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Noether analysis for the hidden symmetry responsible for an infinite set of nonlocal currents

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In general two-dimensional principal chiral models we exhibit a parametric infinitesimal transformation which is defined for all field configurations and leaves the action invariant. This "hidden" symmetry (i.e., the invariance of action) leads, through a Noether-type analysis, to a parametric conservation law. Expanding in the parameter, we find a systematic procedure to write down the infinitesimal transformations responsible for higher nonlocal currents and thus complete the derivation of the infinite set of nonlocal currents as Noether currents.

I. INTRODUCTION

In the last few years, an infinite set of nonlocal conservation laws has been found in classical two-dimensional chiral models.[1-9] The existence of these nonlocal charges is a signal for the presence of a highly nontrivial hidden symmetry in these models.

From earlier derivations of these nonlocal currents which made use of either the equations of motion[2] or the solution-generating "dual symmetry"[1,4,8] or the "linearized" (i.e., inverse-scattering) equations,[3,5,6] the hidden symmetry was shown to exist in the solution subset of all field configurations.[7] Naturally there arises the following question: Does such a symmetry exist in the entire space of field configurations which leads, via Noether's theorem, to the infinite set of nonlocal currents?

Dolan and Roos[10] have given a partial answer to the question. They wrote down explicitly two nonlocal infinitesimal transformations, from which the first two nonlocal currents were derived as Noether currents. However, they were not able to simply generalize their results for arbitrary higher nonlocal currents.

In this paper we give a complete answer to this question for principal chiral models. In the spirit of using a parametric conservation law to summarize the infinite set of nonlocal currents,[3-8] we use a parametric infinitesimal transformation to summarize the infinite set of infinitesimal transformations responsible for the nonlocal currents. To find an appropriate form for the parametric transformation, we first present in Sec. II a new derivation for the conservation laws starting from a parametric symmetry transformation in the solution set. Then by generalizing it to arbitrary field configurations, we obtain the desired transformation which is displayed in Sec. III. In the same section, we also give a proof for the invariance of the action, thus showing that the parametric infinitesimal transformation is indeed a symmetry of the whole space of field configurations. In the subsequent sections (Secs. IV and V), using different methods, we show how to obtain nontrivial nonlocal conservation laws from the hidden symmetry. In particular, we find a systematic expansion for obtaining the infinitesimal transformations responsible for arbitrary higher nonlocal currents, so that we can claim that we have completed the derivation of the infinite set of nonlocal currents as Noether currents.

Finally, conclusions are summarized in Sec. VI, and a discussion about the interpretation of our transformation and possible generalizations of our results are also presented.

Our calculations are performed in Minkowski space-time. The metric used is $\eta_{00} = -\eta_{11} = 1$, $\epsilon_{01} = -\epsilon_{10} = 1$, $\epsilon^{01} = -\epsilon^{10} = -1$. All formulas in this paper may be extended to Euclidean space without difficulty.

II. A NEW ON-SHELL DERIVATION

To exhibit more explicitly the connection between nonlocal conserved charges and a hidden symmetry which exists at least on-shell, we present here a new derivation of nonlocal currents starting from an on-shell symmetry transformation, which will give us some hints to generalize the symmetry off-shell.

For the principal chiral models, the Lagrangian density is

$$\mathcal{L}(x) = \tfrac{1}{16}\,\mathrm{tr}\{\partial_\mu g(x)\partial^\mu g^{-1}(x)\}\,, \tag{2.1}$$

where $g(x) \in G$, a matrix Lie group. Defining

$$A_\mu(x) = g^{-1}\partial_\mu g\,, \tag{2.2}$$

the equations of motion obtained from Eq. (2.1) are

$$\partial^\mu A_\mu = 0\,. \tag{2.3}$$

As a pure gauge potential, $A_\mu(x)$ satisfies

$$\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0\,. \tag{2.4}$$

It is easy to see that under the global transformation

$$\delta g = -gT\,, \tag{2.5}$$

where $T=\alpha^a T_a$ (with infinitesimal constants α^a) belongs to the Lie algebra of the group G, the Lagrangian density (2.1) is invariant:

$$\delta\mathcal{L} = -\tfrac{1}{8}\mathrm{tr}\{A_\mu\partial^\mu(g^{-1}\delta g)\}=0\,. \tag{2.6}$$

This invariance gives rise to the conservation of A_μ. If the constant generator T in Eq. (2.5) is replaced by a space-time-dependent $T(x)$, the Lagrangian density is generally not invariant, as seen from Eq. (2.6).

However, if we assume $T(x)$ to be the following particular matrix function of space-time

$$T(x)=U(x;l)T\,U(x;l)^{-1}\,, \tag{2.7}$$

with $U(x;l)\in G$ satisfying the so-called "inverse-scattering equations" (l being a complex parameter)[6]

$$\begin{aligned}\partial_0 U &= \frac{l}{1-l^2}(lA_0-A_1)U\,,\\ \partial_1 U &= \frac{l}{1-l^2}(lA_1-A_0)U\,,\end{aligned} \tag{2.8}$$

then we can show that for those field configurations which satisfy Eqs. (2.3) and (2.4), the Lagrangian density (2.1) is changed by a total divergence under the infinitesimal transformation

$$\delta g = -gT(x)\equiv -gU(x)TU(x)^{-1}\,. \tag{2.9}$$

In fact, in this case we have

$$\begin{aligned}\delta\mathcal{L} &= -\tfrac{1}{8}\mathrm{tr}\{A_\mu\partial^\mu(g^{-1}\delta g)\}\\ &= \tfrac{1}{8}\mathrm{tr}\{A_\mu\partial^\mu(UTU^{-1})\}\\ &= \tfrac{1}{8}\mathrm{tr}\{[U^{-1}A_\mu U, U^{-1}\partial^\mu U]T\}\,.\end{aligned} \tag{2.10}$$

From Eq. (2.8) we can express $U^{-1}A_\mu U$ in terms of $U^{-1}\partial_\mu U$,

$$U^{-1}A_\mu U=\frac{1}{l}\epsilon_{\mu\nu}U^{-1}\partial^\nu U-U^{-1}\partial_\mu U\,. \tag{2.11}$$

Upon substituting Eq. (2.11) into Eq. (2.10), $\delta\mathcal{L}$ can be expressed as a total divergence

$$\delta\mathcal{L}=\tfrac{1}{8}\partial^\mu\,\mathrm{tr}\left\{\frac{2}{l}\epsilon_{\mu\nu}U^{-1}\partial_\nu UT\right\}. \tag{2.12}$$

This means that the action is invariant under the transformation (2.9).

It should be pointed out that since the integrability conditions of Eq. (2.8) are just Eq. (2.4) and the equations of motion Eq. (2.3), the transformation (2.9) is defined only for those field configurations $g(x)$, which satisfy Eqs. (2.3) and (2.4), and only for them is $\delta\mathcal{L}$ given by Eq. (2.12). Because of this, we call this symmetry, Eq. (2.9), an on-shell symmetry.

The derivation of nonlocal conserved currents from this on-shell symmetry is straightforward. Using the equations of motion Eq. (2.3), we easily obtain another expression for $\delta\mathcal{L}$:

$$\delta\mathcal{L}=-\tfrac{1}{8}\partial^\mu\,\mathrm{tr}\{A_\mu g^{-1}\delta g\}=\tfrac{1}{8}\partial^\mu\,\mathrm{tr}\{U^{-1}A_\mu UT\}\,. \tag{2.13}$$

Now from Eqs. (2.12) and (2.13) we get the conservation law

$$\partial^\mu J_\mu(x;l)=0\,, \tag{2.14}$$

where the parametric conserved current J_μ is

$$\begin{aligned}J_\mu(x;l) &= U(x;l)^{-1}A_\mu U(x;l)\\ &\quad -\frac{2}{l}\epsilon_{\mu\nu}U(x;l)^{-1}\partial^\nu U(x;l)\,,\end{aligned} \tag{2.15}$$

which summarizes the usual infinite number of nonlocal conserved currents. Performing a Taylor expansion in l around $l=0$, we obtain the desired infinite set of nonlocal charges given in Ref. 1, the first two being

$$\begin{aligned}Q_1 &= \int_{-\infty}^{+\infty}dx\,A_0(x,t)\,,\\ Q_2 &= \int_{-\infty}^{+\infty}dx\int_{-\infty}^{x}dx'\,A_0(x,t)A_0(x',t)\\ &\quad -\int_{-\infty}^{+\infty}dx\,A_1(x,t)\,.\end{aligned} \tag{2.16}$$

III. THE OFF-SHELL HIDDEN SYMMETRY

The above derivation is not in the spirit of Noether's theorem, because in a derivation of the conservation law in the manner of Noether one needs an off-shell symmetry, i.e., a symmetry of the whole space of field configurations. We will show in this section that we can indeed improve our derivation to exhibit the existence of an off-shell symmetry which is responsible for the infinite set of nonlocal charges.

To generalize the transformation (2.9) off-shell, it is sufficient to require that $U(x)$ should satisfy one of the inverse-scattering equations (2.8), e.g.,

$$\partial_1 U=\frac{l}{1-l^2}(lA_1-A_0)U\,. \tag{3.1}$$

Thus, we are led to consider the following nonlocal infinitesimal transformation defined for all field configurations:

$$\delta g=-gU(x;l)TU(x;l)^{-1} \tag{3.2}$$

with the space-time function $U(x)$ given as follows:

$$U(x;l)=P\exp\left(\frac{l}{1-l^2}\int_{-\infty}^{x'}dy[lA_1(y,t)-A_0(y,t)]\right). \tag{3.3}$$

To show that (3.2) is really an off-shell symmetry, we need to prove that for arbitrary $g(x)$, the change in $\mathcal{L}$ under (3.2) can still be expressed as a total divergence without using the equations of motion. Hence in this derivation, use of only Eqs. (3.1) and (2.4) is allowed. We start with Eq. (2.10), i.e.,

$$\delta\mathcal{L}=\tfrac{1}{8}\mathrm{tr}\{[U^{-1}A_0U,U^{-1}\partial_0U]T-[U^{-1}A_1U,U^{-1}\partial_1U]T\}. \tag{3.4}$$

From Eq. (3.1) we have

$$U^{-1}\partial_1U=\frac{l}{1-l^2}(lU^{-1}A_1U-U^{-1}A_0U). \tag{3.5}$$

Making use of this equation, we can rewrite $\delta\mathcal{L}$ as follows:

$$\begin{aligned}\delta\mathcal{L}&=\tfrac{1}{8}\mathrm{tr}\left\{\frac{1-l^2}{l}[-U^{-1}\partial_1U,U^{-1}\partial_0U]T+l[U^{-1}A_1U,U^{-1}\partial_0U]T+\frac{l}{1-l^2}[U^{-1}A_1U,U^{-1}A_0U]T\right\}\\&=\tfrac{1}{8}\mathrm{tr}\left\{\frac{1-l^2}{l}[U^{-1}\partial_0U,U^{-1}\partial_1U]T+l[U^{-1}A_1U,U^{-1}\partial_0U]T+l[U^{-1}A_1U,U^{-1}A_0U]T+l[U^{-1}\partial_1U,U^{-1}A_0U]T\right\}.\end{aligned} \tag{3.6}$$

Observe that, by using Eq. (2.4), we have

$$[U^{-1}A_1U,U^{-1}A_0U]=U^{-1}(\partial_0A_1-\partial_1A_0)U. \tag{3.7}$$

Then the sum of the last three terms in Eq. (3.6) can be recast into the form $-l\epsilon^{\mu\nu}\partial_\mu(U^{-1}A_\nu U)T$. Finally we get the desired form for $\delta\mathcal{L}$,

$$\delta\mathcal{L}=\tfrac{1}{8}\partial^\mu\,\mathrm{tr}\left\{\left(\frac{1-l^2}{l}\epsilon_{\mu\nu}U^{-1}\partial^\nu U-l\epsilon_{\mu\nu}U^{-1}A^\nu U\right)T\right\}, \tag{3.8}$$

which shows that the action is invariant under the off-shell infinitesimal transformation (3.2), with U given by Eq. (3.3).

In addition to the off-shell invariance of the action, the transformation (3.2) with (3.3) leads to the following two on-shell invariances:

(1) The equations of motion are invariant under δg.

(2) The variation of the energy-momentum density due to δg vanishes on-shell.

The proof follows from the fact that the function U defined by Eq. (3.3) satisfies both inverse-scattering equations (2.8), not just Eq. (3.1), for on-shell field configurations $g(x)$. In fact, writing $\delta g=-gT(x)$, the invariance of the equations of motion (2.3) requires

$$D_\mu\partial^\mu T(x)\equiv\partial_\mu\partial^\mu T(x)+[A_\mu,\partial^\mu T(x)]=0, \tag{3.9}$$

while for the energy-momentum density

$$T_{\mu\nu}=\tfrac{1}{8}\,\mathrm{tr}\{\tfrac{1}{2}g_{\mu\nu}A^\lambda A_\lambda-A_\mu A_\nu\},$$

the vanishing of its variation requires

$$\begin{aligned}\delta T_{00}&=\delta T_{11}\\&=-\tfrac{1}{8}\,\mathrm{tr}\{A_0D_0T(x)+A_1D_1T(x)\}=0,\\\delta T_{01}&=\delta T_{10}\\&=-\tfrac{1}{8}\,\mathrm{tr}\{A_0D_1T(x)+A_1D_0T(x)\}=0.\end{aligned} \tag{3.10}$$

For on-shell $g(x)$, using Eq. (2.8), we have

$$\partial_\mu T(x)\equiv\partial_\mu[U(x)TU(x)^{-1}]=l\epsilon_{\mu\nu}D^\nu T(x),$$

$$D_\mu T(x)=\left[\frac{1}{1-l^2}A_\mu-\frac{l}{1-l^2}\epsilon_{\mu\nu}A^\nu,T(x)\right],$$

from which it is easy to check Eqs. (3.9) and (3.10).

IV. NONLOCAL CURRENTS AS NOETHER CURRENTS

Now we turn to seeing how the off-shell hidden symmetry transformation (3.2) with (3.3) gives rise to the infinite set of nonlocal currents as Noether currents.

The simplest way to derive nonlocal conserved currents seems to be the following. For on-shell $g(x)$, Eq. (2.13) still holds. By equating Eq. (2.13) still holds. By equating Eq. (2.13) and Eq. (3.8) we are led to the "conservation law"

$$\partial^\mu\tilde{J}_\mu\equiv\partial^\mu\left\{U^{-1}A_\mu U+l\epsilon_{\mu\nu}U^{-1}A^\nu U-\frac{1-l^2}{l}\epsilon_{\mu\nu}U^{-1}\partial^\nu U\right\}=0. \tag{4.1}$$

However, from Eq. (3.1) we see that $\tilde{J}_0$ is identically zero even off-shell; similarly J_1 is equal to zero on-shell by Eq. (3.8). Thus the "conserved current" $\tilde{J}_\mu$ turns out to be a trivial current.

Nonetheless, this does not mean that we can obtain only a trivial current from the off-shell hidden symmetry (3.2) with (3.3). Actually, as shown below in this section, we can obtain the usual infinite set of nonlocal conserved currents by dropping some terms, which are total divergences and identically equal to zero, from the expression (3.8) for $\delta\mathcal{L}$.

To this end, we expand the function $U(x;l)$, defined by Eq. (3.3) in powers of the parameter l,

$$U(x;l)=1+\sum_{n=1}^{\infty}l^n\psi^{(n)}(x), \tag{4.2}$$

in which, for convenience of comparison with the known results in the literature, we put

$$\psi^{(1)}=\chi^{(1)},$$
$$\psi^{(2)}=\chi^{(2)}+\tfrac{1}{2}(\chi^{(1)})^2,$$
$$\psi^{(3)}=\chi^{(3)}+\tfrac{1}{2}(\chi^{(1)}\chi^{(2)}+\chi^{(2)}\chi^{(1)}), \tag{4.3}$$
$$\psi^{(4)}=\chi^{(4)}+\tfrac{1}{2}[(\chi^{(2)})^2+\chi^{(2)}(\chi^{(1)})^2-\tfrac{1}{4}(\chi^{(1)})^4]$$
$$+\tfrac{1}{2}(\chi^{(1)}\chi^{(3)}+\chi^{(3)}\chi^{(1)}),\ \ldots .$$

From $UU^{-1}=1$ we obtain the expansion of U^{-1} in l as follows:

$$U(x;l)^{-1}=1-\sum_{n=1}^{\infty} l^n\psi^{(-n)}(x), \tag{4.4}$$

where

$$\psi^{(-1)}=\chi^{(1)},$$
$$\psi^{(-2)}=\chi^{(2)}-\tfrac{1}{2}(\chi^{(1)})^2,$$
$$\psi^{(-3)}=\chi^{(3)}-\tfrac{1}{2}(\chi^{(1)}\chi^{(2)}+\chi^{(2)}\chi^{(1)}), \tag{4.5}$$
$$\psi^{(-4)}=\chi^{(4)}-\tfrac{1}{2}[(\chi^{(2)})^2-\chi^{(2)}(\chi^{(1)})^2-\tfrac{1}{4}(\chi^{(1)})^4]$$
$$-\tfrac{1}{2}(\chi^{(1)}\chi^{(3)}+\chi^{(3)}\chi^{(1)}),\ \ldots .$$

Substituting Eq. (4.2) with (4.3) into Eq. (3.1) and comparing the coefficients of the terms in l^n on both sides, the following equations satisfied by $\chi^{(n)}(x)$ are obtained recursively order by order:

$$\partial_1\chi^{(1)}=A_0,$$
$$\partial_1\chi^{(2)}=-A_1+\tfrac{1}{2}[\chi^{(1)},A_0], \tag{4.6}$$
$$\partial_1\chi^{(3)}=\tfrac{1}{2}[\chi^{(2)},A_0]-\tfrac{1}{2}[\chi^{(1)},A_1]$$
$$+\tfrac{1}{4}[[A_0,\chi^{(1)}],\chi^{(1)}]+\tfrac{1}{2}\chi^{(1)}A_0\chi^{(1)}+A_0,\ \ldots .$$

It is easy to obtain $\chi^{(n)}(x)$ by integrating the right-hand sides of the above equations. Using these $\chi^{(n)}(x)$'s, we define the following infinite set of infinitesimal transformations obtained by expanding the parametric transformation (3.2):

$$\delta^{(n)}g=-g\lambda^{(n)}\quad (n=1,2,3,\ldots) \tag{4.7}$$

$$\lambda^{(1)}=[\chi^{(1)},T],$$
$$\lambda^{(2)}=[\chi^{(2)},T]+\tfrac{1}{2}[\chi^{(1)},[\chi^{(1)},T]], \tag{4.8}$$
$$\lambda^{(3)}=[\chi^{(3)},T]+\tfrac{1}{2}[\chi^{(1)}T\chi^{(1)},\chi^{(1)}]+\tfrac{1}{2}[\chi^{(2)}[\chi^{(1)},T]]$$
$$+\tfrac{1}{2}[\chi^{(1)},[\chi^{(2)},T]].$$

By a direct but very tedious calculation, the variation of $\mathfrak{L}$ for $\delta^{(n)}g$

$$\delta^{(n)}\mathfrak{L}=\tfrac{1}{8}\,\mathrm{tr}(A_\mu\partial^\mu\lambda^{(n)}) \tag{4.9}$$

can be expressed as a total divergence order by order without using the equations of motion:

$$\delta^{(1)}\mathfrak{L}=\tfrac{1}{8}\partial_\mu\mathrm{tr}\{(\tfrac{1}{2}\epsilon^{\mu\nu}[\partial_\nu\chi^{(1)},\chi^{(1)}]-\epsilon^{\mu\nu}A_\nu)T\}, \tag{4.10a}$$
$$\delta^{(2)}\mathfrak{L}=\tfrac{1}{8}\partial_\mu\,\mathrm{tr}\{(\epsilon^{\mu\nu}[\partial_\nu\chi^{(1)},\chi^{(2)}]+\tfrac{1}{6}\epsilon^{\mu\nu}[[\partial_\nu\chi^{(1)},\chi^{(1)}],\chi^{(1)}]-\epsilon^{\mu\nu}[A_\nu,\chi^{(1)}])T\}, \tag{4.10b}$$
$$\delta^{(3)}\mathfrak{L}=\tfrac{1}{8}\partial_\mu\,\mathrm{tr}\{\epsilon^{\mu\nu}([\partial_\nu\chi^{(1)},\chi^{(3)}]+\tfrac{1}{2}[\chi^{(2)},\partial_\nu\chi^{(2)}]+\tfrac{1}{2}[\chi^{(1)},\partial_\nu\chi^{(1)}]+\tfrac{1}{4}[[\partial_\nu\chi^{(1)},\chi^{(1)}],\chi^{(2)}]$$
$$+\tfrac{1}{4}[[\partial_\nu\chi^{(2)},\chi^{(1)}],\chi^{(1)}]+\tfrac{1}{8}[\partial_\nu(\chi^{(1)})^2,(\chi^{(1)})^2]+[\chi^{(2)},A_\nu]-\tfrac{1}{2}[\chi^{(1)},[\chi^{(1)},A_\nu]])T\}. \tag{4.10c}$$

Using the equations of motion (2.3), from (4.9) $\delta^{(n)}\mathfrak{L}$ can be directly written as

$$\delta^{(n)}\mathfrak{L}=\tfrac{1}{8}\partial^\mu\ \mathrm{tr}\{A_\mu\lambda^{(n)}\}. \tag{4.11}$$

Equating Eqs. (4.11) and (4.10a)–(4.10c), respectively, we obtain the conserved currents

$$\partial^\mu J_\mu^{(n)}=0, \tag{4.12}$$

where

$$J_\mu^{(1)}=[A_\mu,\chi^{(1)}]+\epsilon_{\mu\nu}A^\nu-\tfrac{1}{2}\epsilon_{\mu\nu}[\partial^\nu\chi^{(1)},\chi^{(1)}], \tag{4.13a}$$
$$J_\mu^{(2)}=[A_\mu,\chi^{(2)}]+\tfrac{1}{2}[[A_\mu,\chi^{(1)}],\chi^{(1)}]-\epsilon_{\mu\nu}[\partial_\nu\chi^{(1)},\chi^{(2)}]$$
$$-\tfrac{1}{6}\epsilon_{\mu\nu}[[\partial^\nu\chi^{(1)},\chi^{(1)}],\chi^{(1)}]+\epsilon_{\mu\nu}[A^\nu,\chi^{(1)}], \tag{4.13b}$$
$$J_\mu^{(3)}=[A_\mu,\chi^{(3)}]-\tfrac{1}{2}[\chi^{(1)}A_\mu\chi^{(1)},\chi^{(1)}]+\tfrac{1}{2}[[A_\mu,\chi^{(1)}],\chi^{(2)}]+\tfrac{1}{2}[[A_\mu,\chi^{(2)}],\chi^{(1)}]$$
$$-\epsilon_{\mu\nu}([\partial^\nu\chi^{(1)},\chi^{(3)}]+\tfrac{1}{2}[\chi^{(2)},\partial^\nu\chi^{(2)}]+\tfrac{1}{2}[\chi^{(1)},\partial^\nu\chi^{(1)}]$$
$$+\tfrac{1}{4}[[\partial^\nu\chi^{(1)},\chi^{(1)}],\chi^{(2)}]+\tfrac{1}{4}[[\partial^\nu\chi^{(2)},\chi^{(1)}],\chi^{(1)}]$$
$$+\tfrac{1}{8}[\partial^\nu(\chi^{(1)})^2,(\chi^{(1)})^2]+[\chi^{(2)},A^\nu]-\tfrac{1}{2}[\chi^{(1)},[\chi^{(1)},A^\nu]]). \tag{4.13c}$$

These currents are obviously not trivial even on-shell. It can be easily checked that the corresponding nonlocal charges

$$Q^{(n)}=\int_{-\infty}^{+\infty}dx'\,J_0^{(n)}(x',t) \tag{4.14}$$

obtained from Eq. (4.13) are just the usual ones in literature [see also Eq. (2.16)]. In Eq. (4.13) the currents $J_\mu^{(n)}$ are defined for off-shell field configurations, and they reduce to the standard form when the field $g(x)$ are restricted on-shell.

For $n=1,2$, the above formulas Eqs. (4.6)–(4.13) are identical to corresponding ones, appearing in Ref. 9 [Eqs. (2.2)–(2.9)], except for some differences in sign arising from the difference in the signature of space-time. However, by a systematic expansion, we can obtain the infinitesimal transformations and higher nonlocal currents for $n>2$; for example, we have already worked out explicitly the case $n=3$ in detail, in which both the expression for $\lambda^{(3)}$, Eq. (4.8), and the equation for $\chi^{(3)}$, Eq. (4.6), are too complicated to be guessed.

It is interesting to compare Eq. (4.10) with Eq. (3.8). We find that Eqs. (4.10a)–(4.10c) are obtained if we expand the right-hand side of Eq. (3.8) in powers of l and drop terms which are identically zero such as, $\epsilon^{\mu\nu}\partial_\mu\partial_\nu\chi^{(n)}$, $\epsilon^{\mu\nu}(\partial_\mu\chi^{(1)}\chi^{(1)}\partial_\nu\chi^{(1)}+\partial_\nu\chi^{(1)}\chi^{(1)}\partial_\mu\chi^{(1)})$, $\epsilon^{\mu\nu}(\partial_\mu\chi^{(2)}\partial_\nu\chi^{(1)}+\partial_\nu\chi^{(2)}\partial_\mu\chi^{(1)})$, and so on. So in effect, the nontrivial currents $J_\mu^{(n)}$ or Eqs. (4.13a)–(4.13c) can also be obtained from the trivial current J_μ [see Eq. (4.1)] by expanding the latter in powers of l and then dropping terms whose divergence is identically zero.

We point out that we have the freedom to redefine the function $\chi^{(n)}(x)(n\geq 2)$ in terms of which $\delta^{(n)}g$ and $\delta^{(n)}\mathfrak{L}$ are expressed. We choose Eq. (4.3) to define $\chi^{(n)}(x)$ only for convenience in comparing with literature and to write all terms in $\delta^{(n)}\mathfrak{L}$ in the form of a series of commutators. Actually, the simplest way is to define $\chi^{(n)}(x)=\psi^{(n)}(x)$ [see Eq. (4.2)]. The above procedure can be applied to the redefined $\chi^{(n)}$; Eqs. (4.5)–(4.13) will change their appearance, but the physics is the same. Especially, the off-shell current $J_\mu^{(n)}$ may change by divergenceless terms, while the nth nonlocal charges $Q^{(n)}$ does not change on-shell.

V. PARAMETRIC NONLOCAL CURRENT AS NOETHER CURRENT

The merit of the derivation in the last section lies in the fact that it allows one to express the change in $\mathfrak{L}$ as a total divergence order by order, even for off-shell configurations. However, by means of this method one cannot obtain a parametric conserved current which summarizes an infinite set of nonlocal currents.

In order to derive a parametric current from the hidden symmetry (3.2) with (3.3), we observe that to get a conservation law from the symmetry it is sufficient to recast the $\delta\mathfrak{L}$ in Eq. (3.4) into a form which is a total divergence of a vector different from $U^{-1}A_\mu U$ only when on-shell. To this end, let us introduce the function

$$\chi(x)=\int_{-\infty}^{x^1}dx^1\left(\frac{1}{l}U^{-1}\partial_1 U-\frac{l}{1-l^2}U^{-1}(lA_0-A_1)U\right) \quad (5.1)$$

which satisfies

$$\partial_1\chi=\frac{1}{l}U^{-1}\partial_1 U-\frac{l}{1-l^2}U^{-1}(lA_0-A_1)U. \quad (5.2)$$

By a direct and somewhat lengthy calculation, in which only Eqs. (3.1) and (5.2) are used, we can recast the off-shell $\delta\mathfrak{L}$ in Eq. (3.4) into the form

$$\delta\mathfrak{L}=\tfrac{1}{8}\operatorname{tr}\left\{\left(\frac{2}{l}\epsilon^{\mu\nu}\partial_\mu(U^{-1}\partial_\nu U)+\frac{l}{1-l^2}[\chi,U^{-1}(\partial^\mu A_\mu)U]+\partial_\mu\left[\chi,\epsilon^{\mu\nu}U^{-1}\partial_\nu U-\frac{l}{1-l^2}U^{-1}(A^\mu+l\epsilon^{\mu\nu}A_\nu)U\right]\right)T\right\}. \quad (5.3)$$

It is easily seen that the last two terms vanish while on-shell so that $\delta\mathfrak{L}$ then becomes a total divergence. Then equating Eqs. (5.3) and (2.13) gives also the conservation law (2.14), so in some generalized sense, the parametric current (2.15) can also be viewed as a Noether current derived from the hidden symmetry transformation (3.2) with (3.3).

Incidentally, we observe that the total-divergence form for the on-shell $\delta\mathfrak{L}$ is not unique so that we can have several different forms for the parametric conserved current. For instance, on-shell we have

$$\delta\mathfrak{L}=\tfrac{1}{8}\partial_\mu\operatorname{tr}(U^{-1}A^\mu UT) \quad (5.4)$$

$$=\frac{1}{4l}\partial_\mu\operatorname{tr}(\epsilon^{\mu\nu}U^{-1}\partial_\nu UT) \quad (5.5)$$

$$=\frac{-l}{4(1+l^2)}\partial_\mu\operatorname{tr}(\epsilon^{\mu\nu}U^{-1}A_\nu UT) \quad (5.6)$$

$$=-\tfrac{1}{4}\partial_\mu\operatorname{tr}(U^{-1}\partial^\mu UT). \quad (5.7)$$

They are equivalent to each other through the inverse-scattering equation (2.8), i.e.,

$$\left(\partial_\mu+A_\mu-\frac{1}{l}\epsilon_{\mu\nu}\partial^\nu\right)U=0. \quad (5.8)$$

By choosing different pairs from these equations we may get different forms of parametric conservation laws. For example, the J_μ in Eq. (2.15) is obtained by choosing Eqs. (5.4) and (5.5); equating Eq. (5.4) and Eq. (5.6) gives the current as given in Ref. 5; and the conservation law as given in Ref. 3 can be recovered by combining Eqs. (5.5) and (5.7). Other combinations may be used to give some formally new (but essentially old) parametric conservation laws, e.g.,

$$\partial^\mu(U^{-1}\partial_\mu U+\tfrac{1}{2}U^{-1}A_\mu U)=0, \quad (5.9)$$

$$\partial^{\mu}\left(U^{-1}\partial_{\mu}U-\frac{l}{1+l^2}\epsilon_{\mu\nu}U^{-1}A^{\nu}U\right)=0\,, \tag{5.10}$$

$$\partial^{\mu}\epsilon_{\mu\nu}\left(U^{-1}\partial^{\nu}U+\frac{l}{1+l^2}U^{-1}A^{\nu}U\right)=0\,. \tag{5.11}$$

VI. CONCLUSIONS AND DISCUSSION

We have shown that in two-dimensional principal chiral models there exists a (off-shell) hidden symmetry which shifts the Lagrangian density by a total divergence and is responsible for the existence of an infinite sequence of nonlocal conserved charges. The infinitesimal transformation for the (off-shell) hidden symmetry is exhibited by Eq. (3.2) with (3.3). Since the matrix function U given by Eq. (3.3) depends on A_{μ} nonlocally, this infintesimal transformation (3.2) transforms the field $g(x)$ nonlinearly and nonlocally. Moreover, because the function U contains a complex parameter l, the infinitesimal transformation is also dependent on l, so that it leads to a parametric Noether current. Expanding the latter in l gives an infinite set of usual nonlocal currents. The derivation of the first two nonlocal currents as Noether currents in Ref. 9 becomes part of our discussion, appearing as the special case corresponding to the first two terms in the expansion of our parametric infinitesimal transformation. Our formalism also provides a systematic way to obtain the infinitesimal transformations for all higher nonlocal currents.

The infinitesimal transformation (3.2) with (3.3) proposed here has an interesting physical interpretation. We know that $\delta g=-gT$ represents a global isospin rotation. So our transformation $\delta g=-gU(x)TU(x)^{-1}$ represents a local isospin rotation, which looks like one and the same isospin rotation if seen at various space-time points from the local isospin frames obtained by doing a gauge rotation $V(x)=U(x)^{-1}$. Equation (3.3) implies that these local isospin frames are parallel to each other[11] along the equal-time lines with respect to the gauge potential $\tilde{A}_{\mu}=-l(lA_{\mu}+\epsilon_{\mu\nu}A^{\nu})/(1-l^2)$. For on-shell configurations, $\tilde{A}_{\mu}$ is also curvature-free[3,6] so the local isospin frames at all space-time points are parallel to each other with respect to $\tilde{A}_{\mu}$. However, we have not succeeded in understanding why just this potential appears in the symmetry transformation. Further implications and the geometrical interpretation of the transformation (3.2) deserve more attention.

We emphasize that the gauge rotation $U(x)^{-1}$ in the transformation is not an arbitrary one. It is a particular function of space-time, nonlocally dependent on the field $g(x)$ by Eq. (3.1). While on-shell it satisfies the inverse-scattering equations (2.8), in which the functions $U(x;l)$ can also be viewed as the dual transformation operators.[4,5] In this way, our off-shell parametric "hidden" symmetry can be viewed as the off-shell generalization of the usual on-shell, dual symmetry. It would be very interesting to generalize the off-shell dual symmetry to four-dimensional non-Abelian gauge theories.

It is easy to generalize the method and results of this paper to the discussion of hidden symmetry in more general cases such as supersymmetric chiral models[6,12] and nonlinear σ models on symmetric spaces.[4,8] The details for these generalizations will be published elsewhere. The discussion concerning the group structure of the symmetry and related problems are in progress.

Note added. After this paper was submitted, we became aware of a paper by T. Curtright and C. Zachos [Phys. Rev. D (to be published)] that parallels some of the discussion of the present paper for the O(N) Gross-Neveu model using a different approach.

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