12) into (B2), and performing parameter integrations, we obtain

$$I_a'=(i\pi^2/\kappa^2)[(1-\ln 2)(1-\tfrac{1}{3}\vec{\sigma}_1\cdot\vec{\sigma}_2)-\ln(\kappa^2/\lambda^2)\vec{\sigma}_1\cdot\vec{\sigma}_2]+O(|\vec{k}|)\,. \tag{B4}$$

Similarly, the integral for the annihilation crossed box diagram

$$I_b' = \int dl \frac{b' + b_\mu' l_\mu + b_{\mu\nu}' l_\mu l_\nu}{(l^2+\lambda^2)(l^2 - 4l_0 p_0 - 4p_0{}^2 + \lambda^2)[l^2 - \vec{l}\cdot(\vec{k}+\vec{s}) - 2l_0 p_0][l^2 + \vec{l}\cdot(\vec{k}-\vec{s}) - 2l_0 p_0]} \tag{B5}$$

gives

$$I_b' = \frac{i\pi^2}{\kappa^2}\int_0^1 du_1 \int_0^{u_1} du_2 \int_0^{u_2} du_3 \left(\frac{b_{\mu\mu}'}{2D'} + \frac{b' - 2ib_4'\kappa(u_1 - \frac{1}{2}u_2) - 4b_{44}'\kappa^2(u_1 - \frac{1}{2}u_2)^2}{\kappa^2 D'^2}\right) + O(|\vec{k}|)\,, \tag{B6}$$

and then, on substituting (5.16),

$$I_b' = (i\pi^2/\kappa^2)[(1-\ln 2)(1 + \tfrac{1}{3}\vec{\sigma}_1\cdot\vec{\sigma}_2) + \ln(\kappa^2/\lambda^2)\vec{\sigma}_1\cdot\vec{\sigma}_2] + O(|\vec{k}|)\,. \tag{B7}$$

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[12] For convenience, we shall sometimes replace the quark field ψ by $\psi\eta$ such that ψ is an anticommuting Dirac spinor, and η is a commuting color-space spinor.

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[15] See the Appendix in Ref. 10.