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Construction of exact multimonopole solutions

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The newly found exact axially symmetric multimonopole solutions of arbitrary topological charge are constructed and proven to be real. Arguments supporting the regularity of the solutions are presented.

I. INTRODUCTION

Ward[1] made a major breakthrough in our understanding of the monopole sector of classical Yang-Mills gauge theories by presenting for the first time an exact axially symmetric monopole solution of topological charge two. Inspired by Ward's work, the present authors[2,3] recently found exact axially symmetric monopole solutions of arbitrary topological charge. The purpose of this paper is to give a construction of the solutions presented in Ref. 2 with a proof of their reality and to discuss the singularity problem. We present a number of checks and arguments in favor of the complete regularity, though they do not yet amount to a rigorous proof.

To recapitulate, let us define in four-dimensional Euclidean space (x_1, x_2, x_3, x_4) the gauge potentials A_μ^a where $a=1,2,3$ and $\mu=1,2,3,4$. The gauge-field strength is defined by

$$F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + e\epsilon^{abc}A_\mu^b A_\nu^c, \tag{1.1}$$

where e is an arbitrary constant, the gauge coupling constant. The problem, simply stated, is to solve the self-duality equations

$$F_{\mu\nu}^a = +\tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho}F_{\lambda\rho}^a \tag{1.2}$$

(our convention is $\epsilon_{1234}\equiv+1$) for the gauge potentials A_μ^a subject to the following requirements:

(i) In all gauges, A_μ^a are static (independent of x_4): $\partial_4 A_\mu^a = 0$. In this case A_4^a is referred to as the Higgs field.

(ii) In some gauge, A_μ^a are all real nonsingular functions of (x_1, x_2, x_3).

(iii) The gauge-invariant quantity $h^2 \equiv A_4^a A_4^a$ has the asymptotic form

$$h^2 = f^2 - \frac{2fn}{er} + O(r^{-2}) \text{ as } r\to\infty, \tag{1.3}$$

where $r^2 \equiv x_1^2 + x_2^2 + x_3^2$, f is an arbitrary constant with dimensions of inverse length, and n is a positive integer called the topological charge. We assume that $ef > 0$. (To compare our formulas with Ward's, one should set $f=1$, $e=2$.) The energy E of the monopole is then

$$E \equiv \int \tfrac{1}{4}F_{\mu\nu}^a F_{\mu\nu}^a d^3x = \tfrac{1}{2}\int \nabla^2 h^2 d^3x = 4\pi fg, \tag{1.4}$$

where we have defined the magnetic charge g to be (n/e). From now on we will use units in which $e=f=1$.

In Ref. 2 it was found that requirements (i) and (iii) can be easily satisfied whereas (ii) is very difficult to implement and for $n=3$ was originally done using the symbol-manipulating computer program MACSYMA. For $n>3$ it was found that requiring the gauge field be real and nonsingular on the x_3 axis gives a unique candidate for the n-monopole solution. One must still prove that this candidate gives real nonsingular gauge fields over all three-dimensional Euclidean space, and this is the very purpose of this paper.

The n-monopole solutions are constructed from $n\times n$ determinants that are, however, too cumbersome to evaluate for any practical purpose, such as a direct check of requirement (ii). We have realized, however, that in order to prove reality and nonsingularity one does not have to explicitly compute these determinants. In particular the proof of reality is extremely simple, being a straightforward generalization of Ward's original proof.

The singularity problem, however, is much more involved and amounts to checking that a certain $n\times n$ determinant is nonvanishing everywhere in three-dimensional Euclidean space. Since we are dealing with an elliptic (albeit nonlinear) problem, a proper way to handle it consists of looking for two separate classes of solutions. The first class is regular at infinity except possibly for a line singularity along the x_3 axis. The second class is regular on the x_3 axis (and on the $x_3=0$ plane) and it is real analytic around it.

The proper way to analyze regularity at infinity is to start from the class of solutions first introduced in Ref. 2 involving determinants of some function Δ_l where $-n<l<n$ that we have found to be expressible in terms of derivatives of modified spherical Bessel functions of negative order. Using Δ_l we are able to exactly compute the Abelian part h^A of the norm h of the Higgs field A_4^a:

$$h^A = 1 - \sum_{k=1}^{n} \frac{1}{r_k}, \tag{1.5a}$$

$$r_k^2 = x_1^2 + x_2^2 + (x_3 - q_k)^2, \tag{1.5b}$$

where q_k are arbitrary complex constants constrained only by the requirement that h^A be real. Equation (1.5a) is valid everywhere up to exponentially damped corrections of order e^{-2r_k} around each singularity of h^A.

On the other hand, the proper way to analyze regularity on the x_3 axis and the $x_3=0$ plane is to start from a class of functions $\tilde{\Delta}_l$ where $-n<l<n$ and which are power-series expandable around the origin. We have found $\tilde{\Delta}_l$ to be expressible in terms of integrals of modified cylindrical Bessel functions. Our fundamental result is the proof that for our unique candidate n-monopole solution

$$\Delta_l = \tilde{\Delta}_l \quad \text{for } -n<l<n. \tag{1.6}$$

The miraculous equality expressed in Eq. (1.6) is the heart of our proof of reality and provides the matching we were looking for between solutions belonging to the two different classes. A full proof of regularity would now require showing that the domains of regularity of the two (coincident) solutions cover all three-dimensional space.

In the course of our analysis, we have found a number of interesting and elegant mathematical structures that are discussed in the appendices. First of all, Eq. (1.6) is a yet unknown property of Bessel functions and second, we can prove the equivalence of Toeplitz determinants involved in our solution to a special case of U(n) group integrals over the invariant Haar measure. Finally, this allows us to establish a formal equivalence of these determinants to the partition function of two-dimensional lattice quantum chromodynamics (QCD) and as a consequence we can also discuss the large-n limit of monopoles.

II. FORMULATION OF THE SELF-DUALITY EQUATIONS[4]

We begin by defining the 2×2 matrix-valued fields

$$A_\mu \equiv \frac{\sigma^a}{2i} A_\mu^a \text{ and}$$
$$F_{\mu\nu} \equiv \frac{\sigma^a}{2i} F_{\mu\nu}^a = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{2.1}$$

where σ^a are the Pauli matrices. For real gauge fields, A_μ and $F_{\mu\nu}$ are anti-Hermitian traceless matrices. We now analytically continue A_μ into complex space where x_1, x_2, x_3, x_4 are complex. The self-duality equations (1.2) are then valid also in complex space, in a region containing real space where the x's are real. Now consider the four new complex variables p, $\bar{p}$, q, and $\bar{q}$ defined by

$$\sqrt{2}\,p \equiv x_1 + ix_2, \quad \sqrt{2}\,\bar{p} \equiv x_1 - ix_2,$$
$$\sqrt{2}\,q \equiv x_3 - ix_4, \quad \sqrt{2}\,\bar{q} \equiv x_3 + ix_4. \tag{2.2}$$

The self-duality equations (1.2) then reduce to

$$F_{pq}=0, \quad F_{\bar{p}\bar{q}}=0, \quad \text{and } F_{p\bar{p}} + F_{q\bar{q}} = 0. \tag{2.3}$$

The equations $F_{pq}=0$ and $F_{\bar{p}\bar{q}}=0$ can be immediately integrated, since they are pure gauge, to give

$$A_p = D^{-1}D_p, \quad A_q = D^{-1}D_q,$$
$$A_{\bar{p}} = \bar{D}^{-1}\bar{D}_{\bar{p}}, \quad A_{\bar{q}} = \bar{D}^{-1}\bar{D}_{\bar{q}}, \tag{2.4}$$

where D and $\bar{D}$ are arbitrary 2×2 complex matrix functions of p, $\bar{p}$, q, and $\bar{q}$ with determinant $=1$ and $D_p \equiv \partial_p D$, etc. Gauge transformations are the transformations

$$D \to \bar{V}(\bar{p}, \bar{q}) D \mathcal{L}, \quad \bar{D} \to V^{-1}(p,q)\bar{D}\mathcal{L}, \tag{2.5}$$

where $\mathcal{L}$ is an arbitrary complex matrix function of p, $\bar{p}$, q, and $\bar{q}$ whereas $\bar{V}$ (V) is an arbitrary complex matrix function of $\bar{p}, \bar{q}$ (p, q) and we take the determinants of $\mathcal{L}$, V, and $\bar{V}$ to be one. Under the gauge transformation (2.5) the gauge potential A_μ and gauge-field strength $F_{\mu\nu}$ transform as

$$A_\mu \to \mathcal{L}^{-1}A_\mu\mathcal{L} + \mathcal{L}^{-1}\partial_\mu\mathcal{L}, \quad F_{\mu\nu} \to \mathcal{L}^{-1}F_{\mu\nu}\mathcal{L}. \tag{2.6}$$

The energy density $F_{\mu\nu}^a F_{\mu\nu}^a = -2\,\mathrm{Tr}(F_{\mu\nu}F_{\mu\nu})$ is invariant under gauge transformations. Let us define a matrix J by

$$J \equiv D\bar{D}^{-1}, \tag{2.7}$$

then the remaining self-duality equation $F_{p\bar{p}} + F_{q\bar{q}} = 0$ becomes

$$(J^{-1}J_p)_{\bar{p}} + (J^{-1}J_q)_{\bar{q}} = 0. \tag{2.8}$$

Under the gauge transformations (2.5), J transforms as

$$J \to \bar{V}(\bar{p}, \bar{q}) J V(p,q). \tag{2.9}$$

Since J is an arbitrary complex 2×2 matrix function with determinant one it can be parametrized as

$$J = \begin{pmatrix} \dfrac{1}{\phi} & \dfrac{\bar{\rho}}{\phi} \\ \dfrac{\rho}{\phi} & \dfrac{\phi^2 + \rho\bar{\rho}}{\phi} \end{pmatrix}, \tag{2.10}$$

where ϕ, ρ, and $\bar{\rho}$ are arbitrary and independent complex functions of p, $\bar{p}$, q, and $\bar{q}$. The self-duality equations (2.8) in terms of ϕ, ρ, and $\bar{\rho}$ become

$$(\partial_p\partial_{\bar{p}}+\partial_q\partial_{\bar{q}})\ln\phi+\frac{(\rho_p\bar{\rho}_{\bar{p}}+\rho_q\bar{\rho}_{\bar{q}})}{\phi^2}=0\,, \tag{2.11a}$$

$$\left(\frac{\rho_p}{\phi^2}\right)_{\bar{p}}+\left(\frac{\rho_q}{\phi^2}\right)_{\bar{q}}=0\,,\quad \left(\frac{\bar{\rho}_{\bar{p}}}{\phi^2}\right)_{p}+\left(\frac{\bar{\rho}_{\bar{q}}}{\phi^2}\right)_{q}=0\,. \tag{2.11b}$$

To construct the gauge potentials A_μ from J requires a selection of gauge [i.e., there are an infinite number of ways of factoring (2.10) in the form (2.7)]. We will work exclusively in Yang's R gauge which is defined by $J\equiv R\bar{R}^{-1}$ where

$$R=\begin{pmatrix}\frac{1}{\sqrt{\phi}} & 0\\ \frac{\rho}{\sqrt{\phi}} & \sqrt{\phi}\end{pmatrix},\quad \bar{R}=\begin{pmatrix}\sqrt{\phi} & -\frac{\bar{\rho}}{\sqrt{\phi}}\\ 0 & \frac{1}{\sqrt{\phi}}\end{pmatrix}. \tag{2.12}$$

The gauge potentials in the R gauge take the form

$$A_u=\begin{pmatrix}-\frac{\phi_u}{2\phi} & 0\\ \frac{\rho_u}{\phi} & +\frac{\phi_u}{2\phi}\end{pmatrix},\quad A_{\bar{u}}=\begin{pmatrix}\frac{\phi_{\bar{u}}}{2\phi} & -\frac{\bar{\rho}_{\bar{u}}}{\phi}\\ 0 & -\frac{\phi_{\bar{u}}}{2\phi}\end{pmatrix}, \tag{2.13}$$

where $u\equiv p,q$. Now because of requirement (i) the gauge potentials (2.13) must be x_4 independent and we require this to be true in all gauges which implies that the gauge transformation matrix $\mathcal{L}$ must be x_4 independent:

$$\partial_4\mathcal{L}=0 \tag{2.14}$$

by virtue of (2.6). In particular $h^2\equiv A_4^aA_4^a=-2\,\mathrm{Tr}(A_4A_4)$ is gauge invariant.

Even if the gauge potentials A_μ derived from (2.13) [e.g., $A_4=(-i/\sqrt{2})(A_q-A_{\bar{q}})$] are static, they will not in general be anti-Hermitian as demanded by requirement (ii). For real gauge fields $A_\mu^\dagger\doteq-A_\mu$ (the symbol $\doteq$ is used for equations valid only for real values of x_1, x_2, x_3, x_4) we have from (2.4) and (2.7)

$$A_\mu^\dagger\doteq-A_\mu\Rightarrow\bar{D}\doteq(D^\dagger)^{-1}\Rightarrow J=D\bar{D}^{-1}\doteq DD^\dagger\,, \tag{2.15}$$

i.e., for real gauge fields J is a positive-definite 2×2 Hermitian matrix. A necessary and sufficient condition to meet requirement (ii) (we must, of course, also check that the gauge fields are nonsingular functions of x_1, x_2, and x_3) is that we be able to find matrices $\bar{V}(\bar{p},\bar{q})$ and $V(p,q)$ in Eq. (2.9) such that $\bar{V}(\bar{p},\bar{q})R\bar{R}^{-1}V(p,q)$ is a positive-definite Hermitian matrix. If we can find such matrices V and $\bar{V}$ then the gauge transformation matrix $\mathcal{L}$ in (2.6) is simply a square root of the matrix,

$$\mathcal{L}\mathcal{L}^\dagger=(R^\dagger\bar{V}^\dagger V^{-1}\bar{R})^{-1}\,, \tag{2.16}$$

and this matrix $\mathcal{L}$ will make A_μ and $F_{\mu\nu}$ anti-Hermitian as demanded by requirement (ii).

III. SOLUTION OF THE SELF-DUALITY EQUATIONS (2.11): THE ATIYAH-WARD ANSATZ[5,6]

Let us define $(2n+1)$ functions Δ_l where $-n\leq l\leq n$ which satisfy the following equations:

$$\partial_p\Delta_l=-\partial_{\bar{q}}\Delta_{l+1}\,,\quad \partial_q\Delta_l=\partial_{\bar{p}}\Delta_{l+1}\,. \tag{3.1}$$

It is now convenient to define the following notation:

$H^{j\times j}_{k\pm l+m}\equiv$ determinant of the $j\times j$ matrix whose entry in the kth row and lth column is given by $\Delta_{k\pm l+m}$ where the Δ's are defined by (3.1) and j, k, l are positive integers whereas m can be any positive or negative integer or zero. (3.2)

We can now state the nth Atiyah-Ward ansatz $\mathfrak{A}_n$ as follows. The self-duality equations (2.11) are solved by $({}_n\phi, {}_n\rho, {}_n\bar{\rho})$ where

$${}_n\phi=(-1)^{n+1}\frac{H^{n\times n}_{k+l-n-1}}{H^{n-1\times n-1}_{k+l-n}}=\frac{H^{n\times n}_{k-l}}{H^{n-1\times n-1}_{k-l}}\,, \tag{3.3a}$$

$${}_n\rho=-\frac{H^{n\times n}_{k+l-n-2}}{H^{n-1\times n-1}_{k+l-n}}=(-1)^n\frac{H^{n\times n}_{k-l-1}}{H^{n-1\times n-1}_{k-l}}\,, \tag{3.3b}$$

$${}_n\bar{\rho}=+\frac{H^{n\times n}_{k+l-n}}{H^{n-1\times n-1}_{k+l-n}}=(-1)^{n+1}\frac{H^{n\times n}_{k-l+1}}{H^{n-1\times n-1}_{k-l}}\,. \tag{3.3c}$$

A useful relation that follows from Jacobi's theorem on subdeterminants of adjugate matrices is

$${}_n\phi^2+{}_n\rho_n\bar{\rho}=-\frac{H^{n+1\times n+1}_{k+l-n-2}}{H^{n-1\times n-1}_{k+l-n}}=\frac{H^{n+1\times n+1}_{k-l}}{H^{n-1\times n-1}_{k-l}}\,. \tag{3.4}$$

In Ref. 2 it was shown that the Δ's are given by

$$\Delta_0={}_1\phi=e^{ix_4}\Lambda_0\,, \tag{3.5a}$$

$$\Delta_k=(-1)^ke^{ix_4}(\sqrt{2}p)^{-k}(1-\partial_3)^k\Lambda_k\quad(k=1,2,\ldots,n)\,, \tag{3.5b}$$

$$\Delta_{-k}=(-1)^ke^{ix_4}(\sqrt{2}\bar{p})^{-k}(1+\partial_3)^k\Lambda_k\quad(k=1,2,\ldots,n)\,, \tag{3.5c}$$

where

$$\Lambda_0=\sum_{k=1}^n\alpha_k\frac{\sinh r_k}{r_k}\,, \tag{3.6a}$$

$$\Lambda_k=\bar{p}^{-1}\partial_p\Lambda_{k+1}=p^{-1}\partial_{\bar{p}}\Lambda_{k+1}\quad(k=0,1,2,\ldots,n)\,, \tag{3.6b}$$

$$r_k^2=x_1^2+x_2^2+(x_3-z_k)^2\,, \tag{3.6c}$$

and α_k and z_k are arbitrary complex constants constrained only by the requirement that Λ_k be a real function of x_1, x_2, and x_3 for $k=0,1,2,\ldots,n$.

IV. THE ARBITRARY n-MONOPOLE SOLUTION

In Ref. 2 it was shown that to obtain a monopole solution of arbitrary topological charge $n \geq 1$ one should take the $\mathfrak{a}_n$ ansatz with

$$\Lambda_0 = \sum_{k=1}^{n} \alpha_k \frac{\sinh r_k}{r_k}, \tag{4.1a}$$

where

$$r_k^2 = x_1^2 + x_2^2 + (x_3 - z_k)^2, \tag{4.1b}$$

$$z_k = \left[\frac{(n+1)}{2} - k\right](i\pi), \tag{4.1c}$$

$$\alpha_k = \frac{(n-1)!}{(k-1)!(n-k)!}. \tag{4.1d}$$

Note that $z_k - z_{k-1} = i\pi$ and α_k are the binomial coefficients. From now on we will assume that α_k and z_k are given by (4.1d) and (4.1c), respectively.

Choosing the matrices $\bar{V}(\bar{p}\bar{q})$ and $V(pq)$ to be

$$\bar{V}(\bar{p}\bar{q}) = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad V(pq) = \begin{bmatrix} 0 & \gamma^{-1}(\sqrt{2}p)^{-n} \\ -\gamma(\sqrt{2}p)^n & 0 \end{bmatrix}, \tag{4.2}$$

and requiring that $\bar{V}R\bar{R}^{-1}V$ be a positive-definite Hermitian matrix for $({}_n\phi, {}_n\rho, {}_n\bar{\rho})$ constructed from (4.1), it was found that in order to have real gauge fields,

$$\gamma \text{ must be a real constant}, \tag{4.3a}$$

$$-\gamma^2(2p\bar{p})^n({}_n\phi^2 + {}_n\rho_n\bar{\rho})e^{-2ix_4} = \gamma^2[(2\pi)^{n-1}(n-1)!]^2 = +1. \tag{4.3b}$$

It was also asserted in Ref. 2 that the following determinants

$${}_1\phi_2\phi \cdots {}_j\phi = (-1)^{j(j-1)/2} H^{j\times j}_{k+l-j-1} = H^{j\times j}_{k-l} \tag{4.4}$$

are nonvanishing for $j=1,2,\ldots,n$ and thereby ensuring the gauge fields are nonsingular. Equation (4.4) allows us to write the "superposition formula" of Ref. 2 as follows:

$$h^2 \equiv A_4^a A_4^a = 1 - \nabla^2 \ln H^{n\times n}_{k-l}. \tag{4.5}$$

V. EXPLICIT FORMULAS FOR Λ_n AND INTEGRAL REPRESENTATIONS FOR Δ_l

Let us denote by $I_\nu(x)$ the modified Bessel function of x of the first kind of order ν. Explicitly,

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)}, \tag{5.1}$$

from which it follows that for real x and $\nu > -1$, $x^{-\nu}I_\nu(x)$ is a nonvanishing positive function of x.

Using Eqs. (3.6) the formulas for Λ_k given in Ref. 2 can be compactly written as follows:

$$\Lambda_j = \left(\frac{\pi}{2}\right)^{1/2} \sum_{k=1}^{n} \alpha_k r_k^{\,j-1/2} I_{1/2-j}(r_k) \quad (j=0,1,2,\ldots,n), \tag{5.2}$$

where α_k and r_k are given by (4.1).

Let us now define the real variables s and θ as follows:

$$\sqrt{2}p = x_1 + ix_2 = se^{i\theta}, \quad \sqrt{2}\bar{p} = x_1 - ix_2 = se^{-i\theta}, \tag{5.3a}$$

so that

$$s^2 = x_1^2 + x_2^2, \quad \theta = \tan^{-1}(x_2/x_1). \tag{5.3b}$$

There exists a very useful integral representation for $(\sinh r)/r$ which is[7]

$$\frac{\sinh(x_1^2+x_2^2+x_3^2)^{1/2}}{(x_1^2+x_2^2+x_3^2)^{1/2}} = \frac{1}{2}\int_{-1}^{1} e^{-x_3 t} I_0(s(1-t^2)^{1/2})\,dt. \tag{5.4}$$

Using Eqs. (4.1), (5.2), and (5.4) one finds

$$\Lambda_0 = \frac{1}{2}\sum_{k=1}^{n} \alpha_k \int_{-1}^{1} e^{-(x_3-z_k)t} I_0(s(1-t^2)^{1/2})\,dt \tag{5.5a}$$

$$= \frac{1}{2}\int_{-1}^{1} \left[2\cos\left(\frac{\pi t}{2}\right)\right]^{n-1} e^{-x_3 t} I_0(s(1-t^2)^{1/2})\,dt. \tag{5.5b}$$

From (5.5b) it is obvious that Λ_0 is a nonvanishing positive function for all values of (x_1, x_2, x_3), i.e., over all three-dimensional Euclidean space.

Let us now define the following functions:

$$\tilde{\Delta}_l \equiv \frac{1}{2}(-1)^l e^{ix_4 - il\theta} \int_{-1}^{1} \left[2\cos\left(\frac{\pi t}{2}\right)\right]^{n-1} e^{-x_3 t}\left(\frac{1+t}{1-t}\right)^{l/2} I_l(s(1-t^2)^{1/2})\,dt, \tag{5.6}$$

where $-n \leq l \leq n$. It is shown in Appendix A that the functions Δ_l as defined in (3.5) are given by

$$\Delta_l = \tilde{\Delta}_l \quad \text{for } -n < l < n, \tag{5.7a}$$

$$\Delta_n = \tilde{\Delta}_n - (\sqrt{2}p)^{-n}[(2\pi)^{n-1}(n-1)!]e^{-\sqrt{2}q}, \tag{5.7b}$$

$$\Delta_{-n} = \tilde{\Delta}_{-n} - (\sqrt{2}\bar{p})^{-n}[(2\pi)^{n-1}(n-1)!]e^{\sqrt{2}\bar{q}}. \tag{5.7c}$$

VI. EXACT EQUIVALENCE OF POTENTIALS CONSTRUCTED FROM Δ_l TO THOSE CONSTRUCTED FROM $\tilde{\Delta}_l$

Equations (5.7) prove that within the range $-n \leq l \leq n$ the functions Δ_l and $\tilde{\Delta}_l$ are equal to each other except at the end points $l=n$ and $l=-n$. We will now prove that the gauge potentials constructed from $({}_n\phi, {}_n\rho, {}_n\bar{\rho})$ using (3.3) are exactly equivalent to those constructed from $({}_n\tilde{\phi}, {}_n\tilde{\rho}, {}_n\tilde{\bar{\rho}})$ using (3.3) where the Δ's are replaced by $\tilde{\Delta}$'s.

Using Eqs. (3.3) and (5.7) it is easy to verify that

$$ {}_n\phi = {}_n\tilde{\phi}\,, \tag{6.1a}$$

$$ {}_n\rho = {}_n\tilde{\rho} + (\sqrt{2}\bar{p})^{-n}(2\pi)^{n-1}(n-1)!\,e^{\sqrt{2}\bar{q}}\,, \tag{6.1b}$$

$$ {}_n\bar{\rho} = {}_n\tilde{\bar{\rho}} - (\sqrt{2}p)^{-n}(2\pi)^{n-1}(n-1)!\,e^{-\sqrt{2}q}\,, \tag{6.1c}$$

from which it follows

$$ {}_n\phi = {}_n\tilde{\phi}\,, \quad {}_n\rho_u = {}_n\tilde{\rho}_u\,, \quad {}_n\bar{\rho}_{\bar{u}} = {}_n\tilde{\bar{\rho}}_{\bar{u}}\,, \tag{6.2}$$

for $u=p, q$ and thus, by virtue of Eq. (2.13), proving the assertion made in the beginning of this section.

VII. PROOF OF REALITY

In this section we will prove that the complex gauge potentials constructed from $({}_n\tilde{\phi}, {}_n\tilde{\rho}, {}_n\tilde{\bar{\rho}})$ using (3.3) where the Δ's are replaced by $\tilde{\Delta}$'s are in fact complex gauge equivalent to real gauge potentials. Our proof is a straightforward generalization of Ward's original proof for the charge-2 monopole.

Let us define three new complex variables ω_1, ω_2, and ω as follows (we warn the reader that our notation differs radically from Ward's):

$$\sqrt{2}\,\omega_1 = (\bar{q} - p\zeta)\,, \quad \sqrt{2}\,\omega_2 = -(q + \bar{p}\zeta^{-1})\,, \tag{7.1a}$$

$$\omega = \omega_1 - \omega_2 = x_3 - \frac{s}{2}(e^{i\theta}\zeta - e^{-i\theta}\zeta^{-1})\,, \tag{7.1b}$$

where ζ is to be regarded as a complex parameter ranging over the extended complex line, i.e., the Riemann sphere.

Atiyah and Ward in their original paper[5] defined a so-called transition matrix $\mathcal{G}^{(n)}(\omega_1, \omega_2, \zeta)$ by

$$\tilde{\mathcal{G}}^{(n)}(\omega_1, \omega_2, \zeta) = \begin{bmatrix} \zeta^n & \tilde{\Omega}^{(n)}(\omega_1, \omega_2, \zeta) \\ 0 & \zeta^{-n} \end{bmatrix}, \tag{7.2a}$$

where

$$\tilde{\Omega}^{(n)}(\omega_1, \omega_2, \zeta) = \sum_{k=-\infty}^{\infty} \tilde{\Delta}_k \zeta^{-k}\,, \tag{7.2b}$$

and argued that solutions of the self-duality equations (2.11) can be extracted from (7.2) using techniques of algebraic geometry and twistor theory. It was Corrigan, Fairlie, Goddard, and Yates[6] who showed how to explicitly extract $({}_n\tilde{\phi}, {}_n\tilde{\rho}, {}_n\tilde{\bar{\rho}})$ from (7.2) leading eventually to the nth Atiyah-Ward ansatz $\mathfrak{A}_n$ as defined in Eq. (3.3) with Δ's replaced by $\tilde{\Delta}$'s.

In the context of monopoles, Ward[1] argued that in order for the gauge potentials extracted from (7.2) to be static and real it was sufficient to assume there exist two 2×2 matrices $\tilde{Q}_L^{(n)}$ and $\tilde{Q}_R^{(n)}$ such that

$$\tilde{Q}_L^{(n)}\tilde{\mathcal{G}}^{(n)}(\omega_1, \omega_2, \zeta)\tilde{Q}_R^{(n)} = \mathcal{G}^{(n)}(\omega, \zeta)\,, \tag{7.3a}$$

$$\tilde{Q}_L^{(n)} \text{ is analytic away from } \zeta = 0\,, \tag{7.3b}$$

$$\tilde{Q}_R^{(n)} \text{ is analytic away from } \zeta = \infty\,, \tag{7.3c}$$

$$\det \mathcal{G}^{(n)}(\omega, \zeta) = +1\,, \tag{7.3d}$$

$$[\mathcal{G}^{(n)}(\omega, \zeta)]^\dagger = \mathcal{G}^{(n)}(\omega^*, -\zeta^{*-1})\,. \tag{7.3e}$$

We now proceed to compute $\tilde{\mathcal{G}}^{(n)}(\omega_1, \omega_2, \zeta)$ for the n-monopole solution (4.1) with $\tilde{\Delta}_l$ given by (5.6). Substituting (5.6) into (7.2b) and using the relation

$$\sum_{k=-\infty}^{\infty} I_k(x)\zeta^{-k} = e^{x(\zeta+\zeta^{-1})/2} \tag{7.4}$$

gives

$$\tilde{\Omega}^{(n)}(\omega_1, \omega_2, \zeta) = \frac{(n-1)!\,\pi^{n-1}}{2}\,\frac{[e^{2\omega_1} + (-1)^n e^{2\omega_2}]}{\prod_{k=1}^{n}(\omega - z_k)}\,. \tag{7.5}$$

Equation (7.5) could also have been derived as follows. The essence of the n-monopole solution (4.1) is contained in the following "splitting rule":

$$\tilde{\Omega}^{(n)}(\omega_1, \omega_2, \zeta) = \tilde{\Omega}^{(n-1)}\left(x_3 + \frac{i\pi}{2}\right) + \tilde{\Omega}^{(n-1)}\left(x_3 - \frac{i\pi}{2}\right), \tag{7.6}$$

so that Eq. (7.5) could have been derived by just knowing $\tilde{\Omega}^{(1)}$ given in Ward's paper.[1]

We now choose the matrices $\tilde{Q}_L^{(n)}$ and $\tilde{Q}_R^{(n)}$ to be

$$\tilde{Q}_L^{(n)} = \begin{bmatrix} e^{-\omega_2} & 0 \\ 0 & e^{\omega_2} \end{bmatrix}, \tag{7.7a}$$

$$\tilde{Q}_R^{(n)} = \begin{bmatrix} 0 & -e^{\omega_1} \\ e^{-\omega_1} & \zeta^n \dfrac{2e^{-\omega_1}}{\pi^{n-1}(n-1)!}\prod_{k=1}^{n}(\omega - z_k) \end{bmatrix}, \tag{7.7b}$$

from which it follows that

$$\mathcal{G}^{(n)}(\omega,\zeta)=\begin{bmatrix}\frac{(n-1)!\,\pi^{n-1}}{2}\frac{[e^{\omega}+(-1)^n e^{-\omega}]}{\prod_{k=1}^{n}(\omega-z_k)} & (-1)^n\zeta^n e^{-\omega}\\ \zeta^{-n}e^{-\omega} & 2e^{-\omega}\frac{\prod_{k=1}^{n}(\omega-z_k)}{(n-1)!\,\pi^{n-1}}\end{bmatrix}, \tag{7.8}$$

which one easily checks satisfies all of Ward's requirements in Eq. (7.3). Thus, the reality of the gauge fields resulting from (4.1) is guaranteed.

VIII. GENERAL STATIC SELF-DUAL GAUGE FIELDS WHICH ARE REAL BUT NOT NECESSARILY NONSINGULAR

In Sec. VII we proved that the n-monopole solution as defined by Eq. (4.1) is complex gauge equivalent to real gauge fields. In this section we find a general class, which includes the solution (4.1), of self-dual gauge fields which are both static and real but not necessarily nonsingular.

More specifically, we take a general Atiyah-Ward transition matrix $\hat{\mathcal{G}}^{(n)}$:

$$\hat{\mathcal{G}}^{(n)}(\omega_1,\omega_2,\zeta)=\begin{bmatrix}\zeta^n & \hat{\Omega}^{(n)}(\omega_1,\omega_2,\zeta)\\ 0 & \zeta^{-n}\end{bmatrix} \tag{8.1}$$

and ask what is the general form of $\hat{\Omega}^{(n)}(\omega_1,\omega_2,\zeta)$ consistent with the requirement that the gauge fields be static and real (not necessarily nonsingular). Let us first consider the requirement that the gauge fields be static. In Ref. 2 it was shown that if

$${}_n\phi=e^{ix_4}Q_\phi,\quad {}_n\rho=e^{ix_4}Q_\rho,\quad {}_n\bar{\rho}=e^{ix_4}Q_{\bar{\rho}}, \tag{8.2a}$$

$$Q_\phi, Q_\rho, Q_\rho \text{ are functions of } x_1, x_2, x_3 \text{ only}, \tag{8.2b}$$

then the gauge fields are guaranteed to be static. By virtue of Eqs. (7.2b) and (3.3), the transition matrix (8.1) can give static monopole solutions of the form (8.2) only if

$$\text{static} \Rightarrow \hat{\Omega}^{(n)}(\omega_1,\omega_2,\zeta)=e^{\omega_1+\omega_2}F(\omega), \tag{8.3}$$

where $F(\omega)$ is an arbitrary function of $\omega=\omega_1-\omega_2$.

We now turn to the requirement that the gauge fields be (complex gauge equivalent to) real which means that we must be able to satisfy all of Ward's conditions (7.3). To satisfy Ward's conditions (7.3b) and (7.3c) we choose the matrices $\hat{Q}_L^{(n)}$ and $\hat{Q}_R^{(n)}$ to be

$$\hat{Q}_L^{(n)}=\begin{bmatrix}e^{-\omega_2} & 0\\ 0 & e^{\omega_2}\end{bmatrix}, \tag{8.4a}$$

$$\hat{Q}_R^{(n)}=\begin{bmatrix}0 & -e^{\omega_1}\\ e^{-\omega_1} & \zeta^n e^{-\omega_1}q_0\prod_{k=1}^{n}(\omega-q_k)\end{bmatrix}, \tag{8.4b}$$

where $q_0, q_1, \ldots, q_n$ are arbitrary complex constants with $q_0\neq 0$. It is then easy to check that in order to satisfy Ward's conditions (7.3a), (7.3d), and (7.3e) the function $F(\omega)$ must necessarily be

$$\text{real} \Rightarrow F(\omega)=\frac{e^{\omega}+(-1)^n e^{-\omega}}{q_0\prod_{k=1}^{n}(\omega-q_k)} \tag{8.5a}$$

and

$$q_0^*\prod_{k=1}^{n}(\omega^*-q_k^*)=q_0\prod_{k=1}^{n}(\omega^*-q_k). \tag{8.5b}$$

Equation (8.5b) implies that q_0 must be real and $q_1, q_2, \ldots, q_n$ are either real or occur in complex-conjugate pairs. Thus, we have found a general class of static and real self-dual gauge fields that are derivable from the transition matrix (8.1) built from Eqs. (8.3) and (8.5).

It is the final requirement that the gauge fields be nonsingular which uniquely fixes the constants $q_1, q_2, \ldots, q_n$ to be given by Eq. (4.1c), i.e.,

$$\text{nonsingular} \Rightarrow q_k=z_k=\left[\left(\frac{n+1}{2}\right)-k\right](i\pi)\quad \text{for } k=1,2,\ldots,n. \tag{8.6}$$

The next three sections of this paper are devoted to proving Eq. (8.6). Note that by virtue of Eq. (A3), only for the choice $q_k=z_k$ does $F(\omega)$ become an entire function of ω.

IX. MULTIMONOPOLE SOLUTIONS OUTSIDE THE CORE REGION

In order to discuss the singularity properties of our candidate solutions, we can consider the problem of finding solutions that are regular at spatial infinity separately from the problem of regularity at the origin.

In particular, there exists a general class of axisymmetric solutions that can be derived from

$$\Lambda_0=\sum_{j=1}^{n}\beta_j\frac{\sinh r_j}{r_j}, \tag{9.1}$$

where

$$r_j^2 = s^2 + (x_3 - q_j)^2 \quad (9.2)$$

and q_j, β_j are now arbitrary parameters (with the only request that their choice preserves the reality of Λ_0). This form of the solution is naturally suggested by the analysis given in the previous section.

We are going to show that these solutions correspond to n topological charges concentrated in finite regions around locations in the two-dimensional x_3-s plane.

The analysis of the self-duality equations in the unitary gauge[8] clearly shows that, due to the presence of two different kinds of "physical" (gauge-invariant) field components, massive and massless, it is possible to define two regions in space: the "core" where the magnetic charge is localized and the exponentially damped massive field components take values that are significantly different from zero, and the "Abelian" region where the damping is effective, and only the long-range massless fields survive.

It is easy to check that these surviving fields obey the (scalar and vector) Laplace equation up to exponentially depressed corrections. In particular, the norm of the Higgs field $h = (A_4^a A_4^a)^{1/2}$ can be split into

$$h = h^A + O(e^{-2r_j}) \quad (9.3)$$

and h^A satisfies

$$\nabla^2 h^A = 0 . \quad (9.4)$$

Our aim is to derive the exact form of h^A for the most general axisymmetric solution, i.e., (9.1), corresponding to the $\mathfrak{A}_n$ ansatz.

We may define

$$\Delta_l = \sum_{j=1}^{n} \beta_j \Delta_l^{(j)}(r_j, q_j), \quad (9.5)$$

and in Appendix A we show that

$$\Delta_l^{(j)} = e^{ix_4}\left[\left(\frac{x_3 - q_j - r_j}{\sqrt{2}p}\right)^l \frac{e^{r_j}}{2r_j} - \left(\frac{x_3 - q_j + r_j}{\sqrt{2}p}\right)^l \frac{e^{-r_j}}{2r_j}\right]. \quad (9.6)$$

It is straightforward to identify the exponentially damped contribution coming from each $\Delta_l^{(j)}$ and to define the Abelian counterpart of Eq. (9.6):

$$\Delta_l^{(j)A} = e^{ix_4} e^{-il\theta}\left[\left(\frac{x_3 - q_j - r_j}{s}\right)^l \frac{e^{r_j}}{2r_j}\right]. \quad (9.7)$$

Let us notice that (a) $\Delta_l^{(j)A}$ as defined in Eq. (9.7) still satisfies the Helmholtz equation and the recursion relations Eqs. (3.1) so that they define a bonafide (albeit singular) solution of the self-duality equations. (b) It is natural to think of the Abelian solution as arising from the limit when the characteristic scale (or inverse mass) in the monopole problem is set to zero: it is then natural to regard Eq. (9.7) as the corresponding limit of Eq. (9.6).

In order to determine the gauge-invariant fields we have to evaluate the fundamental Toeplitz determinant

$$H_{k-l}^{n\times n} = \det\Delta_{k-l} . \quad (9.8)$$

While this is a hopeless task when Δ_l is given by Eqs. (9.5) and (9.6), dramatic simplifications occur when the form Eq. (9.7) is assumed.

First of all, let us remember that, when each column of a determinant is written as a sum of columns, as happens in our problem since

$$\Delta_{k-l}^A = \sum_{j=1}^{n} \beta_j \Delta_{k-l}^{(j)A}, \quad (9.9)$$

the determinant itself may be decomposed into a sum of determinants,

$$\det\Delta_{k-l}^A = \sum_{\{j_l \cdots j_n\}} \det\beta_{j_l}\Delta^{(j_l)A}{}_{k-l}, \quad (9.10)$$

where the sum runs over all possible choices $1 \le j_k \le n$. However, since

$$\Delta_m^{(j)A} = e^{i(l-m)\theta}\left(\frac{x_3 - q_j - r_j}{s}\right)^{m-l} \Delta_l^{(j)A}, \quad (9.11)$$

whenever the same value of j_k appears in two different columns linear dependence implies a vanishing determinant. This special feature allows us to restrict the sum in Eq. (9.10) to the determinants where each value from 1 to n appears once and only once among the j_k, that is,

$$\det\Delta_{k-l}^A = \sum_{\mathrm{perm}(j_l)} \det\beta_{j_l}\Delta_{k-l}^{(j_l)A} . \quad (9.12)$$

By using Eq. (9.7) and simple properties of Toeplitz determinants (and by removing the irrelevant e^{ix_4} factors) we find

$$\det\Delta_{k-l}^A = \prod_{j=1}^{n}\left(\beta_j \frac{e^{r_j}}{2r_j}\right) \sum_{\mathrm{perm}(j_l)} \det[(x_3 - q_{j_l} - r_{j_l})^{k-l}] \quad (9.13)$$

the determinants in Eq. (9.13) can be given a Vandemonde structure by extracting common factors in the columns and, by further deleting an overall numerical factor that does not affect the logarithmic derivatives, we are led to the form

$$\det\Delta_{k-l}^A = \prod_{j=1}^{n}\left(\frac{e^{r_j}}{r_j}\right) \sum_{\mathrm{perm}(j_l)} \prod_{l=1}^{n} (x_3 - q_{j_l} - r_{j_l})^{1-l} \det[(x_3 - q_{j_l} - r_{j_l})^{k-1}] . \quad (9.14)$$

But notice that by insertion of a factor $(-1)^\delta(-1)^\delta$ where δ is the number of permutations we can rearrange the previous expression into the form

$$\det\Delta^A_{k-l} = \prod_{j=1}^{n}\left(\frac{e^{r_j}}{r_j}\right)\sum_{\text{perm}(j_l)}(-1)^\delta\prod_{l=1}^{n}(x_3-q_{j_l}-r_{j_l})^{1-l}\det[(x_3-q_l-r_l)^{k-1}], \tag{9.15}$$

and the sum after permutation defines a new Vandemonde determinant, such that

$$\det\Delta^A_{k-l} = \prod_{j=1}^{n}\left(\frac{e^{r_j}}{r_j}\right)\det[(x_3-q_k-r_k)^{1-l}]\det[(x_3-q_l-r_l)^{k-1}]. \tag{9.16}$$

A trivial property of Vandemonde determinants now leads to

$$\det\Delta^A_{k-l} = \prod_{j=1}^{n}\left(\frac{e^{r_j}}{r_j}\right)\prod_{i<j}\left(1-\frac{x_3-q_i-r_i}{x_3-q_j-r_j}\right)\left(1-\frac{x_3-q_j-r_j}{x_3-q_i-r_i}\right). \tag{9.17}$$

Let us now recall that

$$(h^A)^2 = 1-\nabla^2\ln\det\Delta^A_{k-l} \tag{9.18}$$

$$=1-\nabla^2\left[\sum_{j=1}^{n}(r_j-\ln r_j)\right]-\nabla^2\left[\sum_{i<j}\ln\left(1-\frac{x_3-q_i-r_i}{x_3-q_j-r_j}\right)\left(1-\frac{x_3-q_j-r_j}{x_3-q_i-r_i}\right)\right]. \tag{9.19}$$

It is straightforward to check that

$$\nabla^2 r_j = \frac{2}{r_j} \text{ and } \nabla^2\ln r_j = \frac{1}{r_j^2}, \tag{9.20}$$

and it does not take too much effort to verify that

$$\nabla^2\ln\left(1-\frac{x_3-q_i-r_i}{x_3-q_j-r_j}\right)\left(1-\frac{x_3-q_j-r_j}{x_3-q_i-r_i}\right) = -\frac{2}{r_ir_j}. \tag{9.21}$$

Collecting all results we finally obtain

$$(h^A)^2 = \left(1-\sum_{k=1}^{n}\frac{1}{r_k}\right)^2 \tag{9.22}$$

and our strikingly simple final result for the Abelian (Higgs) field outside the core region is

$$h^A = 1-\sum_{k=1}^{n}\frac{1}{r_k}. \tag{9.23}$$

It is trivial to check that this is indeed a solution of Eq. (8.2), and to find the asymptotic limit

$$h^A \underset{r\to\infty}{\sim} 1-\frac{n}{r} \tag{9.24}$$

corresponding to an n-monopole solution, as expected on general grounds.

It is important to observe that when we deal with axisymmetric monopoles the massless fields are completely described in terms of h^A, the scalar potential related to the curl-free magnetic field $\vec{H}$; in cylindrical coordinates ($\mu, \nu=1,2$):

$$H_\mu = \partial_\mu h, \tag{9.25}$$

and ψ^A, the axial component of the vector potential (or stream function), related to the divergenceless magnetic induction field $\vec{B}$:

$$B_\nu = \frac{1}{r}\epsilon_{\mu\nu}\partial_\mu\psi, \tag{9.26}$$

and in the Abelian region $\vec{B}=\vec{H}$, relating ψ^A to h^A through

$$\frac{1}{r}\epsilon_{\mu\nu}\partial_\mu\psi^A = \partial_\nu h^A. \tag{9.27}$$

It is easy to integrate Eq. (9.27) when h^A is expressed by Eq. (9.23), thus obtaining

$$\psi^A = -\sum_{k=1}^{n}\frac{x_3-q_k}{r_k}. \tag{9.28}$$

ψ^A may differ from ψ only by exponentially damped corrections. Actually, in the $n=1$ case the relationship

$$\psi^A = \psi = -\frac{x_3}{r} \tag{9.29}$$

holds everywhere.

Two considerations are at hand:

(a) h^A is real due to the presence of complex-conjugate pairs for each complex quantity appearing in the equations;

(b) h^A blows up on the x_3 axis when q_k is real and

$$x_3 = q_k \tag{9.30}$$

and it blows up on the $x_3=0$ plane when q_k is imaginary and

$$s = |q_k|. \tag{9.31}$$

The Abelian solution appears to be characterized by pointlike singularities. In order to remove these singularities, we must replace Eq. (9.7) with Eq. (9.6), noticing that the "smoothed" functions $\Delta_l^{(j)}$ are obtained from their Abelian counterparts through the symmetrization

$$\Delta_l^{(j)}(r_j) = \Delta_l^{(j)A}(r_j)+\Delta_l^{(j)A}(-r_j). \tag{9.32}$$

It is easy to check that this choice is uniquely dictated by the request of regularity around each location, that is, when $r_j \to 0$.

Our derivation of the Abelian solution immediately implies that all the exponentially damped corrections we have introduced involve at least one factor e^{-2r_j}. When $p \neq 0$, each of these factors is relevant only in a finite region around the corresponding singularity of the Abelian solution.

We have found that a sensible way of defining the core consists in considering the circles of radius π defined in the s-x_3 plane and having their centers at the points defined by Eqs. (9.30) or (9.31). Our candidate solutions are characterized by

$$q_k = z_k = i\pi\left(\frac{n+1}{z} - k\right), \quad k = 1, \ldots, n \tag{9.33}$$

such that each of the locations is outside all other elementary cores and the overall core region is a pancake-shaped volume having a radius equal to $\pi(n+1)/2$ in the $x_3 = 0$ plane.

As one can easily check from the definition equation (9.6) our correction automatically introduces (for $n > 1$) a line singularity along the x_3 axis due to the blowing up factors: $(x_3 - q_j \pm r_j)/\sqrt{2}p$.

Let us notice that this singularity was fictitious in the Abelian case, because it corresponded to the Dirac string, and it did not appear in the gauge-invariant quantities. However, for arbitrary locations this cancellation will not in general occur in the smoothed configurations we have now constructed. It will be our task to show that no such singularity occurs in our candidate solutions.

X. MULTIMONOPOLE SOLUTIONS INSIDE THE CORE

When we consider the behavior of the solutions in finite domains, such as the core region, we expect that their regularity properties are reflected in the possibility of expressing them by power-series expansions in the coordinate variables. It turns out that the approach based on Eqs. (3.6) and the introduction of the functions Δ_l is not appropriate, just because it was appropriate for the discussion of the asymptotic behavior we gave in the previous section. It is natural, instead, to start from Eqs. (7.2) and introduce the generating function $\tilde{\Omega}^{(n)}$ that, under the reality constraint, turns out to be a meromorphic function of $\omega_1 + \omega_2$ and ω, as shown in Sec. VIII.

This property of $\tilde{\Omega}^{(n)}$ is critical to the purpose of expressing our results as power-series expansions; the analyticity domains of $\tilde{\Omega}^{(n)}$ are the natural candidates for the analyticity domains of the solutions.

At this stage we do not know the convergence properties of these expansions in the general case with arbitrary locations q_k. We have, however, developed a number of techniques in order to analyze the singularity behavior of our candidate solutions.

In Appendix B we present some direct approaches that lead to a proof of the absence of singularities within a cylinder of radius $s = \frac{1}{2}$ around the x_3 axis and in a disc of radius $s = (n+1)\pi/2$ in the $x_3 = 0$ plane. In Appendix C we discuss a more general approach to the properties of Toeplitz determinants that leads to the representation

$$H^{j \times j}_{k-l} = \int_0^{2\pi} \frac{1}{j!} \prod_{k=1}^{j} \frac{d\psi_k}{2\pi} \times \prod_{k<l} 4\sin^2\left(\frac{\psi_k - \psi_l}{2}\right) \prod_{i=1}^{j} \tilde{\Omega}^{(n)}(e^{i\psi_i}). \tag{10.1}$$

Let us sketch the procedure by which starting from Eq. (10.1) we can ensure the absence of singularities in the core region for our solutions.

First of all, it is apparent that a strong sufficient condition for the positivity of the integral in Eq. (10.1) is the positivity of the integrand, that is,

$$\mathrm{Re} \prod_{i=1}^{j} \tilde{\Omega}^{(n)}(e^{i\psi_i}) \neq 0, \tag{10.2}$$

where

$$\tilde{\Omega}^{(n)}(e^{i\psi}) = e^{-s\cos\psi} \frac{e^{x_3 - is\sin\psi} + (-1)^n e^{-x_3 + is\sin\psi}}{\prod_k (x_3 - is\sin\psi - q_k)}. \tag{10.3}$$

In Eq. (10.3) we have dropped irrelevant factors of e^{ix_4} and set $\theta = 0$ since $H^{j\times j}_{k-l}$ does not depend on θ.

By considering the $x_3 = 0$ plane, we immediately see that the choice $q_k = z_k$ ensures that

$$\tilde{\Omega}^{(n)}(e^{i\psi}, x_3 = 0) = e^{-s\cos\psi} \frac{e^{-is\sin\psi} + (-1)^n e^{is\sin\psi}}{\prod_k (-is\sin\psi - z_k)} \tag{10.4}$$

is real and nonvanishing in the interval

$$s \leq \frac{n+1}{2}\pi.$$

In order to move out of the s plane, let us consider the first order of an expansion of Eq. (10.3) in powers of x_3:

$$\tilde{\Omega}^{(n)}(e^{i\psi}) = \hat{\Omega}^{(n)}(e^{i\psi}, x_3 = 0)\left\{1 + x_3\left[[\tanh(-is\sin\psi)]^{(-1)^n} - \sum_{k=1}^{n} \frac{1}{-is\sin\psi - z_k}\right] + O(x_3^2)\right\}. \tag{10.5}$$

Let us observe that

$$(\tanh x)^{(-1)^n} = \sum_{k=-\infty}^{\infty} \frac{1}{x - z_k} . \tag{10.6}$$

As a consequence, it is easy to check that the purely imaginary function

$$[\tanh(-is\sin\psi)]^{(-1)^n} - \sum_{k=1}^{n} \frac{1}{-is\sin\psi - z_k} \equiv \sum_{k\neq 1,\ldots,n} \frac{1}{-is\sin\psi - z_k} \tag{10.7}$$

is a smooth, bounded function in the interval

$$s \leq \frac{n}{2}\pi . \tag{10.8}$$

It is then possible to show that Eq. (10.2) holds in a finite region outside the $x_3 = 0$ plane and the width of this region is approximated by the condition

$$x_3 < 1 . \tag{10.9}$$

Let us notice however, that the condition Eq. (10.2) ensures us that the determinant not only is nonvanishing in its region of validity, but also it is different from zero by a finite amount.

Then by continuity the region where $H_{k-l}^{j\times j}$ is nonvanishing can be further enlarged well beyond the limit given by Eq. (10.9). It is appropriate to observe that, due to the strong analyticity properties of $\tilde{\Omega}^{(n)}$, all functions are smooth and well behaved.

This analysis could be extended to higher orders of the series expansion in x_3 and the bounds for positivity could be made more rigorous. However, a fundamental consideration is due at this stage: If we believe that regular solutions to the self-duality equations have to exist, at least in a finite domain like the core region, the arguments we have developed lead to a unique determination of the free parameters, such that *if* the solution exists, it is bound to be the one we have discussed.

In this context, let us observe that the results presented in Appendix C provide the following estimate of the $n\times n$ Toeplitz determinant when $s > n$ and $x_3 < s$:

$$H_{k-l}^{n\times n} \cong \left(\frac{e^{s+x_3^2/2s}}{2s}\right)^n 2^{n(n-1)} \prod_{k=1}^{n-1} \binom{n-1}{k} \left(\sin^2 \frac{k\pi}{2s}\right)^{n-k} \tag{10.10}$$

show that our solutions are regular and smooth on and around the s plane for all values of s, quite independently of the restriction to the core region. More generally, it is apparent from the study of the one- and two-monopole solutions (see Appendix B) that all of our estimates are extremely conservative. As a matter of fact, we would like to express the following conjecture: we believe that the convergence radius of solutions constructed from a generating function $\tilde{\Omega}^{(n)}$ coincides with the convergence radius of $\tilde{\Omega}^{(n)}$, such that when $\tilde{\Omega}^{(n)}$ is an entire function, as it happens to be for our regular solutions, the convergence region is the whole space. This conjecture, if proven, would also provide an extremely powerful tool in the analysis of the positivity properties of Toeplitz forms and U(n) group integrals.

Finally, we expect that discussions similar to the one we have presented in this section can be extended to the (generally complex and singular) class of solutions generated by

$$\Omega(e^{i\psi}) = e^{ix_4 - s\cos\psi}\frac{1}{2}\int_{-1}^{1} dt\, e^{-t(x_3 - is\sin\psi)} p(t), \tag{10.11}$$

where $p(t)$ is some arbitrary weight function, positive in the interval $-1 \leq t \leq 1$ in order to ensure positivity of $\tilde{\Delta}_0$. It would be interesting to know which constraints the request of singularity in some finite region imposes on the form of $p(t)$. Let us recall that the known regular solutions have

$$p(t) = \left(2\cos\frac{\pi}{2}t\right)^{n-1} .$$

XI. DISCUSSION OF REGULARITY

In Sec. IX we have constructed solutions of the self-duality equations in terms of matrix elements Δ_l. These solutions are regular at spatial infinity and well behaved everywhere outside the core region, except possibly for a line singularity along the x_3 axis.

In Sec. X we have constructed solutions in terms of the matrix elements $\tilde{\Delta}_l$ and we have shown that by a special choice of parameters $q_k = z_k$ these solutions turn out to be regular on the $x_3 = 0$ plane and continuous and smooth around the plane. Moreover, we have explicitly shown that these solutions are regular on and around the x_3 axis.

However, as we have shown in Appendix A, for the special choice of parameters characterizing our unique solution

$$\Delta_l(q_k = z_k) = \tilde{\Delta}_l(q_k = z_k), \quad -n < l < n . \tag{11.1}$$

Equation (11.1) implies that our solution shares the properties of solutions belonging to both classes. The only missing step toward the completion of a formal proof of regularity is now the determination of quantitative bounds on the respective regions of regularity ("inner" and "outer" core radius) such that these regions may be shown to cover all three-dimensional space. While this turns out to be a rather formidable mathematical task, we would like to stress that a number of different pieces of evidence leave little doubt about the final result. Let us recall among them the unique determination of the solutions and the explicitly checked regularity of the two- and three-monopole configurations.

Finally, let us observe that (as shown in Appendix D) the determinant $H_{k-l}^{n\times n}$ associated with the n-monopole solution in the limit $n\to\infty$ is a positive convex function with a minimum *nonzero* value at the origin. Based on the analysis of the $n=1$, 2, and 3 determinants, we conjecture this property to be true for any finite n, thus ensuring regularity of our solutions.

XII. DISCUSSION

We have presented here a construction for static, finite-energy, axisymmetric (and mirror symmetric[9]) n-monopole solutions of the Yang-Mills-Higgs equations in the limit of vanishing Higgs potential.

These solutions are now well understood: we know that the gauge-invariant quantities are algebraic combinations of elementary functions and we have complete control over the fields outside the core region.

However, the task of writing down explicit expressions for the fields in all space is extremely cumbersome and not even in any sense useful in order to get a better understanding of the solutions when $n\geq 3$.

It is thus gratifying that, by use of simple symmetry arguments, we are able to dramatically simplify the exact form of the norm h of the Higgs field on the x_3 axis and on the $x_3=0$ plane.

The norm h of Higgs field is given by

$$h(x_3,s)=\left[\frac{{}_n\tilde\phi_3^{\,2}}{{}_n\tilde\phi^2}+\frac{2\,{}_n\tilde\rho_q\,{}_n\bar{\tilde\rho}_{\bar q}}{{}_n\tilde\phi^2}\right]^{1/2} \tag{12.1}$$

and in general is not a rational function. However, as we will now show, h does become a rational function on the x_3 axis and on the $x_3=0$ plane.

On the x_3 axis

$${}_n\tilde\rho(x_3,s=0)={}_n\bar{\tilde\rho}(x_3,s=0)=0\,, \tag{12.2}$$

because $\tilde\Delta_k(x_3,s=0)=0$ for $k\neq 0$, and h reduces to

$$h(x_3,s=0)=|\partial_3\ln{}_n\tilde\phi(x_3,s=0)|=|\partial_3\ln\tilde\Delta_0(x_3,s=0)|\,. \tag{12.3}$$

Equations (A6) and (7.5) imply that

$$\tilde\Delta_0(x_3,s=0)=\frac{1}{2\pi}\int_0^{2\pi}\tilde\Omega^{(n)}(s=0,\theta,x_3,x_4,e^{i\psi})d\psi \tag{12.4a}$$

$$=\frac{(n-1)!\,\pi^{n-1}}{2}e^{ix_4}\frac{[e^{x_3}+(-1)^n e^{-x_3}]}{\prod_{k=1}^{n}(x_3-z_k)}\,, \tag{12.4b}$$

from which it follows that

$$h(x_3,s=0)=\left|(\tanh x_3)^{(-1)^n}-\sum_{k=1}^{n}\frac{1}{x_3-z_k}\right| \tag{12.5a}$$

$$=\left|\sum_{k=-\infty}^{\infty}{}'\,\frac{1}{x_3-z_k}\right|\,, \tag{12.5b}$$

where the $\sum'$ in (12.5b) means to omit the sum for $k=1,2,\ldots,n$. We note that Eq. (12.3) could also have been derived from the superpotential approach[8] by observing that because of the Dirac quantization condition $|\psi(x_3,s=0)|=n$.

Similarly on the $x_3=0$ plane

$${}_n\tilde\phi_3(s,x_3=0)=0\,, \tag{12.6a}$$

$$\frac{{}_n\tilde\rho_q(s,x_3=0)\,{}_n\bar{\tilde\rho}_{\bar q}(s,x_3=0)}{{}_n\tilde\phi^2}=\left|\frac{{}_n\tilde\rho_q(s,x_3=0)}{{}_n\tilde\phi}\right|^2=\left|\frac{{}_n\bar{\tilde\rho}_{\bar q}(s,x_3=0)}{{}_n\tilde\phi}\right|^2\,, \tag{12.6b}$$

because ${}_n\tilde\phi$ is an even function of x_3 and $e^{-in\theta}{}_n\tilde\rho(s,x_3)=-e^{in\theta}{}_n\bar{\tilde\rho}(s,-x_3)$. Thus we find

$$h(s,x_3=0)=\left|\frac{(1+\partial_3)\,{}_n\bar{\tilde\rho}}{{}_n\tilde\phi}\right|_{x_3=0}=\left|\frac{(1-\partial_3)\,{}_n\tilde\rho}{{}_n\tilde\phi}\right|_{x_3=0}. \tag{12.7}$$

In closing, we make the obvious remark that it would be most desirable to be able to construct, using techniques developed in this paper, multimonopole solutions with separated centers, that have been proved to exist.[10]

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APPENDIX A: PROPERTIES OF $\tilde\Delta_l$ AND Δ_l

Following Atiyah and Ward's original approach,[5] we may assume that the fundamental function describing a self-dual solution is $\tilde\Omega(\omega_1,\omega_2,\zeta)$, such that reality (and probably lack of singularities, too) amounts to a simple property of $\tilde\Omega$, as we have shown.

The multimonopole solutions are generated by the "splitting rule"

$$\tilde\Omega^{(n)}(\omega_1,\omega_2,\zeta)=\tilde\Omega^{(n-1)}\left(x_3+\frac{i\pi}{2}\right)+\tilde\Omega^{(n-1)}\left(x_3-\frac{i\pi}{2}\right) \tag{A1}$$

leading to the function

$$\tilde{\Omega}^{(n)}(\omega_1,\omega_2,\zeta)=\frac{(n-1)!}{2}\pi^{n-1}\frac{e^{2\omega_1}+(-1)^n e^{2\omega_2}}{\prod_{k=1}^{n}(\omega-z_k)}$$

$$=(n-1)!\,\pi^{n-1}e^{\omega_1+\omega_2}\left[\frac{e^{\omega}+(-1)^n e^{-\omega}}{2\prod_{k=1}^{n}(\omega-z_k)}\right]. \tag{A2}$$

Well-known properties of complex variable functions allow us to write down the expressions

$$\frac{e^{\omega}-e^{-\omega}}{2\omega}=\prod_{k=1}^{\infty}\left(1+\frac{\omega^2}{k^2\pi^2}\right), \tag{A3a}$$

$$\frac{e^{\omega}+e^{-\omega}}{2}=\prod_{k=1}^{\infty}\left(1+\frac{\omega^2}{(k+\frac{1}{2})^2\pi^2}\right). \tag{A3b}$$

Equations (A3), when inserted in Eq. (A2) together with the definition of z_k, i.e., (4.1c), make the analytic structure of $e^{-(\omega_1+\omega_2)}\tilde{\Omega}^{(n)}$ in the ω plane apparent: this function turns out to be an entire function of ω, with equally spaced zeros everywhere along the imaginary axis at a distance $i\pi$ from each other, except for the n zeros corresponding to the monopole locations z_k, that are removed. This property seems to be critical in ensuring the absence of singularities for the n-monopole solution.

Consistency with Eq. (7.2b) requires the following property to be true:

$$\tilde{\Delta}_l=\frac{1}{2\pi i}\oint\tilde{\Omega}^{(n)}(\omega_1,\omega_2,\zeta)\zeta^l\frac{d\zeta}{\zeta}. \tag{A4}$$

In order to check Eq. (A4) we can make use of the following integral representation of $\tilde{\Omega}^{(n)}$, immediately deducible from Eqs. (A1) and (A2):

$$\tilde{\Omega}^{(n)}=e^{ix_4-(s/2)(e^{i\theta}\zeta+e^{-i\theta}\zeta^{-1})}$$

$$\times\tfrac{1}{2}\int_{-1}^{1}dt\,e^{-t[x_3-(s/2)(e^{i\theta}\zeta-e^{-i\theta}\zeta^{-1})]}\left(2\cos\frac{\pi}{2}t\right)^{n-1}. \tag{A5}$$

We may now perform the ζ integrations by the observation that the integral may be evaluated along the unit circle in the complex ζ plane ($\zeta=e^{i\psi}$):

$$\tilde{\Delta}_l=\frac{1}{2\pi}\int_0^{2\pi}e^{il\psi}\tilde{\Omega}^{(n)}(s,\theta,x_3,x_4,e^{i\psi})d\psi. \tag{A6}$$

By the change of variable $\psi\to\psi-\theta$ we then obtain

$$\tilde{\Delta}_l=e^{ix_4}e^{-il\theta}\tfrac{1}{2}\int_{-1}^{1}dt\,e^{-tx_3}\left(2\cos\frac{\pi}{2}t\right)^{n-1}\frac{1}{2\pi}$$

$$\times\int_0^{2\pi}d\psi\,e^{-s\cos\psi+ist\sin\psi+il\psi}. \tag{A7}$$

The ψ integral corresponds to a known integral representation for the Bessel functions of imaginary argument,[7] and finally we get

$$\tilde{\Delta}_l=e^{ix_4}e^{-il\theta}(-1)^l\tfrac{1}{2}\int_{-1}^{1}dt\,e^{-tx_3}\left(2\cos\frac{\pi}{2}t\right)^{n-1}$$

$$\times\left(\frac{1+t}{1-t}\right)^{l/2}I_l(s(1-t^2)^{1/2})dt, \tag{A8}$$

matching with Eq. (5.6). We mention that the direct Laurent series expansion in Eq. (A5) would have led to

$$\tilde{\Delta}_l=(-1)^l e^{ix_4}e^{-il\theta}\tfrac{1}{2}\int_{-1}^{1}dt\left(2\cos\frac{\pi t}{2}\right)^{n-1}e^{-tx_3}$$

$$\times\sum_{k=0}^{\infty}\frac{(1-t)^k}{k!}\frac{(1+t)^{k+l}}{(k+l)!}\left(\frac{s}{2}\right)^{2k+l}, \tag{A9}$$

coinciding with the power-series expansion of Eq. (A8). Equation (A9) makes it apparent that we are allowed to write

$$\tilde{\Delta}_{\pm l}=(-1)^l e^{ix_4}\frac{e^{\mp il\theta}}{(s)^l}\left(1\mp\frac{\partial}{\partial x_3}\right)^l\tilde{\Lambda}_l, \tag{A10}$$

where $l\geq 0$ and

$$\tilde{\Lambda}_l=\sum_{k=0}^{\infty}\tfrac{1}{2}\int_{-1}^{1}dt\,e^{-tx_3}\frac{(1-t^2)^k}{k!}\frac{(s/2)^{2(k+l)}}{(k+l)!}$$

$$\times\left(2\cos\frac{\pi}{2}t\right)^{n-1}. \tag{A11}$$

By performing the t integrations (again corresponding to representations of Bessel functions) we find

$$\tilde{\Lambda}_l=\sum_{m=1}^{n}\alpha_m\left[\sum_{k=0}^{\infty}2^{k+l}\left(\frac{\pi}{2}\right)^{1/2}\frac{I_{k+1/2}(x_3-z_m)}{(x_3-z_m)^{k+1/2}}\frac{(s/2)^{2(k+l)}}{(k+l)!}\right]. \tag{A12}$$

We shall now exploit a fundamental property of Bessel functions, that is, the relationship

$$\left(\frac{1}{x}\frac{d}{dx}\right)^m\left(\frac{I_\nu(x)}{x^\nu}\right)=\frac{I_{\nu+m}(x)}{x^{\nu+m}} \tag{A13}$$

in order to obtain from Eq. (A12)

$$\tilde{\Lambda}_l=\sum_{m=1}^{n}\alpha_m\left\{\sum_{k=l}^{\infty}\frac{(s/2)^{2k}}{k!}\left(\frac{1}{(x_3-z_m)}\frac{d}{d[(x_3-z_m)/2]}\right)^k\left[\left(\frac{\pi}{2}\right)^{1/2}\frac{I_{1/2-l}(x_3-z_m)}{(x_3-z_m)^{1/2-l}}\right]\right\} \tag{A14a}$$

$$=\sum_{m=1}^{n}\alpha_m\left\{\left(\frac{\pi}{2}\right)^{1/2}\frac{I_{1/2-l}(r_m)}{(r_m)^{1/2-l}}-\sum_{k=0}^{l-1}\frac{(s/2)^{2k}}{k!}\left[\left(\frac{\pi}{2}\right)^{1/2}\frac{I_{1/2-l+k}(x_3-z_m)}{(x_3-z_m)^{1/2-l+k}}\right]\right\}, \tag{A14b}$$

where (A14b) holds for $l \geq 1$.

We now notice that, by recalling Eq. (5.2), we may write

$$\tilde{\Lambda}_l = \Lambda_l - \sum_{k=0}^{l-1} \frac{(s/2)^{2k}}{k!} \Lambda_{l-k}(x_3, s=0) \quad \text{for } l \geq 1 . \tag{A15}$$

But we know that

$$\Lambda_k(x_3, s=0) = 0 \quad \text{when } 1 \leq k < n \tag{A16}$$

and we then obtain

$$\tilde{\Lambda}_l = \Lambda_l \quad \text{when } 0 \leq l < n , \tag{A17a}$$

$$\tilde{\Lambda}_n = \Lambda_n - \Lambda_n(s=0) . \tag{A17b}$$

It now follows trivially from Eq. (A10) that Eqs. (5.7) are satisfied.

We would also like to mention that it is possible to derive rather explicit expressions for Δ_l from the original definitions.

Let us recall that

$$\Delta_l = \sum_{k=1}^{n} \alpha_k \overline{\Delta}_l(r_k, z_k) , \tag{A18}$$

and let us focus on $\overline{\Delta}_l^{(0)} = \overline{\Delta}_l(r, 0)$.

One can show by induction that, for $l \geq 0$,

$$\overline{\Delta}_l^{(0)} = e^{ix_4}\left[\left(\frac{x_3 - r}{\sqrt{2}p}\right)^l \frac{e^r}{2r} - \left(\frac{x_3 + r}{\sqrt{2}p}\right)^l \frac{e^{-r}}{2r}\right], \tag{A19a}$$

$$\overline{\Delta}_{-l}^{(0)} = e^{ix_4}\left[\left(\frac{-x_3 - r}{\sqrt{2}\bar{p}}\right)^l \frac{e^r}{2r} - \left(\frac{-x_3 + r}{\sqrt{2}\bar{p}}\right)^l \frac{e^{-r}}{2r}\right] \tag{A19b}$$

and $\overline{\Delta}_l(r_k, z_k)$ is then obtained by the replacements

$$r \to r_k , \quad x_3 \to x_3 - z_k , \tag{A20}$$

such that Eq. (A18) may immediately be used to get the final explicit result.

Equations (A19) are solutions of Eq. (3.1) and must, therefore, also be solutions of the Helmholtz equation. In order to make this apparent, let us write them in terms of generalized spherical harmonics (Legendre functions):

$$\overline{\Delta}_l = e^{ix_4} e^{-il\theta}\left[(-1)^l P_0^{-l}\left(\frac{x_3}{r}\right)\frac{e^r}{2r} - P_0^l\left(\frac{x_3}{r}\right)\frac{e^{-r}}{2r}\right] , \tag{A21}$$

where $P_0^l(\cos\alpha) = (\cot\alpha/2)^l$. Let us observe that Eq. (A21) develops singularities whenever $l \neq 0$.

The astonishing cancellation of singularities occurring in Eq. (A18) is the key to the existence of regular solutions. The identity that can be extracted from Eqs. (A11) and (A17), when $l < n$,

$$\frac{1}{\sqrt{2\pi}} \int_{-1}^{1} dt\, e^{-itx_3}\left[\frac{s}{(1-t^2)^{1/2}}\right]^l I_l[s(1-t^2)^{1/2}]\left(2\cos\frac{\pi}{2}t\right)^{n-1} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{I_{1/2-l}(r_m)}{(r_m)^{1/2-l}} ,$$

is a yet unknown property of Bessel functions.

APPENDIX B: REGULARITY CHECKS

In order for the gauge fields to be nonsingular we must show that the Toeplitz determinant

$${}_1\phi_2\phi \cdots {}_j\phi = H_{k-l}^{j\times j} \tag{B1}$$

never vanishes for $j = 1, 2, \ldots, n$. For $j = 1$ this is obvious because Λ_0, by virtue of Eq. (5.5b), never vanishes. For $n = 2$ a direct "brute force" proof can be given as follows. For the $n = 2$ monopole we must show that ${}_1\phi_2\phi$ never vanishes. Equations (5.2), (3.5), and (4.1) for $n = 2$ give

$$\begin{aligned} {}_1\phi_2\phi &= \Delta_0^2 - \Delta_1\Delta_{-1} \\ &= \frac{2e^{2ix_4}}{s^2}\left[\frac{(r^2 + \pi^2/4)}{r_1 r_2}\sinh r_1 \sinh r_2 - (1 + \cosh r_1 \cosh r_2)\right] , \end{aligned} \tag{B2a}$$

where

$$\begin{aligned} &r^2 = s^2 + x_3^2 , \quad r_1^2 = s^2 + (x_3 - i\pi/2)^2 , \\ &r_2^2 = s^2 + (x_3 + i\pi/2)^2 . \end{aligned} \tag{B2b}$$

For $s = 0$ we know $\Delta_0^2 - \Delta_1\Delta_{-1} = \Delta_0^2 \neq 0$, so we only consider the case $s > 0$ in which case Eq. (B2b) implies

$$r^2 + \pi^2/4 > r_1 r_2 . \tag{B3}$$

Let us now define

$$r_1 \equiv a + ib , \quad r_2 \equiv a - ib , \tag{B4}$$

where a and b are real. Equation (B3) implies

$$|a| \geq 0 , \quad 0 \leq |b| < \pi/2 . \tag{B5}$$

We are now ready to prove (B2a) never vanishes. The proof is by contradiction. Suppose (B2a) vanishes for some value of s and x_3, then

$$1 + \cosh r_1 \cosh r_2 = \frac{(r^2 + \pi^2/4)}{r_1 r_2} \sinh r_1 \sinh r_2 . \tag{B6}$$

For $a = b = 0$ (i.e., $r_1 = r_2 = 0$) Eq. (B6) implies $\pi^2 = 4$ which is impossible, and, therefore (B2a) cannot vanish for $a = b = 0$.

Using Eqs. (B4), Eq. (B6) can be written as

$$\cos^2 b = \frac{(\pi^2/4) - b^2}{(\pi^2/4) + a^2} \cosh^2 a \quad \text{for } r_1, r_2 \neq 0 . \tag{B7}$$

We now use the inequality

$$\sin|b| > 2|b|/\pi \quad \text{for } 0 < |b| < \pi/2 \tag{B8}$$

to show that Eq. (B7) implies

$$\frac{\pi}{2}\sinh|a| \leq |a| \quad \text{for } |a| > 0 , \tag{B9a}$$

$$\sin^2 b = (2b/\pi)^2 \quad \text{for } |a| = 0 \text{ and } 0 < |b| < \pi/2 , \tag{B9b}$$

which is impossible and thus the assumption (B2a) vanishes is proved to be false.

An identical brute force proof works for the $n = 3$ monopole except the intermediate steps are much more complicated, so much so that originally we had to use the symbol-manipulating computer program MACSYMA. For $n > 3$ this brute force technique becomes impractical and it is more efficient to proceed as follows. Let us define the real functions:

$$\tilde{\delta}_l \equiv e^{-ix_4}\tilde{\Delta}_l(\theta=\pi)$$
$$= \tfrac{1}{2}\int_{-1}^{1}\left[2\cos\left(\frac{\pi t}{2}\right)\right]^{n-1} e^{-x_3 t}\left(\frac{1+t}{1-t}\right)^{l/2} \times I_l(s(1-t^2)^{1/2})dt\,, \tag{B10a}$$

where $-n \leq l \leq n$ and from which it follows that

$$\tilde{\delta}_0 > 0\,,\quad \tilde{\delta}_{\pm l'} > 0 \quad \text{for } s > 0\,,$$
$$\tilde{\delta}_{\pm l'} = 0 \quad \text{for } s = 0, \text{ and } l' = 1, 2, \ldots, n\,. \tag{B10b}$$

It is easy to check that

$$H_{k-l}^{j\times j} = e^{ijx_4}\left|M^{j\times j}\right|\,, \tag{B11a}$$

where the $j\times j$ Toeplitz matrix $M^{j\times j}$ is defined by

$$M^{j\times j} = \begin{pmatrix} \tilde{\delta}_0 & \cdots & \tilde{\delta}_{-j+1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \tilde{\delta}_{j-1} & & \tilde{\delta}_0 \end{pmatrix}. \tag{B11b}$$

Our problem is to prove the determinant $|M^{j\times j}|$ never vanishes for $j = 1, 2, \ldots, n$. To do this, we study the determinant $|M^{j\times j}|$ in different regions of three-dimensional Euclidean space as follows. The determinant $|M^{j\times j}|$, by virtue of (B10b), never vanishes on the x_3 axis where $s = 0$. Let us now consider a solid cylinder of radius $s \leq \frac{1}{2}$ around the x_3 axis. Using the series representation (5.1) for I_ν it is easy to verify that

$$\delta_0 > \delta_{\pm 1} + \delta_{\pm 2} + \cdots + \delta_{\pm(j-1)} \quad \text{for } s \leq \tfrac{1}{2}\,. \tag{B12}$$

Equation (B12) shows that, for $s \leq \frac{1}{2}$, the matrix $M^{j\times j}$ is strictly diagonally dominated and therefore, by virtue of the Gerschgorin circle theorem, cannot have a vanishing eigenvalue. Thus we have proved

$$H_{k-l}^{j\times j} \neq 0 \quad \text{for } s \leq \tfrac{1}{2}\,. \tag{B13}$$

Let us now define the generating function for $\tilde{\delta}$ as

$$\Omega(x_3, s, \zeta) = \sum_{k=-\infty}^{\infty} \tilde{\delta}_k \zeta^{-k} = e^{-ix_4}\tilde{\Omega}(\theta=\pi) \tag{B14a}$$
$$= \frac{(n-1)!\, e^{(s/2)(\zeta+\zeta^{-1})}}{[\Gamma(\frac{1}{2}(n+1))]^2} \times \prod_{k=0}^{\infty}\left\{1 + \frac{[x_3 + \frac{1}{2}s(\zeta-\zeta^{-1})]^2}{[\frac{1}{2}(n+1)+k]^2\pi^2}\right\}. \tag{B14b}$$

In deriving Eq. (B14b) we have used Eq. (A3). Equation (B14a) implies

$$\tilde{\delta}_k = \frac{1}{2\pi i}\oint \frac{d\zeta}{\zeta}\Omega(x_3, s, \zeta)\zeta^k \tag{B15a}$$
$$= \frac{1}{2\pi}\int_0^{2\pi} e^{ik\psi}\Omega(x_3, s, e^{i\psi})d\psi\,, \tag{B15b}$$

where in (B15b) we have chosen the contour of integration to be the unit circle $\zeta = e^{i\psi}$ with $0 \leq \psi < 2\pi$.

Consider now the $x_3 = 0$ plane, in which case the Toeplitz matrix $M^{j\times j}$ becomes a real symmetric $j\times j$ matrix. From (B14b) it is easy to see that

$$\Omega(x_3 = 0, s, e^{i\psi}) \geq 0 \quad \text{for } s \leq \left(\frac{n+1}{2}\right)\pi \tag{B16}$$

and from (B15b) it is easy to see that, for $x_3 = 0$ and $s \leq (n+1)\pi/2$, the real symmetric matrix $M^{j\times j}$ is positive definite [i.e., the quadratic form $\sum_{k,l=1}^{n}\eta_k M_{k-l}^{j\times j}\eta_l > 0$ for any real j-component (not all of which are zero) vector η]. Thus we have proved

$$H_{k-l}^{j\times j} \neq 0 \quad \text{for } x_3 = 0 \text{ and } s \leq \left(\frac{n+1}{2}\right)\pi\,. \tag{B17}$$

APPENDIX C: TOEPLITZ DETERMINANTS AND U(n) GROUP INTEGRATION

As shown in Sec. IV, the discussion of the regularity properties for the n-monopole solutions of the self-duality equations amounts to showing that the determinants $H_{k-l}^{j\times j}$ are nowhere vanishing when $j \leq n$.

According to the definition Eq. (3.2),

$$H_{k-l}^{j\times j} = \det(\Delta_{k-l})\,,\quad k, l = i, \ldots, j \tag{C1}$$

and since

$$\Delta_l = \frac{1}{2\pi}\int_0^{2\pi} d\psi\, e^{il\psi}\tilde{\Omega}^{(n)}(e^{i\psi})\,, \tag{C2}$$

$H_{k-l}^{j\times j}$ are by definition Toeplitz determinants.

It is in general very hard to find simple, necessary, and sufficient conditions for the positivity of a Toeplitz determinant. With this goal in mind, we found it useful to establish a general equivalence between Toeplitz determinants and a class of group integrals over the U(n)-invariant Haar measure. The $U(j)$ group integral is defined as follows:

$$I(F)=\int dU\,F(U)\,, \tag{C3}$$

where dU is the Haar measure in the group $U(j)$ which satisfies

$$dU=d(UV)=d(VU)\,, \tag{C4}$$

where V is an arbitrary unitary matrix.

One can diagonalize U through

$$U=TU_DT\,,\quad U_D=e^{i\psi_i}\delta_{ij}\,, \tag{C5}$$

where T is a unitary matrix. The measure can now be written as

$$dU=dT\,d\mu(\psi)\,, \tag{C6}$$

where $d\mu(\psi)$ was given by Weyl as

$$d\mu(\psi)=\frac{1}{j!}\prod_{i=1}^{j}\frac{d\psi_i}{2\pi}D(e^{i\psi_i})D(e^{-i\psi_i})\,. \tag{C7}$$

$D(x)$ is the Vandermonde determinant

$$D(x)\equiv\det(x_l^{k-1})\equiv\prod_{k<l}(x_k-x_l)\,. \tag{C8}$$

When $F(U)$ is an invariant function over the group,

$$F_I(U)=F_I(U_D)\,, \tag{C9}$$

the integration over dT is trivial and we are left with

$$I(F_I)=\int d\mu(\psi)F_I(U_D)\,. \tag{C10}$$

A further simplification occurs when F_I can be written as the determinant of a function $G(U)$, in which case

$$I(\det G)=\int_0^{2\pi}\frac{1}{j!}\prod_i\frac{d\psi_i}{2\pi}D(e^{i\psi_i})D(e^{-i\psi_i})\prod_i G(e^{i\psi_i})\,, \tag{C11}$$

where $G(e^{i\psi_i})$ is now an ordinary function of one variable. Now by writing down explicit formulas for the determinants

$$\frac{1}{j!}D(e^{i\psi_i})D(e^{-i\psi_i})=\frac{1}{j!}\epsilon_{k_1\cdots k_j}e^{i(k_1-1)\psi_1}\cdots e^{i(k_j-1)\psi_j}\epsilon \quad \cdot_{l_j}e^{-i(l_1-1\psi_1}\cdots e^{-i(l_j-1)\psi_j}$$
$$=\frac{1}{j!}\epsilon_{k_1\cdots k_j}\epsilon_{l_1\cdots l_j}e^{i(k_1-l_1)\psi_1}\cdots e^{i(k_j-l_j)\psi_j} \tag{C12}$$

and by inserting Eq. (C12) into Eq. (C11) we obtain

$$I(\det G)=\frac{1}{j!}\epsilon_{k_1\cdots k_j}\epsilon_{l_1\cdots l_j}\int\prod_i\frac{d\psi_i}{2\pi}e^{i(k_i-l_i)\psi_i}G(e^{i\psi_i})\,. \tag{C13}$$

We may now define

$$G_{k-l}=\int\frac{d\psi}{2\pi}e^{i(k-l)\psi}G(e^{i\psi})\,, \tag{C14}$$

and find

$$I(\det G)=\det G_{k-l}\,, \tag{C15}$$

which is by definition a Toeplitz determinant. As an immediate consequence, we may write

$$H_{k-l}^{j\times j}=I(\det\tilde{\Omega}^{(n)}(U))\,, \tag{C16}$$

that is, we have turned our $j\times j$ determinant into group integrals over the group $U(j)$. A determinant still appears in the right-hand side of Eq. (C16), but since U may be diagonalized the determinant simply reduces to the product of the eigenvalues. Once the equivalence is established, we can use known properties of the Weyl measure, such as the identity

$$D(e^{i\psi_k})D(e^{-i\psi_k})=\prod_{k<l}4\sin^2\left(\frac{\psi_k-\psi_l}{2}\right) \tag{C17}$$

and write the following representation of the $H_{k-l}^{j\times j}$:

$$H_{k-l}^{j\times j}=\int_0^{2\pi}\frac{1}{j!}\prod_{k=1}^{j}\frac{d\psi_k}{2\pi}2^{j(j-1)}\prod_{k<l}\sin^2\left(\frac{\psi_k-\psi_l}{2}\right)\times\prod_{i=1}^{j}\tilde{\Omega}^{(n)}(e^{i\psi_i})\,. \tag{C18}$$

It is apparent from Eq. (C18) that a strong sufficient condition for $H_{k-l}^{j\times j}$ to be nowhere vanishing is

$$\mathrm{Re}\prod_{i=1}^{j}\tilde{\Omega}^{(n)}(e^{i\psi_i})\neq 0\,. \tag{C19}$$

We have made use of the property Eq. (C19) in Sec. X in order to ensure the absence of singularities of the solution in the core region.

We would like to notice here that a (presumably) weaker condition can be extracted from Eq. (C18) by observing that the result is unaltered if we restrict $\tilde{\Omega}^{(n)}(e^{i\psi})$, formally defined as an infinite sum

$$\tilde{\Omega}^{(n)}(e^{i\psi})=\sum_{l=-\infty}^{\infty} e^{-il\psi}\Delta_l \tag{C20}$$

to the finite sum

$$\tilde{\Omega}^{(n)}_{(j)}(e^{i\psi})=\sum_{j+1}^{j-1} e^{-il\psi}\Delta_l\,. \tag{C21}$$

It is straightforward to extract from the representation Eq. (A7) the result

$$\tilde{\Omega}^{(n)}_{(j)}(e^{i\psi})=\frac{1}{2\pi}\int_0^{2\pi} d\theta\,\tilde{\Omega}^{(n)}(e^{i\theta})\,\frac{\sin(j-\frac{1}{2})(\theta-\psi)}{\sin(\theta-\psi)/2}\,. \tag{C22}$$

As a further exercise in the evaluation of Toeplitz determinants through U(j) integration let us compute the large-s behavior of $H^{n\times n}_{k-l}$. First of all, let us consider the representation of Δ_l obtained in Eq. (A7):

$$\Delta_l=e^{ix_4}e^{-il\theta}\tfrac{1}{2}\int_{-1}^{1}dt\,e^{-tx_3}\Big(2\cos\frac{\pi}{2}t\Big)^{n-1}\frac{1}{2\pi}\int_0^{2\pi}d\psi\,e^{-s\cos\psi+ist\sin\psi+il\psi}\,. \tag{C23}$$

For large s the integral is peaked around $\psi=0$ and we can perform a Gaussian integration

$$\Delta_l\cong e^{ix_4}e^{-il(\theta+\pi)}\tfrac{1}{2}\int_{-1}^{1}dt\Big(2\cos\frac{\pi}{2}t\Big)^{n-1}\frac{e^s}{(2\pi s)^{1/2}}e^{-(s/2)t^2+lt-l^2/2s-tx_3}\,. \tag{C24}$$

In turn, when $x_3<s$, the t integral is peaked around $t=(l-x_3)/s$ and a second Gaussian integration gives

$$\Delta_l\cong e^{ix_4}e^{-il(\theta+\pi)}\frac{e^s}{2s}\,e^{+x_3{}^2/2s-x_3l/s}\Big[2\cos\frac{\pi}{2}\,\frac{l-x_3}{s}\Big]^{n-1}. \tag{C25}$$

When we consider a Toeplitz determinant, it is straightforward to show that

$$\det G_{k-l}=\det(a)^{k-l}G_{k-l}\,, \tag{C26}$$

where a is arbitrary.

We may now reexpress our result in the form

$$H^{j\times j}_{k-l}\cong\Big(\frac{e^{s+x_3{}^2/2s}}{2s}\Big)^j\det\Big[2\cos\frac{\pi}{2}\,\frac{k-l-x_3}{s}\Big]^{n-1}, \tag{C27}$$

where we have removed the irrelevant dependence on x_4. It is straightforward to find that the proper choice of $G(e^{i\psi})$ for this problem is

$$G(e^{i\psi})=\sum_{k=0}^{n-1}\binom{n-1}{k}2\pi\delta\Big((2k-n+1)\frac{\pi}{2s}-\psi\Big)e^{-ix_3\psi}\,. \tag{C28}$$

The integration may now be carried out explicitly for arbitrary j and x_3.

We shall restrict our attention here to the case wherein $j=n$, describing the behavior of the solution on the s plane for $s>n$ and in the region $x_3<s$:

$$H^{n\times n}_{k-l}\cong\Big(\frac{e^{s+x_3{}^2/2s}}{2s}\Big)^n\frac{2^{n(n-1)}}{n!}\int\prod_i\frac{d\psi_i}{2\pi}\prod_{i<j}\sin^2\frac{\psi_i-\psi_j}{2}\prod_i G(e^{i\psi_i})\,. \tag{C29}$$

Each ψ_ψ takes n values from $(1-n)\pi/2s$ to $(n-1)\pi/2s$ but the contributions coming from $\psi_i=\psi_j$ are identically zero. We have to pick up only the $n!$ permutations where each ψ_i takes a different value.

Moreover, all resulting factors turn out to have the same form, and collecting all factors we obtain

$$H^{n\times n}_{k-l}=\Big(\frac{e^{s+x_3{}^2/2s}}{2s}\Big)^n 2^{n(n-1)}\prod_{k=1}^{n-1}\binom{n-1}{k}\Big(\sin^2\frac{k\pi}{2s}\Big)^{n-k}. \tag{C30}$$

At very large s, Eq. (C30) reduces to

$$H^{n\times n}_{k-l}=\Big[(n-1)!^{n-1}\frac{e^s}{2s^n}\Big]^n \tag{C31}$$

corresponding to the expected behavior

$$h^2=\Big(1-\frac{n}{s}\Big)^2. \tag{C32}$$

We mention here that the large-x_3 behavior on the x_3 axis ($s=0$) is easily evaluated and turns out to be exactly the same as Eq. (C31) with s replaced by x_3.

A last application of this method leads to an alternative proof of the fundamental result of Sec. IX, Eq. (9.17). Let us consider the class of $n\times n$ determinants where

$$G_l=\sum_{k=1}^{n}C_k(A_k)^l \tag{C33}$$

corresponding to a function

$$G(e^{i\psi})=\sum_{l=-\infty}^{\infty}G_le^{-il\psi}=\sum_{k=1}^{n}2\pi C_k\delta(\psi_k-\psi)\,, \tag{C34}$$

where we have defined the complex numbers ψ_k through

$$e^{i\psi_k}=A_k\,. \tag{C35}$$

Insertion of Eq. (C36) into Eq. (C11) immediately leads to

$$I(\det G)=\left(\prod_{k=1}^{n} C_k\right)\prod_{i<j}\left(1-\frac{A_i}{A_j}\right)\left(1-\frac{A_j}{A_i}\right). \quad \text{(C36)}$$

By choosing

$$C_k=\beta_k\frac{e^{r_k}}{2r_k}\,,\quad A_k=\frac{x_3-q_k-r_k}{s}\,, \quad \text{(C37)}$$

we obtain

$$G_l=e^{il\theta}e^{-ix_4}\Delta_l^A \quad \text{(C38)}$$

and

$$\det\Delta_{k-l}^A\propto\prod_{k=1}^{n}\left(\frac{e^{r_k}}{r_k}\right)\times\prod_{i<j}\left(1-\frac{x_3-q_i-r_i}{x_3-q_j-r_j}\right)\left(1-\frac{x_3-q_j-r_j}{x_3-q_i-r_i}\right). \quad \text{(C39)}$$

APPENDIX D: EQUIVALENCE WITH LATTICE QCD_2 AND THE LIMIT $n\to\infty$

In Appendix C we have shown that

$$H_{k-l}^{j\times j}=I(\det\tilde{\Omega}^{(n)}(U))=\int dU\det\tilde{\Omega}^{(n)}(U). \quad \text{(D1)}$$

By using the explicit form of $\tilde{\Omega}^{(n)}$, Eq. (7.5), it is trivial to show that

$$e^{-ijx_4}H_{k-l}^{j\times j}=\frac{(n-1)!\,\pi^{n-1}}{2}\int dU\,e^{-(s/2)\mathrm{Tr}(U+U^\dagger)}\det\left(\frac{e^{x_3-(s/2)\mathrm{Tr}(U+U^\dagger)}+(-1)^n e^{-x_3+(s/2)\mathrm{Tr}(U-U^\dagger)}}{\prod_{k=1}^{n}[x_3-(s/2)\mathrm{Tr}(U-U^\dagger)-z_k]}\right). \quad \text{(D2)}$$

In turn, this expression is equivalent, thanks to Eqs. (A13), to

$$e^{-ijx_4}H_{k-l}^{j\times j}\propto\int dU\,e^{-(s/2)\mathrm{Tr}(U+U^\dagger)}\prod_{k'}\det\left(ix_3-\frac{is}{2}\mathrm{Tr}(U-U^\dagger)-iz'_k\right), \quad \text{(D3)}$$

where $z'_k=i\pi[(n+1)/2-k']$ as in Eq. (4.1c), but with the restriction that k' be *different* from $1,\ldots,n$.

When $x_3=0$, Eq. (D3) is equivalent to the partition function of lattice two-dimensional QCD with a gauge coupling $g^2=1/s$ and with an infinite number of "fermions" having masses

$$m_{k'}=\left|\left(\frac{n+1}{2}-k'\right)\pi\right|. \quad \text{(D4)}$$

Indeed it is easy to recognize the well-known Wilson action in the exponent $\frac{1}{2}s\,\mathrm{tr}(U+U^\dagger)$ and the fermionic interpretation emerges as a natural way of representing the determinants. Within this interpretation

$$\tau=\ln e^{-inx_4}H_{k-l}^{n\times n} \quad \text{(D5)}$$

plays the role of the free energy (vacuum expectation value of the Hamiltonian) and the statement that only the first n determinants are nowhere vanishing implies that, in this version of QCD_2, a larger group requires suppression of more low-mass particles in the fermionic spectrum if we want the Hamiltonian to be bounded below and the theory to be well defined.

As an immediate by-product of this equivalence, we may study the large-n limit of the monopole solution as a large-n QCD_2 problem.[11] In particular, let us observe that when $n\to\infty$ all fermions are suppressed, and τ reduces to

$$\tau=\ln\int dU\,e^{-(s/2)\mathrm{tr}(U+U^\dagger)}. \quad \text{(D6)}$$

It is apparent from Eq. (D6) that in this limit the solution becomes invariant under translations along the x_3 axis. In order to understand this phenomenon, let us consider the $n\to\infty$ limit of the Abelian result Eq. (9.13):

$$h=1-\sum_{k=1}^{\infty}\frac{1}{r_k}=-2\sum_{k=1}^{\infty}e^{-2k|x_3|}J_0(2ks). \quad \text{(D7)}$$

The Abelian result itself is a sum of terms that are exponentially damped in the x_3 direction, and the core contributions may then suppress all x_3 dependence.

As a check of this result, let us evaluate h along the x_3 axis directly from the known explicit expression Eq. (12.5a). Since

$$\sum_{k=1}^{\infty}\frac{1}{x_3-z_k}=(\tanh x_3)^{(-1)^n} \quad \text{(D8)}$$

we find out

$$h(x_3=0)=0. \quad \text{(D9)}$$

Equation (D6) is the pure Yang-Mills problem first studied by Gross and Witten.[12] According to their result, the $n\to\infty$ limit is (up to an irrelevant additive constant)

$$\tau=\tfrac{1}{4}s^2,\quad s\le n \quad \text{(D10a)}$$

$$\tau=ns-\frac{n^2}{2}\ln\frac{s}{n}-\tfrac{3}{4}n^2,\quad s\ge n. \quad \text{(D10b)}$$

Let us compute the gauge-invariant fields corresponding to Eqs. (D10) through the superpotential equations[8]

$$h^2 = 1 - \nabla^2 \tau \,, \tag{D11a}$$

$$\psi^2 + E^2 = n^2 - s^2 \nabla^2 \tau + 2s \frac{d\tau}{ds} \,, \tag{D11b}$$

$$h\psi = -\partial_3 \tau \,. \tag{D11c}$$

In the (strong-coupling) region $s \leq n$, where the result could have been obtained by a direct application of Szego's theorem to the large-n Toeplitz determinant $H_{k-l}^{n\times n}$, one finds

$$h^2 = 0 \,, \quad \psi^2 = n^2 \,, \quad E^2 = 0 \,, \tag{D12}$$

a trivial solution of the self-duality equation, regular for any finite value of s.

When $s \geq n$ one finds

$$h^2 = 1 - \frac{n}{s} \,, \quad \psi = 0 \,, \quad E^2 = ns \,, \tag{D13}$$

corresponding to a finite amount of energy and topological charge per unit length and satisfying the desired boundary conditions.

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