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Structure of gauge theories with spontaneous symmetry breaking

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It is shown that the state space corresponding to a spontaneously broken global (position-independent) symmetry can be described in two complementary ways. The first is a Hilbert bundle and the second a Hilbert space. The Hilbert space, which is obtained from the bundle via a direct integral, is reducible, i.e., the vacuum state is degenerate. The degeneracy is labeled by a certain integer-valued parameter. It is noted that the above structure continues to be true for a suitably restricted broken local (position-dependent) gauge symmetry in the unitary gauge. The significance of the vacuum degeneracy is discussed.

I. INTRODUCTION

We wish to consider a local gauge theory with a Higgs-type symmetry breaking triggered by the nonvanishing vacuum expectation value of some scalar field. We expect such a theory to share many common characteristics with nonrelativistic many-body theories with "an order parameter," where the phase is broken. To bring out the similarity between the two cases, we consider the description of the underlying state space of a gauge theory. Now, the state space can look nice only in the unitary gauge—the gauge in which the unwanted components of the scalar field have been eliminated and, consequently, where the Yang-Mills fields are no longer massless. In contrast, in renormalizable gauges we have to be prepared to face the indefinite metric. So, we will confine our attention to the unitary gauge. For the same reason, we will restrict ourselves to a "completely broken" gauge theory, where no subsymmetry of the original gauge symmetry remains unbroken. In this way we will avoid the massless Yang-Mills quanta of the unbroken symmetry.

Matters are particularly simple in a "simple" gauge theory, as defined by Weinberg.[1] In such a theory the remaining component of the scalar field in the unitary gauge can be made to align in the same direction at every point of space, the common direction of the field being that of its vacuum expectation value. But the precise choice for this common direction is without any intrinsic significance and thus we have the residual freedom of subjecting the scalar field to global, position-independent symmetry transformations. Even when the gauge theory is not simple, it seems that one always has the freedom of applying global symmetry transformations as long as one does not impose special restrictions, such as that the gauge transformations reduce to identity at spatial infinity. This residual global symmetry transformation is broken by the vacuum expectation value of the scalar field. Thus the problem of understanding the structure of a broken local *gauge* symmetry reduces to the (simpler) problem of understanding the structure of a broken *global* symmetry.

The situation noted above appears to be quite reasonable. Because of the translational invariance, the vacuum expectation value of the scalar field cannot respond to local symmetry transformations but it can, and does, respond to global symmetry transformations. Also, while discussing the state space one usually smears the local fields (with suitable test functions), and these smeared fields, again, do not respond to local symmetry transformations. Thus, one does not know how even to raise the question as to what effect a local gauge transformation has on the state space. In contrast, the action of the global symmetry transformation on the states is well defined.

Along the lines indicated in the foregoing, it will then be shown in the sequel that the underlying state space of the theory can be described in two distinct, but mathematically equivalent, ways. The first way involves a *Hilbert bundle* and the second a *Hilbert space*. A Hilbert bundle is a fiber bundle whose fiber is a Hilbert space and whose "group of the bundle" is the group of all the unitary operators in the Hilbert space. The base space of our Hilbert bundle is the group manifold of the global group G that accompanies the local gauge symmetry (G is the structure group of the space-time-based principal fiber bundle that characterizes[2] the local gauge symmetry). The Hilbert space of the second description is what we call the *physical* Hilbert space. We show that the vacuum state of the physical Hilbert space is degenerate, and the degeneracy can be labeled by a certain integer-valued parameter possessing the following topological significance. The parameter is related, in a prescribed manner, to the mapping-degree of mappings of the group manifold of G to itself. On the other hand, the vacuum state of the Hilbert bundle is unique in the technical sense that each fiber carries a unique Poincaré-invariant ground state.[3] Thus, the mathematical multiplicity (labeled by the points of the base space) of the vacuum in the bundle description does not signify a physical degeneracy. This way of looking at the meaning of vacuum degeneracy is completely equivalent to the traditional way, which is within the context of the following theorem valid for a local, relativistic field theory[4]: any one of the three properties (I) uniqueness of the ground state, (II) irreducibility of the field, (III) existence of the linked cluster decomposition for the ground-state expectation values, implies the two others. The theorem also happens to be valid for a class of nonrelativistic theories that we shall review presently (next section).

This paper is organized as follows. In the next section we present a brief summary of those aspects of nonrelativistic theories with long-range order that most concern us here. In Sec. III, the breakdown of a global symmetry group is analyzed. The picture that emerges from this analysis has implications for local gauge theories that are then spelled out in Sec. IV. The physical significance of the vacuum degeneracy is also discussed there. In Sec. V we make concluding remarks.

II. NONRELATIVISTIC THEORIES

Nonrelativistic many-body theories that display a breakdown of the (constant) phase transformation have been known for quite some time. Below is a brief summary of those features of the theory that are most relevant for our purpose.

The common structure of these theories emerged from the pioneering work of Haag[5] (on the BCS model) and of Araki and Woods[6] (on the Bose gas), and is as follows. The ground state is such that it can be labeled either by a continuous parameter α, with $0 \leq \alpha < 2\pi$, or by an integer-valued parameter n. The state space that corresponds to these two descriptions, we denote by $B(\alpha)$ and H, respectively. Then, H is a Hilbert space which is the union of all H_n, where H_n is the Hilbert space that results from the nth ground state. The field is represented (the algebra generated by the smeared fields is represented) irreducibly in $B(\alpha)$ but reducibly in H, the linked-cluster expansion is valid in $B(\alpha)$ but not in H, the ground state is unique in $B(\alpha)$ but degenerate in H. Thus $B(\alpha)$ possesses nice features that are not shared by H. Nevertheless, H is the *physical* Hilbert space; unphysical, phase-variant quantities have zero ground-state expectation value in H [and nonzero values in $B(\alpha)$]. The physical significance of the ground-state degeneracy (in H) was explained by Haag[5] and by Araki and Woods[6] as follows. Ground states that differ from each other by the presence of *a finite* number of zero-momentum bosons represent the same physical state; since the ground state, in any event, contains an infinite number (in the infinite-volume limit) of *zero-momentum* bosons. Notice that for superconductivity the condensate is one of zero-momentum Cooper pairs.

In a recent reexamination of Haag's paper,[5] it was pointed out[7] that the space $B(\alpha)$ is a Hilbert bundle based on the circle and that the ground-state index n has the "topological" significance that it denotes an element of the fundamental group of the circle. Exactly the same analysis can be carried out for the Araki Woods paper and leading to the same conclusion. We make a final remark. As explained in Ref. 7, the phase transformation is implemented in $B(\alpha)$ as a bundle mapping which acts essentially as a translation of the base space, and not as a unitary operator on the fiber. This is a natural way of implementing a "broken symmetry," as has been explained by Borchers and Sen[8] in a different context.

In the next section, we will analyze the breakdown of a global symmetry in such a manner as to bring out, to the fullest degree possible, the analogy with the structure of nonrelativistic theories, summarized above.

III. BREAKDOWN OF A GLOBAL SYMMETRY

Relativistic local field theories with spontaneous symmetry breakdown were considered by many authors in the past. Of these, the model that is closest to the spirit of the Haag-Araki-Woods theory is the one due to Lopuszanski and Reeh,[9] based on a one-parameter group of symmetries. Here again the existence of two complementary descriptions provided by the reducible Hilbert space and the irreducible Hilbert bundle (although the authors do not explicitly mention the word bundle, it is clear that their irreducible space *is* a Hilbert bundle) is noted. Here, we wish to analyze the structure of the underlying state space of a theory with a broken non-Abelian symmetry. We focus our attention exclusively on the intuitive, geometrical aspects of the problem, as we have done in Ref. 7 for the BCS model. The problem is interesting in its own right, quite irrespective of its possible relevance for local gauge theories.

We consider a Lorentz-invariant, local field theory possessing a global symmetry group G. That is, G is a group of automorphisms of the algebraic structure generated by the fields. We require G to commute with the Poincaré group; in other words, G is an internal symmetry. It is sufficient to restrict G to an SU(n) group and we will do so; although our results remain valid for any compact, Lie group. Let the field theory have a scalar field $\Phi(x)$ transforming, in general, as a reducible representation of G such that the vacuum expectation value $\langle 0|\Phi|0\rangle$ is nonvanishing and is such that the group is completely broken. Focusing our attention on the vacuum expectation value, we will find out what structure the state space should have, provided it exists. We begin our discussion with a simple model, where G is the group SU(2).

A. A simple model based on SU(2)

Let G be the group SU(2). The vacuum expectation value of $\Phi(x)$ can be characterized by a set of four real numbers $(\bar{a}_1, \bar{a}_2, \bar{b}_1, \bar{b}_2)$ satisfying the constraint

$$\bar{a}_1^{\,2}+\bar{a}_2^{\,2}+\bar{b}_1^{\,2}+\bar{b}_2^{\,2}=N, \tag{1a}$$

where N is some fixed real number. We set $\bar{a}_i = a_i\sqrt{N}$ and $\bar{b}_i = b_i\sqrt{N}$ $(i=1,2)$ so that

$$a_1^{\,2}+a_2^{\,2}+b_1^{\,2}+b_2^{\,2}=1\,. \tag{1b}$$

Thus the orbit of the vacuum expectation value is a three-sphere. It is, in fact, the group manifold of SU(2). We denote by q the collection (a_1, a_2, b_1, b_2) and also label the vacuum state by q. Thus

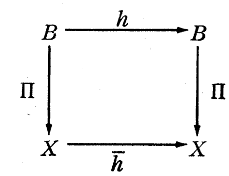
$$\langle 0|\Phi(x)|0\rangle = \langle 0|\Phi(0)|0\rangle \equiv \langle q|\Phi|q\rangle = Nq\,. \tag{2}$$

We may look upon q as a number; it is, in fact, a unit quaternion. We may go further and obtain an explicit picture of q by introducing a 2×2 matrix (over complex numbers) basis for the quaternionic units. Then q looks like

$$q=\begin{pmatrix}\alpha & \beta\\ -\beta^* & \alpha^*\end{pmatrix},\quad \alpha=a_1+ia_2,\ \ \beta=b_1+ib_2 \tag{3}$$

where α^* is the complex conjugate of α. The above matrix is unitary with determinant equal to one.

We consider the structure of the state space that corresponds to the above description of the vacuum. The Hilbert space that results from the vacuum $|q\rangle$ as a result of the application of "creation operators" (suitable polynomials in the smeared fields) we call H_q. Thus we have a family of Hilbert spaces labeled by the continuous parameter q. All H_q's are exact copies of each other and thus of some $\bar{H}$. Linear combinations and inner products are *not* defined between states that sit over distinct values of q. It is clear that these properties define a Hilbert bundle; $\bar{H}$ is the (abstract) fiber, H_q the fiber over q, and the base space is the manifold of q which is a three-sphere. We denote the bundle by $B(q)$. We note that each vacuum $|q\rangle$ is Poincaré invariant (G commutes with the Poincaré group), and each fiber H_q has a unique vacuum. We consider the action of G on $B(q)$. Clearly, G acts as a left translation on the base space; let $\bar{h}(g)$, $g\in G$, denote the corresponding base map. Then the action of g on $B(q)$ corresponds to a well-defined bundle map $h(g)$ such that the following diagram



is commutative.[10] Here Π denotes the projection from the bundle to the base space X. We can also write down an explicit bundle mapping formula for $h(g)$, along the lines of Ref. 11, but we will not do it here.

The foregoing description corresponds to a perfectly legal conventional field theory such as described, for instance, by Reeh.[12] The precise choice for the base point q is without any physical consequence and one can thus work with an arbitrarily chosen fixed q and the corresponding fiber H_q and do field theory (for instance, a Wightman field theory) there. Thus the bundle structure is physically superfluous, but its existence has to be kept in mind to see the underlying mathematical structure. We ask: Can we now construct a physical Hilbert *space* in which the symmetry breaking is no longer apparent? To do this, we have to "wash out" the dependence on the group; that is, we have to do an integration over X. *This direct integral* is obtained as follows.

Now, our Hilbert bundle has the property of being equivalent, in the group of the bundle, to a product and this follows from a very general result stated by Borchers and Sen,[8] which is, that *any Hilbert bundle based on a paracompact base space is equivalent to a product.* On the other hand, the group manifold of any Lie group is known to possess the property of being paracompact.[13] Thus our conclusion follows. Since the bundle $B(q)$ is a product, we are assured of the existence of (continuous, global) cross sections of the bundle. A point b of B can written as a pair (q,f), where $f\in\bar{H}$ and $q\in S^3$ (S^3 is the three-sphere). Then $(q,f)\to f_q$ defines a cross section of the Hilbert bundle. Let f_q and g_q be cross sections and let $(f_q,g_q)=(f,g)_q$ denote their inner product in H_q. Define now a new inner product [,] by the rule

$$[f,g]=\int_{S^3}(f,g)_q\,d\mu(q)\,, \tag{4}$$

where $d\mu(q)$ is the normalized measure on S^3 [the Haar measure of SU(2)], and the integration is carried out over the entire group manifold. It now follows from the general theory that with respect to the inner product [,] the linear space of the cross sections gets endowed with the structure of a Hilbert space.[14] We call this Hilbert space H. Let $|\Omega\rangle$ denote the vacuum state constructed from $|q\rangle$ via direct integration. Then, we obtain from (4)

$$\langle\Omega|\Omega\rangle \equiv [\Omega,\Omega] = \int_{S^3} \langle q|q\rangle d\mu(q) = \int_{S^3} d\mu(q) = 1, \qquad (5)$$

where we have used the fact that $|q\rangle$ is normalized. Thus $|\Omega\rangle$ is normalized also. We next derive from (2) and (4) that

$$\langle\Omega|\Phi|\Omega\rangle = N\int_{S^3} q\, d\mu(q) = 0. \qquad (6)$$

Although the above result follows from general theorems, it is instructive to give a direct proof. For this purpose we use the picture of q given by Eq. (3) and take the following form for the measure[15]:

$$d\mu = \frac{dt\, d\phi\, d\psi}{4\pi^2}, \qquad (7a)$$

where

$$t = \alpha\alpha^*, \quad \alpha = \sqrt{t}\, e^{i\phi}, \quad \beta = (1-t)^{1/2} e^{i\psi}, \qquad (7b)$$

to conclude that

$$\int_{S^3} \alpha\, d\mu = \int_{S^3} \beta\, d\mu = 0. \qquad (8)$$

Thus Eq. (6) stands proved. We may note, at this stage, that the passage from the q vacuum to the Ω vacuum corresponds to the usual procedure of redefining the "physical field" of $\Phi(x)$ by subtracting from $\Phi(x)$ its vacuum expectation value. The real significance of relation (6) lies elsewhere and can be seen as follows. Let us assume *provisionally* that there exists an operator U with the following properties: (1) U is unitary, (2) U commutes with all the field variables and thus also with the elements of the Poincaré group, and (3) U satisfies the relations

$$\langle q|U|q\rangle = \frac{1}{N}\langle q|\Phi|q\rangle = q, \qquad \langle\Omega|U|\Omega\rangle = 0. \qquad (9)$$

Then, we interpret our relation (6) to signify that the state $|\Omega_1\rangle$ in H defined by

$$|\Omega_1\rangle = U|\Omega\rangle \qquad (10)$$

is orthogonal to the state $|\Omega\rangle$; that is,

$$\langle\Omega|\Omega_1\rangle = 0. \qquad (11)$$

Proceeding inductively and using unitarity of U, we easily see that we have the string of vacuum states (in H)

$$|\Omega_n\rangle = U^n|\Omega\rangle, \quad n = 0, \pm1, \pm2, \ldots \qquad (12)$$

obtained by the repeated application of U to $|\Omega\rangle$. All these vacuums are Poincaré invariant and represent one and the same physical state. The Hilbert space that corresponds to $|\Omega_n\rangle$ we call H_n. Thus H is the big Hilbert space which is the union of all H_n's. To see the significance of the integer n, let us look at the picture of U in the fiber H_q [Eq. (9)]. The mapping $q \to q^n$ maps S^3 to itself and all mappings $S^3 \to S^3$ are known to fall into *homotopy classes* characterized by the mapping degree n. In fact, n denotes an element of the Hurewicz group $\Pi_3(G)$, with $G = \mathrm{SU}(2)$. To conclude this part of our discussion, we have to exhibit the existence of the operator U. It is given by

$$U = \frac{1}{N}\lim_{V\to\infty}\frac{1}{V}\int_V \Phi(x)d^3x, \qquad (13)$$

where V is the three-dimensional volume. If our theory is quantized according to the canonical commutation relations then the above U commutes with the field variables $\Phi(x)$, $\dot{\Phi}(x)$ and also with all fields other than $\Phi(x)$ that might be present. We consider the representation of U in the bundle $B(q)$. Since for each fiber H_q there exists a unique vacuum state, we can appeal to the general theorem of Ref. 4 to conclude that the field is irreducibly represented in H_q and therefore U must be a c number (multiple of the identity operator in H_q). In other words, U is equal to its vacuum expectation value

$$U = \langle q|U|q\rangle = q. \qquad (14)$$

Notice that in the second step in the above, use has been made of Eqs. (2) and (13) and the translational invariance of the vacuum. It now follows rapidly that

$$\langle q|Q'U^\dagger UQ|q\rangle = \langle q|Q'Q|q\rangle, \qquad (15)$$

where Q' and Q are arbitrary polynomials in the fields. We are now in a position to show that U is a nontrivial unitary operator in H. Indeed, from (4) and (15) it follows that

$$\langle\Omega|Q'U^\dagger UQ|\Omega\rangle = \int_{S^3}\langle q|Q'U^\dagger UQ|q\rangle d\mu(q) = \langle\Omega|Q'Q|\Omega\rangle. \qquad (16a)$$

In an exactly similar way

$$\langle\Omega|Q'UU^\dagger Q|\Omega\rangle = \langle\Omega|Q'Q|\Omega\rangle. \qquad (16b)$$

Thus U is a unitary operator. Since it is not a c number in H, it follows that the fields are represented reducibly in H. This concludes our discussion of the SU(2)-based model.

B. The general case

We want to show that the structure uncovered in the foregoing example remains valid, with appropriate modifications, for the general case where the symmetry group G is some SU(m). The general case is of intrinsic interest. Equation (2) of the foregoing example has to be replaced now by

$$\langle 0|\Phi|0\rangle = Nx\,, \tag{17}$$

where N is some real number and x denotes a point on the group manifold of G. We may, if we wish, get an explicit picture of x in terms of a certain $m\times m$ unitary, unimodular matrix u, in exact analogy with (3), or equivalently, in terms of some orthogonal matrix (over reals). We will also label the vacuum by x. Repeating our previous arguments, we easily arrive at the following conclusion: *the state space of our theory is a Hilbert bundle based on the group manifold of G.* At this stage it is useful to get a feel for the geometry of the manifold. The relevant result is given by Hopf.[16] *The group manifold of G is a "twisted product" of a certain number λ of odd spheres, where λ is the rank of G. An odd sphere is an* odd-dimensional sphere, such as S^{2m+1} (m is a positive integer here). Twisted product means a "topological product insofar as the cohomology theory is concerned" (it is not generally a homeomorphism, however). Thus

$$\mathrm{SU}(m) = S^3\times S^5\times S^7\times\cdots\times S^{2m-1}, \quad m>2 \tag{18}$$

where we have denoted the manifold of SU(m) also by SU(m) and the equality sign means a twisted product. Another way of visualizing the above is to consider SU(m) as a principal fiber bundle,[17] over the sphere S^{2m-1}, with structure group SU($m-1$) and proceed recursively. The local product structure of the twisted product is now quite obvious.

The construction of the physical Hilbert space H proceeds as before, namely, via Eq. (4) with the only change that we have to replace q by x and $d\mu(q)$ by $d\mu(x)$ and integrate over the group manifold of G. Of course, $d\mu(x)$ is the Haar measure of G. Equation (6) remains valid, when appropriately generalized. This last statement follows from the Schur orthogonality relation for unitary *matrices:*

$$\int_G u_{ij}(x)\bar{v}_{kl}(x)d\mu(x) = 0\,, \tag{19}$$

where u and v are two nonequivalent matrices (u_{ij} and v_{ij} are matrix elements). The desired result follows upon taking for v the corresponding unit matrix. All the steps of our foregoing model now go through *in toto;* the unitary operator U is defined the same way as before [Eq. (13)]. Thus we see that the vacuum state in H is degenerate, which is labeled, again, by an integer n.

To see the significance of the vacuum degneracy parameter n, we consider maps $U\to U^n$, which look in the Hilbert bundle simply as (bundle) mappings $x\to x^n$. Now, the general class of mappings of the type $g\to g^n$, $g\in G$, G is any compact Lie group, was studied by Hopf.[16] Following Hopf, we call such maps power maps. Now let p_n be the power map

$$p_n(x) = x^n \tag{20}$$

and let d_n denote the corresponding mapping degree, defined in the usual way.[18] Then, Hopf proves the result

$$d_n = n^\lambda\,, \tag{21}$$

where λ is the rank of G. For our case $G=\mathrm{SU}(m)$ λ is simply $(m-1)$. The significance of the integer n that we have been seeking is contained in the above Hopf relation. To form a complete mental picture, we remember the intuitive significance of mapping degree as the number of times the manifold wraps around itself (under the map). It is also known that homotopy classes of maps are characterized by the mapping degree.

The physical significance of the vacuum degeneracy is read off from Eqs. (12) and (13). The vacuum states $|\Omega_n\rangle$ are all Poincaré invariant and represent one and the same physical state. But $|\Omega_n\rangle$ differs from $|\Omega\rangle$ by the presence of n spurions that carry zero energy and momentum. From the presence of the spurion alone we cannot draw any conclusion regarding the possible existence of massless Goldstone modes.[19] However, the Goldstone theorem, as is well known, can be proved provided we make use of the *additional* assumption that the symmetry is generated by conserved currents.[20] In the latter event, it seems permissible to proceed a step further and identify the spurion with an infrared Goldstone boson. On the other hand, we will, in the next section, encounter a situation in which the spurion coexists with a spectrum free of massless particles.

IV. GAUGE THEORIES

The unitary gauge has been defined by Weinberg[1] to be the one in which the "unwanted" components of the scalar field $\Phi(x)$ have been eliminated (i.e., transferred to the Yang-Mills fields). For a simple gauge theory the scalar field, moreover, is brought to the form

$$\Phi(x) = N\xi(x)q \ , \tag{22}$$

where q denotes a point on the group manifold of G (q does not depend on x), $\xi(x)$ is the surviving component of the scalar field, and N some real number. The vacuum expectation value of the real field $\xi(x)$ is here normalized to unity. Since the precise value of q has no intrinsic significance (only N has), we are thus left with the residual freedom of subjecting $\Phi(x)$ to global, position-independent symmetry transformations. The corresponding global symmetry group is broken by the vacuum expectation value of $\Phi(x)$. Therefore, our previous results apply to the present case and there is nothing more to discuss as far as the mathematical structure of the theory is concerned. Our conclusion remains valid even when the gauge theory is not "simple" since the breaking of a residual global symmetry is present here also.

The vacuum degeneracy, as remarked earlier, implies the presence of spurions. However, the spurion does not have any *physical* significance.

We should note the precise nature of the residual global symmetry. Inspection of Eq. (22) shows that the group acts on the field $\Phi(x)$ via left translations. Thus there is symmetry. However, the symmetry is *not generated by conserved currents possessing the property of being local with respect to the fields*. Consequently, the proof of the Goldstone theorem does not go through in the present case.[21]

The analysis of this paper does not apply to real life gauge theories such as the standard electroweak model,[22] where one symmetry always remains unbroken. It remains an open question whether our methods could be generalized to include more realistic situations.

V. CONCLUDING REMARKS

In conclusion, we make the following remarks.

(1) In the theory of induced representations,[14] it is shown that the inducing construction can be carried through provided that the base space of the Hilbert bundle is locally compact. However, we have seen that the direct integral construction of the vacuum state given by the analog of Eq. (5) for the general case is possible only when the total measure of the group manifold $\mu[G]$ is finite, that is, when G is compact. This has happened because we have demanded that both the vacuum $|x\rangle$ and the vacuum $|\Omega\rangle$ be normalizable. It is to be noted that all useful broken "gauge groups" in physics are compact.

(2) All compact Lie groups have nontrivial "topological quantum numbers" associated with them. Thus we should always expect the latter to enter the description of broken symmetry.

(3) The structure that has emerged from our analysis bears a certain degree of parallelism with the instanton phenomenon,[23] with the problem of magnetic monopoles as discussed by Arafune, Freund, and Goebel[24] and also with Michel's treatment of topological classification of defects,[25] although the physical origins of these effects are entirely different. It is also interesting to note that many of these later developments are anticipated in the work of Finkelstein and co-workers.[26]

(4) We should mention yet another possibility of describing the state space, now involving a Hilbert bundle based on the circle into which the Hilbert space H can be decomposed. The circle here is the Pontrjagin dual to the additive group of integers that labels vacuums in H. The θ vacuum that resides in the circle-based bundle is formally similar to the quantum-chromodynamic vacuum.[27]

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