Three-dimensional O(N) theories at large distances

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The infrared structure of charged, O(N)-invariant field theories in three dimensions is analyzed. The zero-mass limit, appropriate for a mean-field-theory description of phase transitions in statistical physics, is studied using the $1/N$ expansion. The infrared divergences of the loop expansion are then eliminated, and it is shown how the low-momentum behavior is completely governed by an infrared-stable fixed point of the renormalization group. In particular, the fixed point, which appears to leading order in $1/N$, is shown to persist to higher orders by virtue of cancellations among terms which are singular in the zero-momentum limit. Some new computations of anomalous dimensions are presented and transcribed into the corresponding critical indices of statistical mechanics. The structure of the effective potential of the theory is summarized. It can be computed order by order in the $1/N$ expansion and its only minimum is at the origin.

I. INTRODUCTION

Quantum field theories in three Euclidean dimensions have interesting and unusual properties and can be important for a variety of physical problems. Consider, for example, a four-dimensional gauge theory at very high temperature.[1] Its long-distance behavior is determined completely by the corresponding theory in three dimensions.[2-4] For a non-Abelian theory such as quantum chromodynamics (QCD), the three-dimensional version contains interacting massless fields which lead to severe infrared divergences in the loop expansion.[3,4] These divergences, due in effect to the super-renormalizability of the theory, appear in the Green's functions of the theory at two loops and beyond. When the external momenta are large, they can be connected with the appearance of new operators in the operator-product expansion.[3,5]

Another class of three-dimensional theories, of more immediate physical interest, are the mean-field theories of the Ginzburg-Landau type. They are formulated in terms of an N-component, charged or neutral, field $\phi_a(\vec{x})$ and used to compute the critical indices associated with second-order phase transitions. At the critical temperature $T=T_c$, the physical mass (the inverse correlation length) goes to zero and again infrared divergences appear in the loop expansion. In order to deal with this problem, two methods have been used to circumvent the loop expansion. One is the $1/N$ expansion where N is the number of components in ϕ_a and the other is the ϵ expansion where $\epsilon=4-d$, d being the dimensionality of space.

In this paper we shall discuss several aspects of the $1/N$ expansion for Ginzburg-Landau theories in three dimensions. The work has grown out of the earlier efforts of R. Pisarski and one of us[3] to analyze finite-temperature and three-dimensional QCD. There the $1/N$ expansion for, say, an SU(N) gauge group is still far from tractable since it involves all planar diagrams to leading order. In order to get some feeling for nonperturbative behavior in a three-dimensional theory, a simpler model was examined. The model involved an Abelian gauge field $A_\mu(\vec{x})$ coupled to a massless, charged, N-component field $\phi_a(\vec{x})$. It was analyzed in the $1/N$ expansion which involves a much simpler set of diagrams to leading order than in SU(N) gauge theories.

This paper continues that analysis. In addition to the gauge coupling, a $(\vec{\phi}^\dagger\cdot\vec{\phi})^2$ coupling is also included, which makes the model a genuine Ginzburg-Landau theory. In a three-dimensional model, this coupling is not required for ultraviolet renormalizability but, as we shall see, it *is* necessary for vacuum stability. Although this theory has been extensively analyzed in the $1/N$ expansion,[6] many of its important properties have not, to our knowledge, been described in the literature.

We shall consider the theory only with the physical mass of the field ϕ_a set equal to zero. This corresponds to working at $T=T_c$, that is, "on the critical surface." The theory contains a nonzero, infrared-stable fixed point in both the gauge and $(\vec{\phi}^\dagger\cdot\vec{\phi})^2$ couplings and the objects of main interest are the anomalous dimensions and critical indices associated with this fixed point. Some of these have to do with the behavior of correlation functions at $T=T_c$ and others have to do with the approach to T_c. However, it is sufficient to work at $T=T_c$ to determine all the critical indices. This well-known fact will be reviewed briefly at the end of Sec. III.

It is perhaps worthwhile to list some of the questions which generated this investigation and which arose during the course of it. Each of them will be treated in some detail and at least partially answered in Secs. II–V.

1. In a general sense, both this work and much of Ref. 3 developed in an effort to learn more about the structure of massless three-dimensional

field theories. For the class of O(N) theories being considered here, some new things have been learned. Whether any of these provide insight into SU(N) gauge theories remains to be seen.

2. Perhaps the main question to be addressed in this paper is the nature of the $1/N$ expansion. Are higher orders truly of order $1/N$, $1/N^2$, etc., for arbitrarily small momentum? The answer is yes, although not obviously so. After the cancellation of singular terms, the running coupling constants and associated β functions can be computed to higher orders. However, a convention dependence enters and the running coupling constants can be defined, for example, so that their infrared-stable fixed points are exactly given by the leading-order values.

3. The anomalous dimensions and critical indices associated with the fixed point are not convention dependent. As a part of our general analysis, we have computed some of them which we have not seen reported before in the literature.

4. In any dimension above three, it is not hard to show that the addition of higher-dimension interactions such as $(\vec{\phi}^\dagger\cdot\vec{\phi})^3$ or $(\vec{\phi}^\dagger\cdot\vec{\phi})^4$ will not affect the infrared structure of the theory. These are so-called "irrelevant" operators. However, in three dimensions, the interaction $(\vec{\phi}^\dagger\cdot\vec{\phi})^3$ is exactly renormalizable and its effect on the infrared structure is not so clear. We shall consider the addition of such a term to the Lagrangian and show that, *in the presence* of the $(\vec{\phi}^\dagger\cdot\vec{\phi})^2$ interaction, it is irrelevant even in three dimensions.

5. The cancellation of singular terms in the higher-order contributions to the coupling-constant renormalization has important consequences for the effective potential[7] of the theory. This cancellation results in a vanishing of the $(\vec{\phi}^\dagger\cdot\vec{\phi})^2$ term, thus strongly changing the character of the effective potential from its classical behavior. We find that in the $1/N$ expansion the effective potential starts off like $(\vec{\phi}^\dagger\cdot\vec{\phi})^3$ for small $\vec{\phi}$. On the other hand, we see no sign of instability of the classical, symmetric vacuum.

The paper is organized as follows. In Sec. II, the theory is defined and the elements of the $1/N$ expansion are reviewed. The behavior of the theory to leading order in the $1/N$ expansion is described and interpreted in the language of the renormalization group. An infrared-stable fixed point is shown to exist and the associated anomalous dimension of the scalar field ϕ_a is computed.

In Sec. III, we consider higher orders in the $1/N$ expansion. It is shown that singular dynamical factors arise in the infrared but that they cancel in the running coupling constants and β functions, verifying that $1/N$ is the correct expansion parameter. The convention dependence which enters the nonsingular corrections to the β functions at order $1/N$ is described. The anomalous dimension of $\vec{\phi}^\dagger\cdot\vec{\phi}$ is computed and that, along with the anomalous dimension of $\vec{\phi}$, is used to tabulate the various critical indices of the corresponding statistical-mechanical system.

Section IV examines the consequences of adding a $(\vec{\phi}^\dagger\cdot\vec{\phi})^3$ term to the Lagrangian. It is shown how the $(\vec{\phi}^\dagger\cdot\vec{\phi})^2$ term controls the infrared behavior and renders the $(\vec{\phi}^\dagger\cdot\vec{\phi})^3$ term irrelevant even though it is just renormalizable, entering with a dimensionless coupling constant. Finally, we point out that an effective $(\vec{\phi}^\dagger\cdot\vec{\phi})^3$ vertex will be induced even if it is not inserted by hand. No attention need be paid to this for purposes of computing critical indices but it is an important piece of the effective potential of the theory.

In Sec. V, our results are summarized and some comparisons with the ϵ expansion are offered. The reliability of our results and the second-order character of the phase transition depend on the stability of the symmetric vacuum $\vec{\phi}=0$. To verify the stability, we have analyzed the effective potential of the theory in the $1/N$ expansion. The structure of the potential is briefly described here. A more detailed study of the effective potential in three-dimensional O(N) theories will be presented in a separate publication.

II. ELEMENTS OF THE $1/N$ EXPANSION

In this section we will define the model, establish the notation used in the remainder of the paper, and review the idea of the $1/N$ expansion. Running coupling constants are introduced and the existence and meaning of fixed points of the renormalization group are discussed. Finally, we compute the anomalous dimension of the charged scalar field to leading order in $1/N$ which is just the critical exponent η of the corresponding statistical-mechanics problem.

The Lagrangian density for the Euclidean theory is

$$\mathcal{L}=\tfrac{1}{4}F_{ij}^{\,2}+\sum_{a=1}^{N}|(\partial_i+ieA_i)\phi_a|^2+m_0^{\,2}\vec{\phi}^\dagger\cdot\vec{\phi}+\frac{\lambda}{2N}(\vec{\phi}^\dagger\cdot\vec{\phi})^2\,, \tag{2.1}$$

where $\vec{\phi}$ is an N-component vector of complex fields and i,j run from one to three. All expressions and computational results will be restricted to the Euclidean theory in this paper. The coupling constants λ and $\alpha\equiv Ne^2$ both have dimensions of mass. The theory is super-renormalizable.

We shall restrict our attention to those values of the parameters (m_0,α,λ) which lead to the vanishing of the physical mass of the scalars and which determine the so-called critical surface. The bare mass $m_0(\alpha, \lambda, \Lambda)$, which depends linearly on an ultraviolet cutoff Λ, is adjusted at each order of approximation to make the physical mass vanish. The inverse scalar propagator can then be written in the form

$$G^{-1}(k) = k^2 - \tilde{\Pi}_s(k), \tag{2.2}$$

where $\tilde{\Pi}_s(k)$ is the subtracted scalar self-energy

$$\tilde{\Pi}_s(k) = \Pi_s(k; m=0, \alpha, \lambda) - \Pi(0; m=0, \alpha, \lambda). \tag{2.3}$$

Since the coupling constants have dimensions of mass, this massless theory is plagued with infrared divergences. The effective loop expansion parameters are α/k and λ/k, leading to infrared-divergent Green's functions already at the two-loop level.[3,4] One scheme which leads to infrared-finite results is an expansion in the dimensionless parameter $1/N$ with α and λ fixed. Each order in the $1/N$ expansion sums an infinite class of Feynman graphs which in turn leads to infrared-finite amplitudes at the next level of approximation.

To leading order in $1/N$, only those graphs are included which contain one closed loop for every additional coupling factor of α/N or λ/N. The two possibilities are the corrections to the gauge propagator shown in Fig. 1, and the corrections to the $(\vec{\phi}^\dagger \cdot \vec{\phi})^2$ vertex shown in Fig. 2. The Feynman rules can be read off from the Lagrangian (2.1) and all computations will be performed in the Landau gauge. The gauge boson propagator is readily computed to leading order with the result[3]

$$D_{ij}^{(1)}(k) = \frac{\delta_{ij} - \hat{k}_i \hat{k}_j}{\vec{k}^2 + (\alpha/16)k}, \tag{2.4}$$

where $k \equiv |\vec{k}|$. Note that there is no tachyonic pole in this propagator and that the interactions soften the infrared behavior from $1/k^2$ to $1/k$.

It is convenient to interpret the result (2.4) in terms of an effective or running coupling constant. The factor which renormalizes the bare propagator is $Z_3(k) \equiv (1+\alpha/16k)^{-1}$. The running coupling constant is formed by starting with the dimensionless factor α/k and then multiplying by $Z_3(k)$. The result is

$$\overline{\alpha}^{(1)}(k) = \frac{\alpha}{k}\left(1 + \frac{\alpha}{16k}\right)^{-1}. \tag{2.5}$$

Note that it is not necessary to consider $\vec{\phi}$ self-energy and vertex corrections since these enter only at next order in the $1/N$ expansion.

The only other graphs which must be computed and summed to leading order in $1/N$ are the corrections to the $(\vec{\phi}^\dagger \cdot \vec{\phi})^2$ vertex shown in Fig. 2. The arrows indicate the flow of "isospin" indices required to produce the necessary factors of N. Each loop contributes the factor $-\lambda/8k$, and the sum, indicated by the isosinglet dashed line, is $-Z_\lambda(k)\lambda/N$ where $Z_\lambda(k) \equiv (1+\lambda/8k)^{-1}$. A running four-point coupling can now be defined. In general, it requires the computation of both the one-particle-irreducible (1PI) four-point and two-point functions. However, the latter first appears only at next order in the $1/N$ expansion. The dimensionless running coupling constant is thus defined to this order by multiplying λ/k by $Z_\lambda(k)$,

$$\overline{\lambda}^{(1)}(k) = \frac{\lambda}{k}\left(1 + \frac{\lambda}{8k}\right)^{-1}. \tag{2.6}$$

Note that to this order, $\overline{\lambda}$ receives no contribution from the gauge coupling and vice versa.

Both $\overline{\alpha}^{(1)}(k)$ and $\overline{\lambda}^{(1)}(k)$ are dominated by lowest-order perturbation theory as $k \to \infty$, vanishing like $1/k$. As $k \to 0$, they approach the constant values 16 and 8, respectively. This behavior can be interpreted in terms of a renormalization-group fixed point by introducing β functions as follows. We write the coupling constants in the form $\overline{\alpha}(k/k_0, \overline{\alpha}_0)$ and $\overline{\lambda}(k/k_0, \overline{\lambda}_0)$, where $\overline{\alpha}_0$ and $\overline{\lambda}_0$ are the values of $\overline{\alpha}$ and $\overline{\lambda}$ at some reference momentum k_0. Then

$$\beta_\alpha(\overline{\alpha}_0) \equiv x\frac{\partial}{\partial x}\overline{\alpha}(x, \overline{\alpha}_0)\Big|_{x=1}, \qquad \beta_\lambda(\overline{\lambda}_0) \equiv x\frac{\partial}{\partial x}\overline{\lambda}(x, \overline{\lambda}_0)\Big|_{x=1} \tag{2.7}$$

and, to leading order in $1/N$,

$$\beta_\alpha(\overline{\alpha}) = -\overline{\alpha} + \tfrac{1}{16}\overline{\alpha}^2, \qquad \beta_\lambda(\overline{\lambda}) = -\overline{\lambda} + \tfrac{1}{8}\overline{\lambda}^2. \tag{2.8}$$

These β functions have zeros at

$$(\overline{\alpha}, \overline{\lambda}) = (0, 0) \tag{2.9a}$$

and at

$$(\overline{\alpha}, \overline{\lambda}) = (\alpha^*, \lambda^*) \equiv (16, 8), \tag{2.9b}$$

the origin being an ultraviolet-stable fixed point and (α^*, λ^*) being infrared stable. Since we are

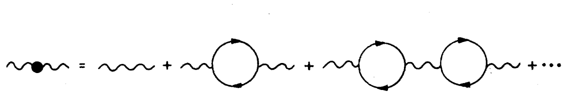

FIG. 1. The leading corrections to the photon propagator in the $1/N$ expansion.

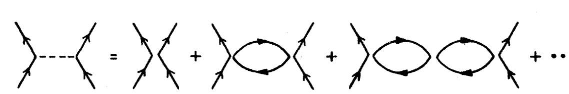

FIG. 2. The leading corrections to the four-point coupling in the $1/N$ expansion. The arrows denote "isospin" flow.

mainly concerned with the infrared structure of the theory, it is the latter fixed point which is the more interesting. Its presence in a statistical-mechanics context signals the existence of a critical surface of second-order phase transitions due to the change of sign of the square of the physical mass of the scalars.

All further interactions in this theory are nonleading in the $1/N$ expansion. As an example, we compute the leading scalar self-energy corrections shown in Fig. 3. A subtraction at zero momentum is to be made and, with that, it has already been observed[3] that the gauge boson contribution is gauge invariant. The result, written explicitly here in Landau gauge, is

$$\tilde{\Pi}_s(k)=\frac{4\alpha}{N}\int\frac{d^3q}{(2\pi)^3}\frac{k^2q^2-(\vec{k}\cdot\vec{q})^2}{(\vec{k}+\vec{q})^2q^2}\frac{1}{q^2+(\alpha/16)q} +\frac{\lambda}{N}\int\frac{d^3q}{(2\pi)^3}\frac{k^2+2\vec{k}\cdot\vec{q}}{(\vec{k}+\vec{q})^2q^2}\frac{1}{1+(\lambda/8q)}\,. \tag{2.10}$$

In the infrared limit $k \ll \alpha/16$, $k \ll \lambda/8$, this becomes

$$\tilde{\Pi}_s(k)=\frac{k^2}{N}\left(-\frac{64}{3\pi^2}\ln\frac{16k}{\alpha}+\frac{4}{3\pi^2}\ln\frac{8k}{\lambda}\right)+O\left(\frac{k^2}{N}\right). \tag{2.11}$$

Therefore, to this order and for small k,

$$G(k)=\frac{1}{k^2}\left[1+\frac{4}{3\pi^2 N}\left(\ln\frac{8k}{\lambda}-16\ln\frac{16k}{\alpha}\right)+\frac{A}{N}\right], \tag{2.12}$$

where A is a constant.

Given the existence of the infrared-stable fixed point (2.9b), the behavior (2.12) can be interpreted as the buildup of an anomalous power behavior of the full scalar propagator for small k. To make this connection, we first remind the reader of the result of a renormalization-group analysis of a general connected n-point function $\overline{\Gamma}_n(\vec{p}_1,\ldots,\vec{p}_n;\overline{\alpha}(\Lambda),\overline{\lambda}(\Lambda);\Lambda)$. Here, Λ is some arbitrary reference scale, $\overline{\alpha}(\Lambda)$ and $\overline{\lambda}(\Lambda)$ are the dimensionless running coupling constants defined at that scale, and the bar indicates that $\overline{\Gamma}_n$ does not contain the δ function for momentum conservation. Suppose that each momentum p is scaled by κ and then κ is allowed to approach zero. Using the fact that $\overline{\Gamma}_n$ must be independent of Λ and the existence of the fixed point (α^*,λ^*), it can be shown[8] that

$$\overline{\Gamma}_n(\kappa\vec{p}_1,\ldots,\kappa\vec{p}_n;\ldots)\sim\kappa^{D_n+n\eta/2}, \tag{2.13}$$

where D_n is the naive scaling dimension of the

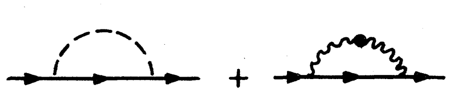

FIG. 3. The leading scalar self-energy corrections in the $1/N$ expansion.

Green's function and η is the anomalous dimension of the field $\phi_a(\vec{x})$, associated with the fixed point.

The naive dimension D_n is given by

$$D_n=d-n\left(1+\frac{d}{2}\right) \tag{2.14}$$

in a general dimension d. For a two-point function, $D_2=-2$ independent of the number of dimensions. For a four-point function in three dimensions, $D_4=-7$, etc. The value of the anomalous dimension η can be determined by comparing the computation (2.12) with the known scaling behavior of the two-point function

$$G(k)=\overline{\Gamma}_2(k)\sim k^{-2+\eta}\,. \tag{2.15}$$

For small η, (2.15) may be expanded,

$$G(k)\sim k^{-2}\left(1+\eta\ln k+\frac{\eta^2}{2}\ln^2 k+\cdots\right), \tag{2.16}$$

and thus, to order $1/N$,

$$\eta=-20/(\pi^2 N)\,. \tag{2.17}$$

Note that the effect of the dynamics is to make the propagator somewhat more singular in the infrared than naive power counting would suggest. This is due to the presence of the gauge interaction which opposes and overwhelms the scalar self-interaction.

In the statistical-mechanics interpretation $G(k)$ is just the two-point correlation function for the order parameter, and the power η governing its small-momentum behavior at the critical temperature is one of the measurable critical exponents.[9]

III. HIGHER ORDERS IN THE 1/N EXPANSION

The anomalous dimension η (2.17) is the first example of a higher-order effect. A program to compute others quickly encounters some of the peculiar features of the $1/N$ expansion. Two of them which must be dealt with in order to compute other anomalous dimensions or higher-order corrections to η are the following:

(a) From our treatment of the leading order it might appear that the expansion parameter is not $1/N$ but α/Nk or λ/Nk. Such terms appear in some of the next-order contributions to $\overline{\lambda}(k)$ and if they were not to cancel they would render the $1/N$ expansion useless in the infrared.

(b) In both $\overline{\lambda}(k)$ and $\overline{\alpha}(k)$, corrections of order $(1/N)\ln k$ appear.[3] These too must cancel if the infrared-stable fixed points are to persist beyond leading order.

We will show that $1/N$ is indeed the expansion parameter and that the only effect of higher-order diagrams is a shift of the fixed point (λ^*, α^*) and the anomalous dimensions by constants of order $1/N$.

A. Gauge coupling renormalization

We begin with the next-order contributions to the photon polarization operator. These are shown in Fig. 4. The diagrams not involving the $(\vec{\phi}^\dagger\cdot\vec{\phi})^2$ interaction were examined in Ref. 3 for both large and small momenta. For the small-momentum limit of interest here, it was observed that there are contributions to the polarization operator of order $\alpha k/16N$ and $(\alpha k/16N)\ln(k/\alpha)$. Recall that the lowest-order contribution Eq. (2.4) is of order $\alpha k/16$. All contributions have now been analyzed with the following result[10]: (1) The polarization tensor is transverse as expected. (2) The $(\alpha k/16N)\ln k$ terms cancel.

The photon propagator through second order in the $1/N$ expansion and for small momentum is therefore

$$D_{ij}^{(2)}(\vec{k})=\frac{\delta_{ij}-\hat{k}_i\hat{k}_j}{\vec{k}^2+(\alpha k/16)(1+A/N)}, \tag{3.1}$$

where A is a numerical constant.[10] The cancellation of the $(1/N)\ln k$ terms, a consequence of the Ward identity of the theory, will be briefly described. The fact that they cancel is important because it indicates that the expansion parameter in the infrared is indeed $1/N$. Thus, if a running coupling constant is defined from Eq. (3.1), it has the form

$$\overline{\alpha}^{(2)}(k)=\frac{\alpha}{k}\left[1+\frac{\alpha}{16k}\left(1+\frac{A}{N}\right)\right]^{-1}. \tag{3.2}$$

In the limit $k\to 0$,

$$\overline{\alpha}^{(2)}(k)\to\frac{16}{1+A/N}, \tag{3.3}$$

so that the infrared-stable fixed point continues to exist and is simply shifted by an amount of order $1/N$. Note that the constant A receives contributions from both the gauge interactions and the $(\vec{\phi}^\dagger\cdot\vec{\phi})^2$ interactions but that the fixed point is independent of both α and λ.

To this order in the $1/N$ expansion, vertex and scalar self-energy contributions also enter the running coupling constant. However, the same Ward identity leading to the cancellation of $\ln k$ terms in the photon propagator (3.1) ensures that similar terms cancel between vertex and self-energy. There can be left over finite parts which can introduce additional (and convention dependent) contributions to $\overline{\alpha}(k)$. We shall return to this point shortly.

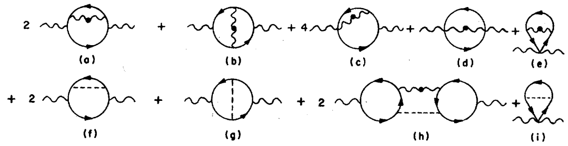

FIG. 4. The next-to-leading corrections to the photon propagator.

The cancellation of logarithmic terms in the propagator (3.1) goes roughly as follows. It might be anticipated that for small external momentum k, any graph of Fig. 4 will be dominated by small internal momenta and therefore that the internal gauge propagator (2.4) can be approximated by the linear term in the denominator. That, in fact, is true for a graph like Fig. 4(d) and therefore, on dimensional grounds, it behaves like αk for small k. However, an attempt to do the same thing for Fig. 4(a) induces a logarithmic ultraviolet divergence in the subgraph. It is of course cut off by the quadratic term in the denominator of the internal propagator (2.4) but then the diagram behaves like $\alpha k\ln(k/\alpha)$. The Ward identity ensures that the induced ultraviolet divergence in the self-energy subgraph will be canceled by a vertex subgraph. In the Landau gauge, the necessary cancellation is provided by Fig. 4(c). With the cancellation, the gauge propagator denominator (2.4) can be reliably approximated by the linear term and from dimensional analysis it can again be seen that no logarithmic factors can enter. A similar cancellation takes place between Figs. 4(f) and 4(g).

B. Renormalization of the four-point coupling

The diagrams contributing to the renormalization of the four-point coupling to next order in $1/N$ are shown in Fig. 5. Recall that the leading contribution (Fig. 2) vanishes proportional to k as $k\to 0$ and that the running coupling constant $\overline{\lambda}^{(1)}(k)$ (Eq. 2.6) was formed by dividing this result by k. The first problem encountered now is that some of the graphs of Fig. 5 do not vanish at zero momentum. An example is Fig. 5(a) which is proportional to λ/N^2. Fortunately, this constant is canceled by similar contributions from Figs. 5(b) and 5(c) and the sum vanishes in the zero-momentum limit.

The three other graphs not vanishing at zero momentum are the set (h, i, j) of Fig. 5. The fact that only the seagull diagrams of the gauge theory fail to vanish is a property of the Landau gauge. Figure 5(h) is especially interesting since, in a theory without the $\lambda(\vec{\phi}^\dagger\cdot\vec{\phi})$ interaction, it would be the leading contribution to the effective potential for small $\vec{\phi}$. It is an *effective* $(\vec{\phi}^\dagger\cdot\vec{\phi})^2$ term which, however, enters with a negative coefficient, signaling the instability of the symmetric vacuum. In fact, the full effective potential in the absence of the $\lambda(\vec{\phi}^\dagger\cdot\vec{\phi})^2$ interaction has no stable minimum at all, and therefore this interaction in the Lagrangian (2.1) is indispensable if the zero-mass

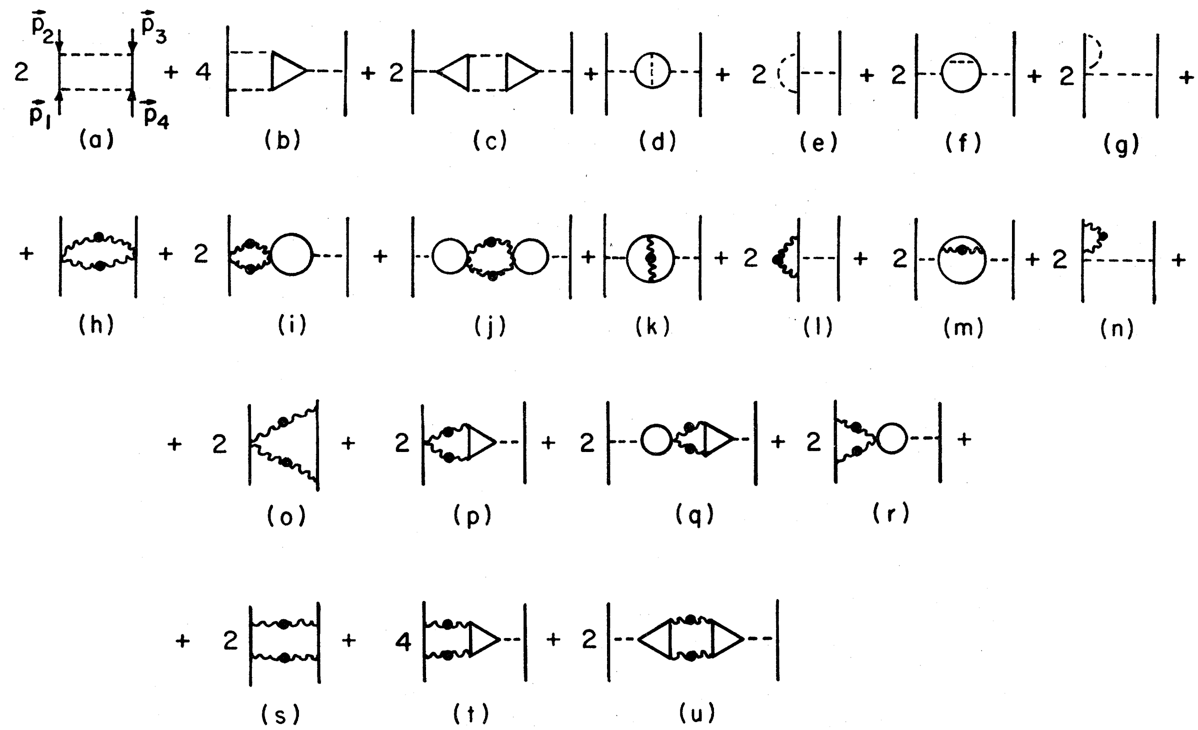

FIG. 5. The next-to-leading corrections to the effective four-point coupling.

theory is to be physically meaningful. For the problem of concern here, it provides graphs 5(i) and 5(j) which precisely cancel graph 5(h) at zero momentum.

The two sets 5(a), 5(b), and 5(c) and 5(h), 5(i), and 5(j) are very similar in their structure and cancellation. In the second set, 5(h) can be viewed as the core, appearing again in 5(i) and 5(j). By adding the three graphs, the core 5(h) is in effect multiplied by a factor which vanishes like k^2 as $k \to 0$. Similarly, 5(a) is the core of the 5(a), 5(b), 5(c) set. The situation is slightly more complicated here since the graphs depend on more than a single external momentum. However, that does not affect the cancellation of the leading, constant terms.

With the constant term gone, attention can be focused on those terms which vanish proportional to one power of the external momentum. The limit can be taken, for example, by starting at the symmetry point $p_1^2 = p_2^2 = p_3^2 = p_4^2 = \frac{3}{4}k^2$, $s = t = u = k^2$, and then scaling k to zero. We adopt this arbitrary procedure for now, and return later to the question of prescription dependence.

There now remain terms which vanish like $k \ln k$ along with others vanishing like k. It is the latter terms which provide the order $1/N$ contributions to $\overline{\lambda}(k)$ once they are divided by the factor k. The $k \ln k$ terms must cancel if the infrared-stable fixed point is to survive to this order and beyond. The relevant graphs are shown in Fig. 5 and the cancellation works as follows.

(a) The set 5(a), 5(b), and 5(c) contains no $k \ln k$ terms in any of the graphs in dimension $d = 3$. For $d \neq 3$, they reappear, but cancel within the set.[11]

(b) In the set 5(h), 5(i), and 5(j), the individual graphs also contain no $k \ln k$ terms in dimension $d = 3$.

(c) Each of the graphs of the set 5(d), 5(e), 5(f), and 5(g) behaves like $k \ln k$. The origin of these terms is similar to that of the $k \ln k$ terms in the photon propagator graphs. That is, they are present because the small-momentum approxmation for the internal propagators does not quite work. It induces an ultraviolet sensitivity in the self-energy or vertex subdiagram which cancels between 5(d) and 5(e) and between 5(f) and 5(g). Note that the external-line corrections 5(e) and 5(g) are crucial here since there is no Ward identity for the scalar coupling.

(d) Precisely the same thing is true for the set 5(k), 5(l), 5(m), and 5(n).

(e) The set 5(o), 5(p), 5(q), and 5(r) is built around a core 5(o) which vanishes like k in the Landau gauge. The additional integrations in 5(p) and 5(q) introduce $k \ln k$ terms which cancel between the two of them.

(f) The final set is 5(s), 5(t), and 5(u) and each member vanishes proportional to two powers of the external momentum.

With all the singular terms having canceled, the general form of the running coupling constant $\overline{\lambda}(k)$ can be written down through order $1/N$. For small k, it becomes

$$\overline{\lambda}^{(2)}(k)=\frac{\lambda}{k}\left[1+\frac{\lambda}{8k}(1+B/N)\right]^{-1}, \tag{3.4}$$

where B is a numerical constant. In the limit $k\to 0$,

$$\overline{\lambda}^{(2)}(k)\to\frac{8}{1+B/N}, \tag{3.5}$$

showing that an infrared-stable fixed point survives in $\overline{\lambda}$ with its value shifted by an amount of order $1/N$.

The existence of the fixed point given by Eqs. (3.3) and (3.5) has important consequences for the infrared structure of the theory. It means that scaling will continue to exist to this order and that corrections to the anomalous dimensions such as η (2.17) can be reliably computed. A proof that the fixed point remains to all orders in the $1/N$ expansion would be a useful addition to the large body of $1/N$ computational work already in existence.

C. Prescription dependence of the running coupling constants

Although the survival of the fixed point is important, its precise value is not. The location of the fixed point depends on the prescription employed to define $\overline{\alpha}$ and $\overline{\lambda}$, i.e., on the choice of coordinate system in the two-dimensional coupling-constant space. The position of the fixed point is not experimentally measurable; only the anomalous dimensions are. They, of course, must be prescription independent.

The origin of the prescription dependence of the fixed point will now be described in a little more detail. The constants A in $\overline{\alpha}(k)$, (3.2), and B in $\overline{\lambda}(k)$, (3.4), will be shown to depend on the point in momentum space at which they are evaluated. It is even possible to make them vanish by an appropriate choice. Consider, for example, diagrams 5(f) and 5(g). We saw that both diagrams contain a leading logarithmic term, and that these terms tend to cancel each other. However, whereas in diagram 5(f) the logarithm depends on the momentum flowing through the diagram, the logarithm of diagram 5(g) depends upon the momentum of the external leg. Hence, even at the symmetry point, both logarithmic terms only cancel up to a difference proportional to $(k/N)\ln(\frac{4}{3})^{1/2}$, which contributes to the constant B. In general, we are not forced to use the symmetry point for renormalization. Any choice for the six invariants p_i^2 ($i=1,\ldots,4$), u, and s (t is then determined by $t=-u-s+\sum_{i=1}^{4}p_i^2$) is allowed, as long as none of them becomes zero. Defining

$$\begin{aligned} &p_i^2=\nu_i^2k^2 \quad (i=1,\ldots,4),\\ &s=k^2,\\ &u=\nu_u^2k^2, \end{aligned} \tag{3.6}$$

B will in general get contributions proportional to $\ln\nu_i$ due to the incomplete cancellation of logarithms depending upon different invariants. Similar remarks apply to other diagrams in Fig.5. It is easy to see that this dependence of B on the ν_i does not cancel and that it could be used, for example, to make B vanish by suitably choosing the ν_i in Eq. (3.6)

In a similar way, the constant A appearing in the running gauge coupling constant (3.2) can have a prescription dependence. In general, $\overline{\alpha}(k)$ depends on vertex corrections and $\vec{\phi}$ self-energy corrections as well as the gauge-boson propagator corrections which went into its definition in Eq. (3.2). Logarithmic singularities will cancel between the vertex and self-energy corrections because of the Ward identity, but there can be left over constant terms which have the same kind of prescription dependence entering $\overline{\lambda}(k)$. It could be used, if desired, to make the constant A vanish.

If it is assumed that N is large enough to ensure the convergence of the expansion, there is no reason that the prescription dependence we have described could not be employed to make all higher-order corrections to the fixed point vanish as well. Equation (2.9b) would then give the position of the fixed point to all orders in $1/N$. Note that this freedom is only sufficient to make the higher-order corrections to $\overline{\lambda}(k)$ and $\overline{\alpha}(k)$ vanish for momenta $k\ll\lambda,\alpha$. The forms (2.5) and (2.6) only become exact in this limit. Thus the β functions are not given exactly by the lowest-order expressions (2.8).

D. The anomalous dimension γ_{ϕ^2}

All the conventionally used critical exponents are experimentally determined by studying the critical scaling behavior of correlation functions containing only the operators $\vec{\phi}$ and $\vec{\phi}^\dagger\cdot\vec{\phi}$.[9] In Sec. II we studied the behavior of the $\vec{\phi}$-$\vec{\phi}$ correlation function to obtain the critical exponent η. We are now going to calculate the $|\vec{\phi}|^2$-$|\vec{\phi}|^2$ correlation function in order to obtain γ_{ϕ^2}, the anomalous dimension of $|\vec{\phi}|^2=\vec{\phi}^\dagger\cdot\vec{\phi}$.

We start by applying a renormalization-group analysis similar to that of Sec. II to the general correlation function of $m|\vec{\phi}|^2$ operators,

$\bar{\Gamma}^{(0,m)}(\vec{p}_i, \ldots, \vec{p}_m; \bar{\alpha}(\Lambda), \bar{\lambda}(\Lambda); \Lambda)$. [The bar again denotes omission of the momentum-conserving δ function, and Λ is some arbitrary reference scale where $\bar{\alpha}(\Lambda)$ and $\bar{\lambda}(\Lambda)$ are defined.] Scaling all the momenta by a factor κ, we obtain from the condition of invariance of the correlation function under redefinition of the renormalization scale Λ

$$\bar{\Gamma}^{(0,m)}(\kappa\vec{p}_1, \ldots, \kappa\vec{p}_m; \ldots) \sim \kappa^{\tilde{D}_m + m\gamma_{\phi^2}} \tag{3.7}$$

in the infrared limit $\kappa \to 0$. Here

$$\tilde{D}_m = d - 2m \tag{3.8}$$

is the canonical dimension of $\bar{\Gamma}^{(0,m)}$, and γ_{ϕ^2} is the anomalous dimension of $|\vec{\phi}|^2$ at the fixed point (α^*, λ^*):

$$\gamma_{\phi^2} = \lim_{\Lambda \to 0} \Lambda \frac{d}{d\Lambda} [\ln Z_{\phi^2}(\bar{\alpha}(\Lambda), \bar{\lambda}(\Lambda))] . \tag{3.9}$$

From (3.7) we obtain for the $|\vec{\phi}|^2$-$|\vec{\phi}|^2$ correlation function on the critical surface, which is conventionally called $\chi(k)$, the small-momentum behavior[12]

$$\chi(k) \sim k^{\rho} = k^{d-4+2\gamma_{\phi^2}} . \tag{3.10}$$

Now we turn to the calculation of $\chi(k)$. The leading contribution is given by the sum of strings of scalar loops, starting with a single loop:

$$\chi^{(1)}(k) = \frac{N}{8k} - \frac{N}{8k}\frac{\lambda}{N}\frac{N}{8k} + \frac{N}{8k}\frac{\lambda}{N}\frac{N}{8k}\frac{\lambda}{N}\frac{N}{8k} - \cdots$$
$$= \frac{N}{8k}\left(1 + \frac{\lambda}{8k}\right)^{-1} . \tag{3.11}$$

Hence $\chi^{(1)}(k)$ has the leading small-momentum behavior

$$\chi^{(1)}(k) \sim \text{const} - k . \tag{3.12}$$

Naively one would be inclined to compare the constant term with the scaling prediction (3.10) and conclude $\rho = 0$. However, this is wrong. In fact, as shown below, it is the term proportional to k which gets logarithmic contributions from higher-order diagrams, and *not* the constant term. Therefore, to leading order, $\rho = 1$ ($\rho = 4 - d$ in d dimensions), yielding a large [*not* $O(1/N)$] anomalous dimension for $|\vec{\phi}|^2$ in $d = 3$ dimensions:

$$\gamma_{\phi^2} = 1 + O(1/N) . \tag{3.13}$$

The appearance of the constant term in $\chi(k)$ is due to the fact that in this theory $\chi(k)$ does not diverge for $k \to 0$. Therefore, in the limit $k \to 0$ the constant term [which for a theory with singular $\chi(k)$ would have been nonleading] surfaces from below the term describing the scaling behavior. In this sense the constant behaves very similar to a mass term in the $\vec{\phi}$-$\vec{\phi}$ correlation function: For small k it would dominate the $k^{-2+\eta}$ scaling behavior found for the critical theory.

The diagrams contributing to $\chi(k)$ in next order are obtained from Figs. 5(c), 5(d), 5(f), 5(j), 5(k), 5(m), 5(q), and 5(u) by simply cutting off the external scalar-field lines. [This corresponds to dividing the corresponding expressions by $(\lambda/N)^2$.] From Sec. III B we know that diagrams 5(c) and 5(j) approach a constant for $k \to 0$ and thus contribute to the constant in $\chi(k)$. The terms proportional to $k \ln k$ in diagrams 5(d), 5(f), 5(m), and 5(q) will modify the linear term in $\chi(k)$ by logarithmic corrections and hence contribute to the anomalous dimension γ_{ϕ^2}. The contributions of the relevant diagrams are

$$\begin{aligned}
\chi^{(c)}(k) &= \frac{8}{\pi^2}\frac{1}{\lambda} + O(k) , \\
\chi^{(d)}(k) &= \frac{64}{\pi^2}\frac{k}{\lambda^2}\ln\frac{8k}{\lambda} + O(k) , \\
\chi^{(f)}(k) &= \frac{64}{3\pi^2}\frac{k}{\lambda^2}\ln\frac{8k}{\lambda} + O(k) , \\
\chi^{(j)}(k) &= \frac{32}{\pi^2}\frac{\alpha}{\lambda^2} + O(k) , \\
\chi^{(m)}(k) &= -\frac{2^{10}}{3\pi^2}\frac{k}{\lambda^2}\ln\frac{16k}{\alpha} + O(k) , \\
\chi^{(q)}(k) &= \frac{2^{10}}{\pi^2}\frac{k}{\lambda^2}\ln\frac{16k}{\alpha} + O(k) .
\end{aligned} \tag{3.14}$$

The other diagrams [5(k) and 5(u)] approach zero like two powers of k as $k \to 0$. Putting these results together we obtain for $\chi(k)$ to this order

$$\chi^{(2)}(k) = \frac{N}{\lambda}\left[1 + \frac{8}{\pi^2 N}\left(1 + \frac{4\alpha}{\lambda}\right)\right] - \frac{N}{\lambda}\frac{8k}{\lambda}\left[1 - \frac{32}{3\pi^2 N}\ln\frac{8k}{\lambda} - \frac{256}{3\pi^2 N}\ln\frac{16k}{\alpha}\right] + \cdots . \tag{3.15}$$

We may now compare this with the scaling law Eq. (3.10). Writing the latter in the form

$$\chi(k) \sim \text{const} - k^{1+2\sigma} , \tag{3.16}$$

where now σ is of order $1/N$, we may expand for large N

$$\chi(k) \sim \text{const} - k(1 + 2\sigma \ln k + \cdots) . \tag{3.17}$$

Comparison with (3.15) then yields

$$\sigma = -\frac{48}{\pi^2 N} . \tag{3.18}$$

The anomalous dimension of $|\vec{\phi}|^2$ at the fixed point therefore is

$$\gamma_{\phi^2} = 1 - \frac{48}{\pi^2 N} . \tag{3.19}$$

E. Other critical indices

We conclude this section by summarizing the critical indices which can all be obtained from η and γ_{ϕ^2} by using the scaling relations given by the solution of the renormalization-group equations for small momenta. As shown, for example, in Ref. 8, one can generalize the scaling relations in Eqs. (2.13) and (3.7) to the case $T \neq T_c$ (massive theory). A general correlation function of n $\vec{\phi}$ fields and m $|\phi|^2$ fields,

$$\overline{\Gamma}^{(n,m)}(\vec{p}_1, \ldots, \vec{p}_n; \vec{q}_1, \ldots, \vec{q}_m; \overline{\alpha}(\Lambda), \overline{\lambda}(\Lambda); \Lambda),$$

in a neighborhood of the fixed point (α^*, λ^*) (i.e., either for small momenta or for small $T - T_c$) scales similar to[8]

$$\overline{\Gamma}^{(n,m)}(\kappa p_i; \kappa q_j; \ldots)$$
$$\sim \kappa^{D(n,m)+n\eta/2+m\gamma_{\phi^2}} f^{(n,m)}(\kappa\xi) \quad (3.20)$$
$$\sim \xi^{-D(n,m)-n\eta/2-m\gamma_{\phi^2}} F^{(n,m)}(\kappa\xi), \quad (3.21)$$

where

$$\xi \sim (T - T_c)^{-\nu} \quad (3.22)$$

is the $\vec{\phi}$-$\vec{\phi}$ correlation length. $D(n,m) = d - n(1 + d/2) - 2m$ is the canonical dimension of $\overline{\Gamma}^{(n,m)}$ and η and γ_{ϕ^2} are the anomalous dimensions of $\vec{\phi}$ and $|\vec{\phi}|^2$, respectively. $F^{(n,m)}$ and $f^{(n,m)}$ are functions of one argument $\kappa\xi$. For $\kappa \to 0$ $F^{(n,m)}$ is regular, and in this limit (3.21) describes the scaling behavior of $F^{(n,m)}$ at zero momentum, as $\xi \to \infty$ and the system approaches the critical surface. $f^{(n,m)}$ in turn is regular as $\xi \to \infty$, and in this limit (3.20) describes the scaling behavior of $F^{(n,m)}$ on the critical surface.

As an instructive example in the use of these scaling relations we now derive the relation between ν and γ_{ϕ^2}. We recall that $\overline{\Gamma}^{(0,2)}$ $(k=0)$ for $T \neq T_c$ is just the specific heat C_V, which is obtained from the free energy per volume Ω/V by differentiating twice with respect to the temperature (this corresponds to two $|\vec{\phi}|^2$ insertions into $\overline{\Gamma}^{(0,0)}$, the sum of all connected vacuum diagrams):

$$C_V = -\frac{\partial}{\partial T}\left(\frac{\partial}{\partial(1/T)}\frac{\Omega}{V}\right)$$
$$= T^2 \frac{\partial^2}{\partial T^2}\overline{\Gamma}^{(0,0)} + 2T\frac{\partial}{\partial T}\overline{\Gamma}^{(0,0)} . \quad (3.23)$$

From Eq. (3.21) we read off the leading scaling behavior

$$\overline{\Gamma}^{(0,0)}(k=0) = \frac{\ln Z}{V} = \frac{\Omega}{V} \sim \xi^{-d} . \quad (3.24)$$

Replacing T by ξ via Eq. (3.22) one finds the leading term

$$C_V \sim \xi^{2/\nu - d} \sim (T - T_c)^{\nu d - 2} . \quad (3.25)$$

Since $C_V \sim (T - T_c)^{-\alpha}$ defines the critical index α, we have

$$\alpha = 2 - \nu d . \quad (3.26)$$

On the other hand, comparing (3.25) with (3.21) for $(n, m) = (0, 2)$, we find

$$\frac{2}{\nu} - d = -d + 4 - 2\gamma_{\phi^2}$$

or

$$\nu = \frac{1}{2 - \gamma_{\phi^2}} . \quad (3.27)$$

In a similar way one derives many other scaling relations.[8] We finally summarize our results for the critical indices in $d = 3$ dimensions:

$$\eta = -\frac{20}{\pi^2 N} + O\left(\frac{1}{N^2}\right),$$

$$\gamma_{\phi^2} = 1 - \frac{48}{\pi^2 N} + O\left(\frac{1}{N^2}\right),$$

$$\nu = \frac{1}{2 - \gamma_{\phi^2}} = 1 - \frac{48}{\pi^2 N} + O\left(\frac{1}{N^2}\right),$$

$$\alpha = 2 - \nu d = -1 + \frac{144}{\pi^2 N} + O\left(\frac{1}{N^2}\right),$$

$$\rho = d - 4 + 2\gamma_{\phi^2} = 1 - \frac{96}{\pi^2 N} + O\left(\frac{1}{N^2}\right),$$

$$\beta = \nu\left(\frac{d}{2} - 1 + \frac{\eta}{2}\right) = \frac{1}{2} - \frac{34}{\pi^2 N} + O\left(\frac{1}{N^2}\right),$$

$$\delta = \frac{d + 2 - \eta}{d - 2 + \eta} = 5 - \frac{40}{\pi^2 N} + O\left(\frac{1}{N^2}\right).$$

We recall that ρ describes the small-momentum behavior of $\chi(k)$ as defined in Eq. (3.10).[12] β describes the vanishing of the order parameter $\langle\vec{\phi}\rangle$ as T approaches T_c from below (i.e., as the square of the scalar mass changes from a negative to a positive value),

$$\langle\vec{\phi}\rangle \sim (T - T_c)^{\beta} .$$

Finally, δ governs the order parameter as a function of an external static scalar field $h\vec{\phi}_{\rm ext}(k=0)$,

$$\langle\vec{\phi}\rangle \sim h^{\delta} .$$

IV. EFFECT OF A ϕ^6 INTERACTION

The Lagrangian (2.1) describes only one of a class of Ginzburg-Landau mean-field theories. If the addition of other interactions allowed by the symmetries does not alter the infrared behavior, then the original Lagrangian is sufficient to compute the critical indices. Other interaction terms will be higher-dimension operators such as $(\vec{\phi}^\dagger \cdot \vec{\phi})^3$, $(\vec{\phi}^\dagger \cdot \vec{\phi})^4$, etc., and in any number of dimensions above three, these are nonrenormalizable with coupling constants having negative mass dimensions. They should therefore be irrelevant[9] in the infrared and should indeed not contribute to critical indices.

In three dimensions, however, the operator

$$\mathcal{L}_6 = g(\vec{\phi}^\dagger \cdot \vec{\phi})^3 \qquad (4.1)$$

is precisely renormalizable and it is perhaps not so clear whether it influences the infrared structure of the theory. This section will be devoted to answering this question in the negative. We have been unable to find any previous discussion of this point directly in three dimensions and in the context of the $1/N$ expansion.

It is important to emphasize that we are talking here about the effect of $\mathcal{L}_6$ *in the presence of the* $(\lambda/N)(\vec{\phi}^\dagger \cdot \vec{\phi})^2$ *interaction.* The theory without the $(\vec{\phi}^\dagger \cdot \vec{\phi})^2$ interaction has been analyzed at some length in connection with the behavior of statistical systems at tricritical points.[13] These occur if, by varying the physical parameters of a system, it is possible to make the physical ϕ mass and the $(\vec{\phi}^\dagger \cdot \vec{\phi})^2$ interaction simultaneously vanish with $g > 0$ in $\mathcal{L}_6$. A recent renormalization-group analysis of this problem has been given by Pisarski.[14]

The manner in which $\mathcal{L}_6$ is treated depends on the order of magnitude of g. We shall assume that it is small enough to justify the use of perturbation theory. To see just how small that is, we first recall that for the $(\vec{\phi}^\dagger \cdot \vec{\phi})^2$ interaction, the coupling constant was taken to be of order $1/N$ in order to make use of a $1/N$ expansion. An analogous approach to the $g(\vec{\phi}^\dagger \cdot \vec{\phi})^3$ interaction could be contemplated by making g of order $1/N^{3/2}$. Then, for example, the class of graphs shown in Fig. 6 will all be of the same order. However, the summation of all these graphs seems to be a rather difficult chore and so we retreat to perturbation theory in g by assuming that $g < 1/N^{3/2}$.

The way in which the $\mathcal{L}_6$ interaction affects the infrared structure of the theory depends on the behavior of the running coupling constant $\bar{g}(k)$ or, equivalently, its β function $\beta_g(\bar{g})$. The dominant contribution to $\bar{g}(k)$ is due to the corrections shown in Fig. 7 which arise from the $(\vec{\phi}^\dagger \cdot \vec{\phi})^2$ interaction. These leading corrections in the $1/N$ expansion determine a $\bar{g}(k)$ when evaluated at a symmetric point as shown in the figure. One finds

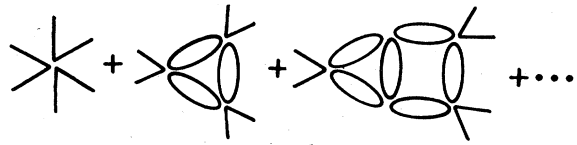

FIG. 6. The leading corrections to the six-point coupling $g(\vec{\phi}^\dagger \cdot \vec{\phi})^3$ in a $1/N$ expansion with g^2N^3 fixed.

$$\bar{g}(k) = g\left(1 - \frac{\lambda/8k}{1+\lambda/8k}\right)^3 \xrightarrow[k \ll \lambda]{} g\left(\frac{8k}{\lambda}\right)^3 + \cdots . \qquad (4.2)$$

Thus $\bar{g}(k)$ vanishes like k^3 as $k \to 0$. A β function can be defined from $\bar{g}(k)$ in analogy to Eq. (2.7), with the result

$$\beta_g(\bar{g}, \bar{\lambda}) = 3\bar{g}\bar{\lambda}/8 . \qquad (4.3)$$

This function governs the evolution of $\bar{g}(k)$. To reproduce the behavior (4.2) as $k \to 0$, we first recall that the β function for $\bar{\lambda}$ Eq. (2.8) depends only on $\bar{\lambda}$ itself to this order. Thus the fixed-point behavior $\bar{\lambda}(k) \to 8$ as $k \to 0$, can be used to first approximation on the right-hand side of Eq. (4.3). Then the renormalization-group equation for $\bar{g}$ can be integrated to recover the result (4.2).

To summarize, in three-dimensional coupling-constant space, to leading order in $1/N$, an infrared-stable fixed point is located at

$$(\alpha^*, \lambda^*, g^*) = (16, 8, 0) . \qquad (4.4)$$

Even if the α and λ couplings were not present in the theory, the origin would be an infrared-stable fixed point for $\bar{g}$. The β function (4.3) contains an additional term of order $\bar{g}^2$ (and higher-order terms as well), and in the absence of the $\bar{\lambda}\bar{g}$ term, it would drive $\bar{g}(k)$ to zero like $1/\ln k$ as $k \to 0$. However, in the model we are considering, the $\bar{\lambda}\bar{g}$ term (4.3) takes over at sufficiently small g and $\bar{g}(k)$ goes to zero like k^3, much more rapidly than an inverse logarithm.

It is because of this rapid approach of $\bar{g}(k)$ to zero that the $\mathcal{L}_6$ interaction has no impact on the computation of critical indices. As an example, consider the correlation function $G(k)$ (2.12). There are diagrams of order $\bar{g}, \bar{g}^2, \bar{g}^3$, etc. After the mass subtraction at $k=0$, it is easy to see that

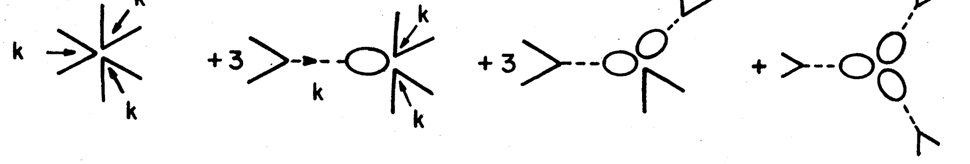

FIG. 7. Leading corrections to the six-point coupling due to the presence of a $(\vec{\phi} \cdot \vec{\phi})^2$ interaction.

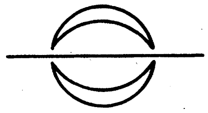

FIG. 8. One of the self-energy corrections due to the six-point coupling.

if the six-point vertices were constants, then the contribution of order $\bar{g}^2$ (see Fig. 8) would behave like $k^2 \ln k$ and therefore contribute to the anomalous dimension η.[15] However, since these vertices vanish like three powers of the momentum, any $G(k)$ graph containing them will vanish like k^2 (not $k^2 \ln k$) as $k \to 0$. The same is true for other correlation functions and thus $\mathcal{L}_6$ is irrelevant for the computation of critical indices.

There is one more remark worth making with regard to the six-point vertex. Even if $\mathcal{L}_6$ is not introduced, an effective six-point vertex will be induced by the $(\vec{\phi}^\dagger \cdot \vec{\phi})^2$ coupling. The leading contribution in the $1/N$ expansion is shown in Fig. 9. The amplitude, on general grounds a function of the three magnitudes k_1, k_2, k_3, is computed to be

$$g_I(k_1, k_2, k_3) = \frac{64}{N^2} \prod_{i=1}^{3} \left(\frac{\lambda}{\lambda + 8k_i} \right). \tag{4.5}$$

This expression has the finite zero-momentum limit $64/N^2$.

The first important point to be made about this induced six-point vertex is that, unlike the fundamental six-point coupling g, it will not be corrected by multiplicative factors which vanish at zero momentum. The kind of corrections pictured in Fig. 7 are already included in $g_I(k_1, k_2, k_3)$. That being the case, it might be worried that higher-order corrections in g_I itself might cause g_I to vanish logarithmically at zero momentum just as a fundamental coupling would in a pure $g(\vec{\phi}^\dagger \cdot \vec{\phi})^3$ theory. If that were to happen, logarithmic corrections to scaling might be anticipated.[13,14]

However, it does not happen. Corrections to g_I must be computed not as an expansion in g_I itself, but as an expansion in $1/N$. When that is done, it can be shown that there are no logarithmic corrections in the zero-momentum limit. The induced six-point coupling $64/N^2$ does not run in this limit; it merely picks up corrections of order $1/N^3, 1/N^4$, etc. The details of this argument are similar to those already used to show that $\bar{\lambda}(k)$ continues to exhibit fixed-point behavior to higher order in the $1/N$ expansion. One class of corrections to g_I is obtained by replacing one of the three dotted lines in Fig. 9 by the next-order corrections shown in Fig. 5. Since we have already shown that all constant and logarithmic terms cancel within the diagrams of Fig. 5, this replacement only changes g_I by a constant of order $1/N^3$. The only other types of next-order corrections to g_I are shown in Fig. 10. Each of these diagrams is logarithmically divergent as the momenta flowing through the diagrams vanish. However, the same cancellation mechanism which worked for the corrections to the four-point coupling leads to a cancellation of these logarithmic terms in Fig. 10, again leaving a constant $1/N^3$ contribution to g_I at zero momentum.

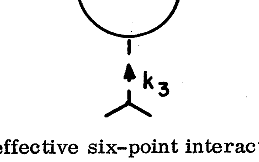

FIG. 9. The effective six-point interaction induced by the $(\vec{\phi}^\dagger \cdot \vec{\phi})^2$ interaction (to leading order in $1/N$).

Assuming that these cancellations can be established to higher orders in $1/N$, the existence of g_I may be happily ignored for the computation of critical indices. Its effects are of course automatically included in the higher-order corrections due to the four-point coupling. However, it plays an important role in the effective potential of the theory. This will be described briefly in Sec. V, and more completely in a separate publication.

V. CONCLUSIONS

The infrared structure of charged, O(N)-invariant theories in three dimensions has been analyzed making use of a $1/N$ expansion. We have considered only the zero-mass case, corresponding to being at the critical temperature in the related statistical-mechanics problem, and we have studied the infrared behavior of the correlation functions of the theory. The expansion parameter $1/N$ appears multiplied by expressions which become singular at zero momentum but these singular terms were shown to cancel. The infrared behavior is then governed by an infrared-stable fixed point of the renormalization group. The anomalous dimensions associated with this fixed point can be computed and we have reported the results of some of these computations. The proof that

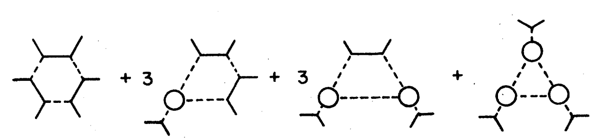

FIG. 10. Some $1/N^3$ corrections to the induced $(\vec{\phi}^\dagger \cdot \vec{\phi})^3$ interactions.

singular terms cancel and that the running coupling constants track into an infrared-stable fixed point has been given through second order in the $1/N$ expansion. We see nothing that would prevent extending the proof to higher orders.

This entire investigation began in an effort to determine the infrared behavior of the gauge boson propagator. The result, given by Eq. (3.1) through second order in $1/N$, corresponds to an infrared-stable fixed point (3.3) for the running gauge coupling constant. In this case, the necessary cancellations are ensured by the Ward identity and they can be expected to take place to all orders. One would then conclude that $D_{ij}(\vec{k}) \propto 1/k$ as $k \to 0$, with the coefficient of proportionality computable order by order in the $1/N$ expansion.

The effect of adding higher-dimension operators to the interaction Lagrangian has been discussed. The most interesting is the renormalizable interaction $\mathcal{L}_6$ (Eq. 4.1), and it has been shown to be irrelevant for the computation of infrared behavior. The anomalous dimensions which describe the theory in this limit are determined completely by the dominant, lower-dimension interactions appearing in the Lagrangian (2.1). Finally, we observed that even if the six-point coupling $\mathcal{L}_6$ is not included in the theory, an effective six-point vertex is induced by the lower-dimension interactions.

We now offer a few qualitative comments on the relation of our results to the ϵ expansion. Whereas in the ϵ expansion critical indices are obtained exactly in the number N of scalar-field components, the $1/N$ expansion yields results which are exact in the number of dimensions. We may use our results to check the convergence of the ϵ expansion in three dimensions (at $\epsilon = 1$). To this end one can expand Wilson's results[9] for the pure ϕ^4 theory additionally in $1/N$ and compare the resulting coefficient of the $1/N$ term (which is a series in ϵ) with our exact coefficient in the limit of vanishing gauge coupling. It is then seen that contributions at least to order ϵ^3 have to be included to get a 20% agreement between the coefficients at $\epsilon = 1$. Therefore, at $\epsilon = 1$ the quantitative agreement of the critical indices in both expansions will generally be bad in low orders. In this sense both approximations are complementary, and in order to get good experimental predictions for systems with moderate values of N in three dimensions, one has to perform the calculations to rather high orders.[16]

However, comparison of both approximations can yield some useful qualitative checks already at low orders. For example, in Sec. II we saw that η (the anomalous dimension of $\vec{\phi}$) is positive for a pure $(\vec{\phi}^\dagger \cdot \vec{\phi})^2$ theory, but after inclusion of the gauge coupling, it is overwhelmed by an additional, larger negative contribution. This is consistent with the negative sign of η obtained in the ϵ expansion in Ref. 17 as opposed to Wilson's positive η for the pure $(\vec{\phi}^\dagger \cdot \vec{\phi})^2$ theory, which can be found in Ref. 9.

The fixed-point structure which we have established order by order in the $1/N$ expansion is evidence that the zero-mass theory defines a surface of second-order phase transitons. In the zero-mass limit, the scalar potential part of $\mathcal{L}$ (2.1) is simply $\lambda/2N|\vec{\phi}|^4$, whose minimum is at the origin $\langle\vec{\phi}\rangle = 0$. The second-order character of the phase transition in fact depends on the ground state moving to the origin at $T = T_c$, but it is not sufficient to examine the Lagrangian to settle this question. The effective potential $V_{\rm eff}$ of the theory should be computed in the $1/N$ expansion and it should then be seen whether its minimum lies at $\langle\vec{\phi}\rangle = 0$.

The effective potential of three-dimensional O(N) theories will be described in a separate paper.[18] We summarize here some of the results of that paper which lead to the conclusion that $V_{\rm eff}$ has only one minimum, at $\langle\vec{\phi}\rangle = 0$.

1. The effective potential can be computed order by order in the $1/N$ expansion. This far-from-obvious result involves the cancellation of a host of singular terms and is closely tied to the behavior of the running coupling constants established in Secs. II–IV.

2. There is no $|\vec{\phi}|^4$ term in the effective potential. Its coefficient vanishes at each order in the $1/N$ expansion as long as the running coupling constant approaches a fixed point. The leading-order graphs are just those of Fig. 2.

3. The induced six-point coupling of Fig. 9 shows up as a positive $|\vec{\phi}|^6$ term. It dominates the potential for small $\vec{\phi}$.

4. The full result for the effective potential is a Taylor series in $|\vec{\phi}|^2/N\lambda$ with finite coefficients calculable to arbitrary accuracy in the $1/N$ expansion. The leading-order terms give

$$V_{\rm eff}(\phi) = \frac{16\pi^2}{3N^2}\,|\vec{\phi}|^6\left[1 - \frac{24\pi^2}{N\lambda}\,|\vec{\phi}|^2 + 4\left(\frac{24\pi^2}{N\lambda}\right)^2 |\vec{\phi}|^4 + \cdots\right]. \quad (5.1)$$

These terms can be summed to a closed form[18] and it can be shown that the only minimum of $V_{\rm eff}$ is at $\langle\vec{\phi}\rangle = 0$.

The full structure of the effective potential will be described in Ref. 18.

A change to a first-order phase transition (reflected by a spontaneous development of a nonvanishing order parameter $\langle\vec{\phi}\rangle \neq 0$) is consistent with the structure (5.1) only if the series in $1/N$ for each coefficient sums up to a value strongly different from the leading result. In particular, a negative coefficient for the $|\vec{\phi}|^6$ term would induce a nonvanishing vacuum expectation value for $\vec{\phi}$. However, this would very likely signal the breakdown of the $1/N$ expansion, and therefore one could not determine the critical number N where this occurs, within the $1/N$ expansion. On the other hand, such a change from a second- to a first-order phase transition has been found in an ϵ expansion for this model[17]; for $d=4-\epsilon$, the change occurs at $N=365.9/2$.

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[1]D. A. Kirzhnits and A. D. Linde, Phys. Lett. 42B, 471 (1972); C. Bernard, Phys. Rev. D 9, 3312 (1974); L. Dolan and R. Jackiw, *ibid.* 9, 3320 (1974); S. Weinberg, *ibid.* 9, 3357 (1974).

[2]S. Weinberg, in *Understanding the Fundamental Constituents of Matter*, proceedings of the 1976 International School of Subnuclear Physics, Erice, edited by A. Zichichi (Plenum, New York, 1977); P. Ginsparg, Nucl. Phys. B170, 388 (1980).

[3]T. Appelquist and R. D. Pisarski, Phys. Rev. D 23, 2305 (1981).

[4]R. Jackiw and S. Templeton, Phys. Rev. D 23, 2291 (1981).

[5]G. 't Hooft, Acta Phys. Austr. Suppl. XXII, 531 (1980).

[6]See, for example, S.-K. Ma, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1976).

[7]S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973); S. Coleman, R. Jackiw, and H. D. Politzer, *ibid.* 10, 2491 (1974).

[8]See, e.g., S. Weinberg, Ref. 2.

[9]K. G. Wilson and J. Kogut, Phys. Rep. 12C, 75 (1974); S.-K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, New York, 1976).

[10]Much of this work has been carried out by S. Nadkarni and will be described in a forthcoming Yale report.

[11]S.-K. Ma, Phys. Rev. Lett. 29, 1311 (1972); Phys. Rev. A 7, 2172 (1973).

[12]In order to avoid confusion with the four-point coupling we have called this exponent ρ instead of λ, the notation used in Ref. 11.

[13]E. K. Riedel and F. Wegner, Phys. Rev. Lett. 29, 349 (1972); F. Wegner and E. K. Riedel, Phys. Rev. B 7, 248 (1973); C. A. Aragão de Carvalho, Nucl. Phys. B119, 401 (1977).

[14]R. Pisarski, private communication.

[15]To be more precise, without the $(\vec{\phi}^\dagger\cdot\vec{\phi})^2$ interaction, $\bar{g}(k)\sim 1/\ln k$ as $k\to 0$. Using this as the effective coupling constant, $G(k)$ contains contributions proportional to $k^2\ln\ln k$, signaling the buildup of a power of a logarithm. This would lead to logarithmic corrections to scaling behavior like those discussed in Refs. 13 and 14.

[16]For a calculation of η to order $1/N^2$, for example, see R. Abe, Prog. Theor. Phys. 49, 1877 (1973).

[17]B. I. Halperin, T. C. Lubensky, and S.-K. Ma, Phys. Rev. Lett. 32, 292 (1974).

[18]T. Appelquist and U. Heinz, in preparation.