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Static Yang-Mills fields in the presence of external sources

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Using the Wu-Yang ansatz, we obtain static non-Abelian solutions in the presence of an assembly of point sources or a localized and extended source. The interaction energy of the system is finite and it indicates a possibility of confinement for two point sources.

I. INTRODUCTION

The success of the Yang-Mills (YM) gauge field theories[1] in electroweak interaction and strong interaction physics has led to the prevalent opinion that the gauge field is the key concept in elucidating the nature of fundamental forces. Thus it is worthwhile to explore every aspect of gauge fields. The classical solutions[2,3] of the sourceless YM field have been under intensive investigations in the past few years after the discovery of the 't Hooft-Polyakov monopole and the Belavin-Polyakov-Schwartz-Tyupkin (BPST) instanton. More recently there has been much interest[4-10] in constructing classical solutions of the YM field equations in the presence of external sources (treated as c numbers), with the hope that correct expressions can be obtained for the interquark potentials. However, most of the non-Abelian solutions derived are restricted to cases in which the external sources are extended. Since extended source distributions may be qualitatively different from point sources[10] and with the view that elementary particles such as quarks behave most probably as point sources, it is of interest to study non-Abelian YM field configurations in the presence of point sources. Solutions with point sources have been discussed in Refs. 4 and 5 but they are of the Abelian Coulomb type. In Ref. 7, a non-Abelian solution with one external point source has been presented and it is time dependent.

In the present paper we construct static solutions in which the space component j_i^a of the external source current j_μ^a is due to the Wu-Yang monopole,[2,11] whereas the external charge density j_0^a (for convenience, we refer to this as the non-Abelian electric source) is prescribed by an assembly of point sources or a localized and extended charge distribution. Our solutions for point sources are non-Abelian and are not gauge equivalent to the solutions found in Refs. 5 and 7. Apart from the self-energy, which must be divergent for point sources, the interaction energy of our solutions is finite. The external sources are also finite in magnitude.

II. GENERAL SOLUTIONS

For simplicity we restrict ourselves to the gauge group SU(2) although we expect no difficulty in generalizing the results to SU(N). In the presence of the external source current j_μ^a, the YM field equations are

$$\partial_\mu F_a^{\mu\nu} + g\epsilon_{abc} A_\mu^b F^{c\mu\nu} = j_a^\nu\,, \tag{1a}$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc} A_\mu^b A_\nu^c\,, \tag{1b}$$

where g is the gauge field coupling constant and our metric is $g_{ii} = -g_{00} = 1$. To construct static solutions, we interlock the isospin and spatial indices of the potential A_μ^a,[2]

$$A_0^a(x) = V(x) n^a, \quad A_i^a = \epsilon_{iaj} n_j A(x)\,, \qquad n_i = x_i/r\,, \quad r^2 = x_i x^i\,. \tag{2}$$

Here $V(x)$ and $A(x)$ are functions of the spatial coordinates x_i only. Substituting the Wu-Yang ansatz (2) into Eq. (1), we have, after some straightforward calculations,

$$\begin{aligned} j_a^i = {}& \epsilon_{ail} n_l \left[-\partial^2 A + \left(gA + \frac{1}{r}\right)\left(\frac{2A}{r} + gA^2 - V^2\right)\right] \\ & - \epsilon_{ail}\left(\frac{3}{r}\partial^l A - \frac{2}{r} n_l n_j \partial^j A\right) \\ & - \epsilon_{jal} n_l \left(\partial_j \partial_i A - \frac{1}{r} n_i \partial_j A\right) \\ & + 3gA\epsilon_{jil} n_a n_l \partial_j A\,, \end{aligned} \tag{3a}$$

$$\begin{aligned} j_a^0 = {}& n_a \left[-\partial^2 V + 2V\left(gA + \frac{1}{r}\right)^2 + 2\left(gA + \frac{1}{r}\right) n_i \partial_i V\right] \\ & + gV(n_a n_i - \delta_{ai})\partial_i A - 2\left(gA + \frac{1}{r}\right)\partial_a V\,. \end{aligned} \tag{3b}$$

If the functions A and V are spherically symmetric, Eqs. (3) become

$$j_a^i = \epsilon_{ail} n_l \left[-\partial^2 A + \left(gA + \frac{1}{r}\right)\left(\frac{2A}{r} + gA^2 - V^2\right)\right], \tag{4a}$$

$$j_a^0 = n_a \left[-\partial^2 V + 2V\left(gA + \frac{1}{r}\right)^2\right]. \tag{4b}$$

By assuming that the function $f(r) = [gA^2 - V^2 + (2A/$

$r)]$ vanishes exponentially, one can in principle solve for $A(r)$ from Eq. (4a) by setting $j_a^i=0$. One then obtains $V(r)$ from $f(r)$ and hence the charge distribution j_a^0. This type of solution has probably been covered in Refs. 6 and 9. Equations (3) can be simplified tremendously if we set $gA=-1/r$. Thus

$$j_a^i=\epsilon_{ial}n_l\partial^2 A=(4\pi/g)\epsilon_{ial}n_l\delta^{(3)}(\vec{x}), \tag{5a}$$

$$j_a^0=-n_a\partial^2 V, \tag{5b}$$

where V is still an arbitrary function of x^i. We note that each of the nonlinear Eqs. (3a) and (3b) involves the functions A and V and by putting a magnetic monopole at the origin, we have decoupled A and V to arrive at Eqs. (5a) and (5b) which are both linear.

Given any function $V(x)$, we can evaluate the electric charge density $q_a(x)=j_a^0(x)$ through Eq. (5b). For the charge distribution $q_a(x)$ thus found and with

$$\begin{gathered} A_0^a(x)=n^aV(x),\quad A_i^a=\epsilon_{iaj}n_j(-1/gr), \\ \partial_\mu A^{a\mu}=\partial_i A^{ai}=0, \end{gathered} \tag{6}$$

we are led to a large family of solutions for the YM equation in the presence of a magnetic monopole together with any arbitrary external non-Abelian electric charge distribution. It can be easily verified that the external current source j_μ^a as given by Eqs. (5) is covariantly conserved, i.e., $(D^\mu j_\mu)^a=0$.

The total energy of the YM fields interacting with the external source is obtained from the time part of the energy-momentum tensor,

$$\begin{aligned} H&=\int d^3x\, T^{00} \\ &=\int d^3x\,[\tfrac{1}{2}(E^2+B^2)+j_a^kA_k^a-\partial_j(E_a^jA_a^0)], \end{aligned}$$

where the non-Abelian electric and magnetic fields are defined by

$$E_i^a=F_{0i}^a,\quad B_i^a=\tfrac{1}{2}\epsilon_{ijk}F^{ajk},$$

and $E^2=E_i^aE_i^a$. For the solution (6), the field strengths are

$$E_i^a=-n^a/\partial_iV \tag{7a}$$

and

$$B_i^a=-n^an_i/(gr^2). \tag{7b}$$

The explicit expression of the electric field is determined by the function $V(x)$ whereas the magnetic field is due to the Wu-Yang monopole at the origin. Irrespective of the form assumed by the electric field, the total energy of the solution (6) is divergent because of the self-energy of the magnetic point source,

$$H=\int d^3x\left[\tfrac{1}{2}\left(\partial_iV\partial^iV+\frac{1}{g^2r^4}\right)-\frac{8\pi}{g^2r}\delta^{(3)}(\vec{x})-\partial_j(V\partial^jV)\right]. \tag{8}$$

Neglecting the self-energy divergence, the total energy of the system can be finite if the function $V(x)$ is chosen suitably so that the charge distribution is smooth and vanishes rapidly at infinity.

The external isospin current for solution (6) can be characterized gauge invariantly as

$$q=(j_a^0j_a^0)^{1/2}=\partial^2V, \tag{9a}$$

$$j=(j_i^aj_i^a)^{1/2}=\frac{4\pi}{g}\delta^{(3)}(\vec{x}). \tag{9b}$$

The integral of each of the above expressions is finite. The total conserved isospin current is

$$J_a^\nu=\partial_\mu F_a^{\mu\nu}=j_a^\nu-g\epsilon_{abc}A_\mu^bF^{c\mu\nu},$$

and the total isospin charge of the system is given by

$$\begin{aligned} I_a&=\int d^3x\,\partial_iF_a^{i0} \\ &=\int d^3x\left[-n^a\partial^2V+\frac{1}{r}(n_an_i-\delta_{ia})\partial_iV\right]. \end{aligned} \tag{10}$$

For spherically symmetric function $V(x)$, which gives rise to spherically symmetric external source distribution, I_a is always zero.

III. EXPLICIT EXAMPLES

We now proceed to discuss a few explicit expressions for the function $V(x)$:

(I) $V(x)$ is a regular function of r and has a finite support, for example $V(x)=\exp(-\alpha r)$, where α is a positive parameter. In other words we have a spherically symmetric source distribution which is localized and extended. In this situation, the electric field is radial,

$$E_i^a=\alpha n^an_ie^{-\alpha r}. \tag{11}$$

Clearly it is screened and vanishes with the external source j_a^0. The energy due to the non-Abelian electric field is finite and is given by

$$H=\frac{1}{2}\int d^3x\,E^2=(\pi/2\alpha). \tag{12}$$

The total conserved isospin is zero. This solution is not gauge equivalent to the screening solution of Sikivie and Weiss[5] and is not included in the spherically symmetric solutions found in Ref. 6 because the gauge-invariant quantity $B_i^aB_i^a$ is not zero here whereas it vanishes in Refs. 5 and 6.

(II) $V(x)=1/r$. The electric field is

$$E_i^a = n^a n_i / r^2 \,. \tag{13}$$

Here we have a situation in which a single external point source at the origin gives rise to non-Abelian electric and magnetic fields of the monopole type. In a sense this single point source could be regarded as a dyon. Of course at this stage we do not have any charge quantization. As expected, the total energy is divergent. The total isospin charge I_a vanishes, which means the external charge j_a^0 is completely screened. The present non-Abelian single-point-source solution is not gauge equivalent to the YM field configuration for one external point source found in Ref. 7 since the gauge-invariant quantities such as $B_a^i B_a^i$ are different.

(III) $V(x) = 1/r_1 \pm 1/r_2$, $r_1 = |\vec{x} - \vec{a}_1|$, $r_2 = |\vec{x} - \vec{a}_2|$, where $\vec{a}_1$ and $\vec{a}_2$ are two constant vectors denoting the locations of the electric point sources. This corresponds to two electric monopoles situated at positions $\vec{a}_1$ and $\vec{a}_2$ and a magnetic monopole at the origin. As Eq. (5b) is linear we could in fact consider $V(x)$ as a linear superposition of any finite number of electric poles at different points, but for convenience, we restrict ourselves to two electric monopoles. The interaction energy between the two electric monopoles can be easily computed and we have

$$\begin{aligned} H_{\text{int}} &= \int d^3x \left[(\mp) \frac{\vec{n}_1 \cdot \vec{n}_2}{r_1^2 r_2^2} \right. \\ &\qquad \left. \pm 4\pi \left(\frac{1}{r_1} \delta^{(3)}(\vec{r}_2) + \frac{1}{r_2} \delta^{(3)}(\vec{r}_1) \right) \right] \\ &= \pm 4\pi / b \,, \end{aligned} \tag{14}$$

where $b = |\vec{a}_1 - \vec{a}_2|$ and $\vec{n}_1, \vec{n}_2$ are, respectively, unit vectors along the directions $\vec{r}_1$ and $\vec{r}_2$. The force between the two electric point sources can be repulsive or attractive depending on the sign chosen in the expression for $V(x)$. The interaction energy given in Eq. (14) is of the Coulomb form and does not indicate the confinement of the two electric point sources. In passing we note that the non-Abelian charge density is

$$j_a^0 = 4\pi n_a [\delta^{(3)}(\vec{r}_1) \pm \delta^{(3)}(\vec{r}_2)]$$

and

$$q = (j_a^0 j_a^0)^{1/2} = 4\pi [\delta^{(3)}(\vec{r}_1) \pm \delta^{(3)}(\vec{r}_2)] \,.$$

(IV) We now consider the external electric charge distribution to be composed of a single point source at the origin and an extended charge distribution, that is, we put

$$V = \frac{1}{r} + e^{-\alpha r} \,. \tag{15}$$

The electric field strength is

$$E_i^a = n^a n_i \left(\frac{1}{r^2} + \alpha e^{-\alpha r} \right),$$

and the interaction energy is

$$H_{\text{int}} = 4\pi \,. \tag{16}$$

This cannot lead to the confinement of a non-Abelian electric point source since the interacting force is zero.

For two electric point sources interacting with a localized and extended charge distribution,

$$V(x) = \frac{1}{r_1} \pm \frac{1}{r_2} + e^{-\alpha r} \,, \tag{17}$$

the interaction energy can again be evaluated,

$$\begin{aligned} H_{\text{int}} = 4\pi \left[\pm \frac{1}{b} + \exp(-\alpha |\vec{a}_1|) \pm \exp(-\alpha |\vec{a}_2|) \right. \\ \left. + (1 \mp 1) \right], \end{aligned} \tag{18}$$

where b is the separation between the two electric point sources and $1/\alpha$ is a parameter which defines the spatial extension of the background charge distribution. The external isospin charge density for expression (17) is

$$\begin{aligned} j_a^0 = 4\pi n_a [\delta^{(3)}(\vec{r}_1) \pm \delta^{(3)}(\vec{r}_2)] \\ - \alpha n_a e^{-\alpha r} \left(\alpha - \frac{2}{r} \right). \end{aligned}$$

We now consider the two external point sources to be in opposite orientations in the isospin space, in other words, we choose the negative sign in Eq. (17). If the parameter α is small enough, the interaction energy (18) can be approximated as

$$H_{\text{int}} = 4\pi \left[-\frac{1}{b} - \alpha(|\vec{a}_1| - |\vec{a}_2|) + 2 \right]. \tag{19}$$

Note that $b = |\vec{a}_1 - \vec{a}_2| \geq |\vec{a}_1| - |\vec{a}_2|$. Thus at short distances, the interaction is dominated by the first term, the Coulomb force, whereas at large separation of the two point sources, the second term is more important. It is then possible to have confinement of two non-Abelian point sources. Note that when the separation b between the two point sources increases, the extension of the background charge distribution must increase correspondingly so that the parameter α is small enough to ensure the validity of the approximation (19). The H_{int} can be treated as the interquark potential and from the discussion above, confinement of two point sources can be achieved only if there exists a background charge distribution.

IV. COMMENTS

Finally we make some brief remarks.

(A) For the solutions discussed by us, the magnetic monopole does not interact with the external electric sources, interaction arises only between the electric sources. The choice of the function $A(x)$ in expression (2) as $gA(x) = -1/r$ has simplified the YM equations in the presence of external sources, but resulted in the decoupling of the magnetic monopole from the non-Abelian electric sources.

(B) The Wu-Yang ansatz[2] has also been used in Ref. 9 to solve Eq. (1) numerically. In Ref. 9 the function $V(x)$ and the charge density j_a^0 are restricted to have a radial symmetry, while the function $A(x)$ used there is more general than our choice here.

(C) It may be of interest to study the case in which $V(x)$ has a cylindrical symmetry.

(D) All the external sources discussed here have finite magnitude.

(E) To attain the confinement of two point sources as discussed in the example (IV), it is not essential that the term in $V(x)$ responsible for the background localized charge distribution be of the form $\exp(-\alpha r)$. It could be any regular exponential function of r, e.g., $\exp(-\alpha r^2)$. However, it is necessary that the parameter α be sufficiently small. The confining force will come into play only if the separation of the two sources as well as the size of the background charge distribution are large.

(F) The expression (6) is not the only one which can decouple the YM Eqs. (1) into two independent equations. The following ansatz will also perform the same job:

$$A_0^a = \epsilon^{ai3}\frac{x_i}{\rho}V(x)\,, \quad A_i^a = \delta^{a3}\epsilon_{ij3}\frac{x_j}{\rho}\frac{1}{g\rho}\,, \tag{20}$$

where $\rho^2 = {x_1}^2 + {x_2}^2$. Instead of Eqs. (15a) and (15b), one now arrives at

$$j_i^a = -\delta^{a3}\frac{2\pi}{g}\epsilon_{i3j}\partial_j(\delta(x_1)\delta(x_2))\,, \tag{21a}$$

$$j_0^a = \epsilon_{aj3}\frac{x^j}{\rho}\partial^2 V\,. \tag{21b}$$

In this case the magnetic field is due to an infinitely long solenoid of infinitesimal cross section,

$$B_i^a = -\frac{2\pi}{g}\delta_{a3}\delta_{i3}\delta(x_1)\delta(x_2)\,. \tag{22}$$

As before the function $V(x)$ will determine the charge density $j_0^a(x)$ and the non-Abelian electric field

$$E_i^a = -\epsilon^{aj3}\frac{x^j}{\rho}\partial_i V\,. \tag{23}$$

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