PHYSICAL REVIEW D VOLUME 24, NUMBER 8 15 OCTOBER 1981

Geon-type solutions of the nonlinear Heisenberg-Klein-Gordon equation

Eckehard W. Mielke
International Centre for Theoretical Physics, Trieste, Italy

Reinhard Scherzer
Institut für Reine und Angewandte Kernphysik der Christian-Albrechts Universität Kiel, 2300 Kiel, Federal Republic of Germany
(Received 3 November 1980)

As a model for a "unitary" field theory of extended particles we consider the nonlinear Klein-Gordon equation—associated with a "squared" Heisenberg-Pauli-Weyl nonlinear spinor equation—coupled to strong gravity. Using a stationary spherical ansatz for the complex scalar field as well as for the background metric generated via Einstein's field equation, we are able to study the effects of the scalar self-interaction as well as of the classical tensor forces. By numerical integration we obtain a continuous spectrum of localized, gravitational solitons resembling the geons previously constructed for the Einstein-Maxwell system by Wheeler. A self-generated curvature potential originating from the curved background partially confines the Schrödinger-type wave functions within the "scalar geon." For zero-angular-momentum states and normalized scalar charge the spectrum for the total gravitational energy of these solitons exhibits a branching with respect to the number of nodes appearing in the radial part of the scalar field. Preliminary studies for higher values of the corresponding "principal quantum number" reveal that a kind of fine splitting of the energy levels occurs, which may indicate a rich, particlelike structure of these "quantized geons."

I. INTRODUCTION

A fundamental theory of matter based on the quark hypothesis[1] has to accommodate an in-built mechanism of (at least partial) confinement of the constituent field in stable particles, otherwise they should be observable at some detectable rate.[2] In order to circumvent the Pauli exclusion principle, these fundamental fermion fields are assumed to obey parastatistics, or equivalently, have to carry, besides flavor, additional color[3] degrees of internal freedom. These color models are distinguished by the binding mechanism of quarks in hadrons, i.e., whether this is mediated by scalar,[4] vector (see, e.g., Ref. 5), or tensor gluons.[6,7]

In quantum chromodynamics (QCD),[5] nowadays the most prominent model for strong interactions, the dynamics of the mediating vector gluons is determined by an action modeled after Maxwell's theory of electromagnetism. The resulting model is a gauge theory of the Yang-Mills type.[8] However, it is known[9] that in such sourceless non-Abelian gauge theories there are no classical glueballs which otherwise would be an indication for the occurrence of confinement in the quantized theory. (The phenomenological consequences of the possible existence of glueballs in QCD have been discussed by Robson.[10]) The reason simply is that nearby small portions of the Yang-Mills fields always point in the same direction in internal space and therefore must repel each other as like charges. Nevertheless, vector gauge fields might be an important ingredient of any model in order to explain saturation.[6]

The confinement itself, according to the proposals of an unconventional scheme termed *color geometrodynamics* (CGMD) (Refs. 11–13) may be achieved by strongly[14-16] interacting massless tensor gluons, their dynamics presumably being determined by Einstein-type field equations. CGMD is a $GL(2N,C)$ gauge model in curved space-time which may be regarded as a generalization of Einstein's gravity theory. The latter corresponds to a gauging of the covering group $SL(2,C)$ of the Lorentz group. Since CGMD is, in general, based on a Riemann-Cartan space-time,[17] Cartan's notion of torsion is known[17] to induce nonlinear spinor terms into the Dirac equation. This has a profound effect on the "fundamental" spinor fields

$$\psi=\{\psi^{(q)}(x)\,|\,q=1,\ldots,N\} \tag{1.1}$$

distinguished by N color (or flavor) internal degrees of freedom. It can be shown also for this $GL(2N,C)$ gauge-invariant generalization[12] that these quark-type fields have to satisfy the Heisenberg-Pauli-Weyl[18-20] nonlinear spinor equation

$$(iL^{\mu}\nabla_{\mu}-\tfrac{3}{8}\epsilon\ell^{*2}\bar{\psi}L^5L_{\mu}\psi L^5L^{\mu}-\mu c/\hbar)\psi=0 \tag{1.2}$$

generalized to a curved[21] space-time (compare also with Ivanenko[22]). Here L^{μ} are space-time-dependent generalizations of the Dirac matrices γ^{α} tensored with the $U(n)$ vector operators λ^j (generalized Gell-Mann matrices). Essentially the modified Planck length

$$\ell^*\equiv(8\pi\hbar G_s/c^3)^{1/2}=(8\pi)^{1/2}\hbar/cM^*\approx10^{-13}\ \text{cm} \tag{1.3}$$

of strong gravity[14-16] occurs also as the coupling constant of the self-interaction in (1.2).

If we transfer the ideas of Mach and Einstein to the microcosmos, the curving up of the hadronic background metric should be self-consistently produced by the stress-energy content $T_\mu{}^\nu(\psi)$ of the spinor fields (1.1) via the Einstein equations

$$G_\mu{}^\nu + \delta^\nu_\mu \Lambda_{\text{conv}} = -\frac{\ell^{*2}}{\hbar c}\, T_\mu{}^\nu \tag{1.4}$$

with cosmological term (we employ the sign conventions of Tolman[23]). In this new geometrodynamical model[11] extended particles owning internal symmetries should be classically described by objects which closely resemble the geons or wormholes of Wheeler.[24] In some sense this approach is also related to the issue to which Einstein and Rosen addressed their 1935 paper.[25]

"Is an atomistic theory of matter and electricity conceivable which, while excluding singularities in the field, makes use of no other fields than those of the gravitational field ($g_{\mu\nu}$) and those of the electromagnetic field in the sense of Maxwell (vector potentials φ_μ)?"

The geon, i.e., a gravitational electromagnetic entity was originally devised by Wheeler[24] to be a self-consistent, nonsingular solution of the otherwise source-free Einstein-Maxwell equations having persistent large-scale features. Such a geon provides a well-defined model for a classical body in general relativity. If spherically symmetric geons would stay completely stable objects they could acquire the possibility to derive their equations of motions solely from Einstein's field equations[26] without the need to introduce field singularities. In a sense this approach also embodies the goals of the so-called unitary field theory.[27,28]

Geons, as we are using the term, are *gravitational solitons,* which are held together by self-generated gravitational forces and are composed of localized fundamental classical fields. The coupling of gravity to neutrino fields has already been considered by Brill and Wheeler.[29] The latter work lays the appropriate groundwork for an extension of their analysis to nonlinear spinor geons satisfying the combined equations (1.2) and (1.4). In this paper, however, we have avoided algebraic complications resulting from the spinor structure as well as from the internal symmetry by considering rather nonlinear scalar fields coupled to gravity. In order to maintain a similar dynamics we assume—as in a previous paper (Ref. 30, hereafter referred to as I)—a self-interaction of these scalar fields which can be formally obtained by "squaring" the fundamental spinor equation (1.2). "Linear" Klein-Gordon geons have been previously constructed.[31] However, we view the additional nonlinearity of the scalar fields as an important new ingredient for our model.

The precise set-up of this theory is given in Sec. II. For the intended construction of localized geons, the stationary, spherical *Ansätze* of I are employed for the scalar fields, whereas the metric is taken in its general spherically symmetric canonical form (Sec. III). As in the case of a prescribed Schwarzschild background—analyzed in I—the curved space-time affects the resulting Schrödinger equation for the radial function essentially via an external gravitational potential.

The stress-energy content of these scalar solutions determines the curvature via Einstein's field equations. In Sec. IV we review the spherical symmetric case and include also a method which enables us to incorporate nonzero-angular-momentum states into this framework by averaging the stress energy of these scalar fields over a spherical shell.

Our geons contain a fixed (quantized) scalar charge. By imposing this restriction (see Sec. V) we not only fix on otherwise undetermined scale of our geons but may also increase their stability. The main concern of Sec. V is, however, to contrast two notions of energy for our gravitational solitons: (1) the field energy of the general-relativisitic scalar waves, and (2) the total gravitational energy of such an isolated system.[32] In order to probe our concepts we construct in Sec. VI a simplified geon by considering radially constant scalar solutions owning the particular constants admitted by the nonlinear self-interaction. Outside a ball of radius ρ_B the scalar fields are discontinuously set to zero. Although this procedure is rather artificial, we thereby obtain a "bag-like"[33] object having inside a portion of an Einstein microcosmos and outside a Schwarzschild manifold as background space-time.

In general, the resulting system of three coupled nonlinear equations for the radial parts of the scalar and the (strong gravitational) tensor fields has to be solved numerically. In order to specify the starting values for the ensuing numerical analysis we derive in Sec. VII asymptotic solutions at the origin and at spatial infinity. Section VII is then devoted to a discussion of the numerical results. Preliminary speculations are offered with the aim to interpret particles as quantum geons. Section VIII concludes the paper with a prospective overview of other developments concerning gravitational solitons.

II. THE MODEL

Following the outline given in the Introduction we may consider as a simplified model a theory consisting of N complex scalar fields

$$\varphi = \{\varphi^{(q)}(x)\,|\, q = 1, \ldots, N\}. \tag{2.1}$$

Their dynamics is governed by the Lagrangian density

$$\mathfrak{L}_{\rm HKG}=\frac{\hbar^2}{2\mu}\sqrt{|f|}\,[f^{\mu\nu}(\partial_\mu\varphi^*)(\partial_\nu\varphi)-U(|\varphi|^2)] \quad (2.2)$$

defined on a curved pseudo-Riemannian space-time with metric tensor $f_{\mu\nu}$. In order to obtain a similar dynamical problem as in the nonlinear spinor theory given by (1.2) the self-interaction potential

$$U(|\varphi^2|)=(\mu c/\hbar)^2\varphi^*\varphi-\frac{3\epsilon\mu c}{16\hbar}\ell^2(\varphi^*\varphi)^2+\frac{3\bar\epsilon}{256}\ell^4(\varphi^*\varphi)^3,\quad \epsilon,\bar\epsilon=0,\pm1 \quad (2.3)$$

is chosen to be similar to that used in I. Such a model has recently been treated in 1+1 dimensions according to quantum-field-theoretical methods.[34] Variation for $\delta\mathfrak{L}_{\rm HKG}/\delta\varphi^*$ yields the nonlinear Klein-Gordon equation

$$[\Box+dU/d(|\varphi|^2)]\varphi=0\,, \quad (2.4)$$

where

$$\Box=\frac{1}{\sqrt{|f|}}\partial_\mu(f^{\mu\nu}\sqrt{|f|}\,\partial_\nu) \quad (2.5)$$

denotes the generally covariant Laplace-Beltrami operator. When (2.4) is explicitly written for the choice (2.3) of the self-coupling it will be referred to as the nonlinear Heisenberg-Klein-Gordon equation

$$\left[\Box-\frac{3\epsilon\mu c}{8\hbar}\ell^2|\varphi|^2+\frac{9\bar\epsilon}{256}\ell^4|\varphi|^4+(\mu c/\hbar)^2\right]\varphi=0\,. \quad (2.6)$$

In I it has been shown that (2.6) is formally similar to that obtained by "squaring" the fundamental spinor equation (1.2). This is part of the motivation for considering a $|\varphi|^6$ term in the corresponding Lagrangian density (2.2).

Although the resulting quantum field theory, contrary to the $|\varphi|^4$ model, would not be renormalizable according to standard criteria of perturbation theory we include in this paper the additional $|\varphi|^6$ self-interaction term. For a semiclassical approach it may be instrumental for the construction of *quasistable*, spherically symmetric and localized solutions. At least in flat space-time, Anderson[35] has shown by means of a phase-space analysis that for

$$-\infty<B_{\rm And}\equiv\frac{\bar\epsilon}{4}(1-\omega^2)<\tfrac{3}{16}\,, \quad (2.7)$$

particlelike (stable) solutions can exist. ω is the ratio between dynamical mass and the "bare" mass μ of this model (see Sec. III). Intuitively, we suspect that the stability of these solutions (and possibly also their degree of confinement) is enhanced by the attractive forces exerted on them via the coupling to strong tensor gluons.[11,7] Geometrically speaking, this would correspond to a curved-background manifold. In our model this curving up of the background is self-consistently produced from the stress-energy tensor[36] (MTW, p. 504)

$$T_{\mu\nu}(\varphi)\equiv-\frac{2}{\sqrt{|f|}}\frac{\delta\mathfrak{L}_{\rm HKG}}{\delta f^{\mu\nu}}=2\delta(|f|^{-1/2}\mathfrak{L}_{\rm HKG})/\delta f^{\mu\nu}-f_{\mu\nu}|f|^{-1/2}\mathfrak{L}_{\rm HKG} \quad (2.8)$$

of the scalar fields φ via Einstein's field equations (1.4). In effect, our geometrodynamical model is then completely determined by the Lagrangian density

$$\mathfrak{L}_{\rm GMD}=\frac{\hbar c}{\ell^{*2}}\sqrt{|f|}\,(R-2\Lambda_{\rm conv})+\mathfrak{L}_{\rm HKG} \quad (2.9)$$

since (2.6) and (2.9) can be derived from it by a variation for $\delta\mathfrak{L}_{\rm GMD}/\delta\varphi^*$ and for $\delta\mathfrak{L}_{\rm GMD}/\delta f^{\mu\nu}$.

III. SPHERICAL SCALAR WAVES IN A CURVED BACKGROUND

As a semiclassical model for a particle we are considering spherical geon-type[24,37] solutions which minimize (2.9). More precisely, we are looking for spherical wave configurations which solve the HKG equation (2.6) in a static, spherically symmetric background space-time which in turn is determined by (1.4).

As is well known[36] (see, e.g., MTW, p. 594 and box 23.3), a canonical form of the general (dimensionless) line element for this background reads

$$d\bar s^2\equiv\frac{2\pi}{\ell^{*2}}ds^2=\frac{2\pi}{\ell^{*2}}f_{\mu\nu}dx^\mu dx^\nu=\frac{2\pi}{\ell^{*2}}e^\nu c^2dt^2-e^\lambda d\rho^2-\rho^2(d\theta^2+\sin^2\theta d\phi^2) \quad (3.1)$$

if the sign conventions of Tolman[23] are adopted. Here $\nu\equiv\nu(\rho)$ and $\lambda\equiv\lambda(\rho)$ are functions which depend solely on the dimensionless Schwarzschild-type (MTW, p. 721) radial coordinate

$$\rho\equiv\frac{(2\pi)^{1/2}}{\ell^*}r=\frac{M^*c}{2\hbar}r,\quad r=|\vec{x}|\,. \quad (3.2)$$

The determinant of this metric is given by

$$\sqrt{|f|}=\left[\frac{\ell^*}{(2\pi)^{1/2}}\right]^3 c\,e^{(\nu+\lambda)/2}\rho^2\sin\theta\,. \quad (3.3)$$

For the construction of spherical scalar waves, we take up the well-known fact that solutions of the free, linear Klein-Gordon equation can be expanded in terms of spherical harmonics $Y_l^m(\theta,\phi)$ which are eigenfunctions,

$$[\Delta_2+l(l+1)]Y_l^m(\theta,\phi) \equiv\left[\frac{1}{\sin\theta}\,\partial_\theta(\sin\theta\,\partial_\theta)+\frac{1}{\sin^2\theta}\,\partial_\phi{}^2+l(l+1)\right]Y_l^m(\theta,\phi)=0\,, \tag{3.4}$$

of the Laplace operator Δ_2 on the two-sphere S^2. Although a nonlinear theory in general does not respect such an expansion on, the nonlinear terms of the field equation (2.6) admit the two distinct separation *Ansätze* of I:

$$\varphi^{(q)}=\frac{4}{\ell}\,(\mu c/\hbar)^{1/2}\left(\frac{4\pi}{3N}\right)^{1/2}e^{-it\omega\mu c^2/\hbar}$$

$$\times\begin{cases}R_l^\omega(\rho)Y_l^{q-l-1}(\theta,\phi) & \text{if } N=2l+1\,, \quad (3.5)\\ R_0^\omega(\rho)Y_0^0(\theta,\phi) & \text{elsewhere}\,. \quad (3.6)\end{cases}$$

Owing to the familiar addition theorem (Landau and Lifschitz, Sec. 26)[38]

$$\sum_{m=-l}^{l} Y_l^m(\theta,\phi)^*Y_l^m(\theta,\phi)=\frac{2l+1}{4\pi}\,, \tag{3.7}$$

a self-interaction given by a polynomial in $|\varphi|^2=\sum_{q=1}^{N}\varphi^{(q)*}\varphi^{(q)}$ remains spherically symmetric as required by separability.

In order to see how space-time curvature affects the wave equation it is instructive to define

$$R_l^\omega(\rho)\equiv\frac{1}{\rho}\,F_l^\omega(\rho^*) \tag{3.8}$$

(in some equations below abbreviated by R or F, respectively) and to introduce Wheeler's[36] "tortoise coordinate" ρ^* (MTW, p 663) via the differential form[37]

$$d\rho^*\equiv e^{(\lambda-\nu)/2}d\rho\,. \tag{3.9}$$

Then the line element (3.1) takes the form

$$d\bar{s}^2=e^\nu\left(\frac{2\pi}{\ell^{*2}}\,c^2dt^2-d\rho^{*2}\right)-\rho^2(d\theta^2+\sin^2\theta\,d\phi^2)\,. \tag{3.10}$$

It resembles the metrical ground form of a space-time with two conformally flat portions. Then a kind of conformal change[39] of the Laplace-Beltrami operator (2.4) may be calculated with the formal result

$$\frac{\ell^{*2}}{2\pi}\,\rho\Box\frac{1}{\rho}=e^{-\nu}\left(\frac{\ell^{*2}}{2\pi c^2}\,\partial_t{}^2-\partial_{\rho^*}{}^2\right) -\frac{1}{\rho^2}\left[\frac{1}{\sin\theta}\,\partial_\theta(\sin\theta\,\partial_\theta)+\frac{1}{\sin^2\theta}\,\partial_\phi{}^2\right] +\frac{1}{\rho}\,e^{-(\nu+\lambda)/2}[\partial_\rho e^{(\nu-\lambda)/2}]\,. \tag{3.11}$$

With respect to the new coordinate ρ^* and the Schrödinger-type wave function $F_l^\omega(\rho^*)$, the structure of the Laplace-Beltrami operator is similar to that of a conformally flat Klein-Gordon operator with an external potential.

Consequently, the stationary *Ansätze* (3.5) and (3.6) together with the property (3.11) cast the HKG equation (2.6) into the Schrödinger-type equation

$$[\partial_{\rho^*}{}^2-V_{\text{eff}}^l(\rho^*)]F =\frac{1}{\beta^2N^2}\,e^\nu\left(\frac{\bar\epsilon}{\rho^4}\,F^4-\frac{2\epsilon}{\rho^2}\,F^2+1-e^{-\nu}\omega^2\right)F \tag{3.12}$$

with an *effective curvature potential* (compare with MTW Ref. 36, p. 868) implicitly given by

$$V_{\text{eff}}^l(\rho^*)=\frac{e^\nu}{\rho^2}\left[l(l+1)-\frac{1}{2}\,\rho e^{-\lambda}\,\partial_\rho(\lambda-\nu)\right]. \tag{3.13}$$

Here and in the following the factor

$$\beta\equiv\frac{M^*}{2N\mu} \tag{3.14}$$

denotes the dimensionless ratio between the Planck mass M^* and the bare mass μ of the N constituent fields. So far we have considered the formal aspects of the theory. For the ensuing numerical calculations, however, it is more convenient to use the equivalent radial equation

$$R''+\left[\tfrac{1}{2}\,\partial_\rho(\nu-\lambda)+\frac{2}{\rho}\right]R'-l(l+1)\frac{e^\lambda}{\rho^2}\,R =\frac{e^\lambda}{\beta^2N^2}\,(\bar\epsilon R^4-2\epsilon R^2+1-e^{-\nu}\omega^2)R \tag{3.15}$$

written explicitly in terms of ρ. (The prime denotes differentiation with respect to ρ.) It may be obtained from (3.12) and (3.13) by resubstitutions, or more directly from (2.6) and the original *Ansätz* (3.5) and (3.6) founded on the background (3.1). It generalizes Eq. (3.9) of Kaup[31] derived there from a linear Klein-Gordon equation.

IV. THE EINSTEIN FIELD EQUATIONS

By applying Machian ideas to the microcosmos, the strong gravitational background will be determined from the stress-energy content of the scalar waves via the Einstein equations (1.4).

With respect to the diagonal metric of (3.1) these equations reduce to (see Tolman,[23] p. 242)

$$-e^{-\lambda}\left(\frac{\nu'}{\rho}+\frac{1}{\rho^2}\right)+\frac{1}{\rho^2}-\Lambda=\frac{\ell^{*4}}{2\pi\hbar c}\,T_r{}^r\,, \tag{4.1}$$

$$-e^{-\lambda}\left(\frac{\nu''}{2}-\frac{\lambda'\nu'}{4}+\frac{\nu'^2}{4}+\frac{\nu'-\lambda'}{2\rho}\right)-\Lambda =\frac{\ell^{*4}}{2\pi\hbar c}\,T_\theta{}^\theta=\frac{\ell^{*4}}{2\pi\hbar c}\,T_\phi{}^\phi\,, \tag{4.2}$$

and

$$e^{-\lambda}\left(\frac{\lambda'}{\rho}-\frac{1}{\rho^2}\right)+\frac{1}{\rho^2}-\Lambda=\frac{\ell^{*4}}{2\pi\hbar c}\,T_0{}^0 . \qquad (4.3)$$

Here

$$\Lambda \equiv \frac{\ell^{*2}}{2\pi}\,\Lambda_{\rm conv} \qquad (4.4)$$

denotes a dimensionless cosmological constant and the prime again means differentiation with respect to ρ. Although this set of equations may look like an overdetermined system, a simple argument shows that this is not the case.

According to second-order variational principles which can be generalized to a curved space-time,[40] there exists the conservation law $\nabla_\nu T_\mu{}^\nu=0$ for the stress-energy tensor (2.8) provided that the matter field equation (2.4) hold. In our case, this law relates the two tangential tensions $T_\theta{}^\theta=T_\phi{}^\phi$ to the radial tensions, i.e., to $T_r{}^r$, $T_0{}^0$, and $dT_r{}^r/dr$. This knowledge of $T_\theta{}^\theta=T_\phi{}^\phi$ is not instrumental for the determination of the gravitational fields from the field equation (2.9). Owing to the contracted Bianchi identity $\nabla_\nu G_\mu{}^\nu=0$ there exists the same relation between the diagonal elements of the Einstein tensor $G_\mu{}^\nu$. Therefore it is enough to consider the two remaining equations (4.1) and (4.3) only (Ref. 41, p. 488).

For the supposed spherically symmetric background (3.1) we notice that the nontrivial *Ansatz* (3.5) would lead to an inconsistency in the gravitational field equations. The reason is that the corresponding stress-energy tensor (2.8), in the scalar case given by

$$T_\mu{}^\nu=\frac{\hbar^2}{\mu}\,(\partial_\mu\varphi^*)(\partial^\nu\varphi)-\delta_\mu{}^\nu|f|^{-1/2}\mathcal{L}_{\rm HKG}\,, \qquad (4.5)$$

would also depend on the angular distribution of the solutions, contrary to the Einstein tensor.

For localized solutions the spherical asymmetry of the scalar waves (3.5) is expected to be negligible sufficiently far away from the center of the geon. Therefore it is physically justifiable not to discard *Ansatz* (3.5) but rather consider the Einstein equations (4.1)–(4.3) with respect to an averaged stress-energy tensor $\langle T_\mu{}^\nu\rangle$ as proposed by Power and Wheeler[41] (see p. 488). Suitable is an average $\langle\,\rangle$ over a spherical shell defined by the property

$$\langle|Y_l^m|^2\rangle\equiv\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}|Y_l^m(\theta,\phi)|^2\sin\theta\,d\theta\,d\phi=\frac{1}{4\pi}\,. \qquad (4.6)$$

The evaluation of $\langle T_\mu{}^\nu\rangle$ will be facilitated by employing the identity

$$\begin{aligned}\langle|\partial_\theta Y_l^m|^2&+\frac{1}{\sin^2\theta}|\partial_\phi Y_l^m|^2\rangle+\langle Y_l^{m*}\Delta_2 Y_l^m\rangle\\&=\left\langle\frac{1}{\sin\theta}\,\partial_\theta(Y_l^{m*}\sin\theta\,\partial_\theta Y_l^m)+\frac{1}{\sin^2\theta}\,\partial_\phi(Y_l^{m*}\sin\theta\,\partial_\phi Y_l^m)\right\rangle\\&=\frac{1}{4\pi}\int_{S^2}\partial^k(Y_l^{m*}\sin\theta\,\partial_k Y_l^m)d^2x=0\,, \end{aligned} \qquad (4.7)$$

which results from the application of Stokes's theorem.

With respect to the *Ansätze* (3.5) and (3.6) the averaged radial tensions come out as

$$\langle T_0{}^0\rangle=\frac{16}{3\beta^2N^2}\,\frac{2\pi\hbar c}{\ell^2\ell^{*2}}\,\omega^2e^{-\nu}R^2-\left\langle\frac{\mathcal{L}_{\rm HKG}}{\sqrt{|f|}}\right\rangle \equiv\frac{2\pi\hbar c}{\ell^{*4}}\,\bar\rho_{00} \qquad (4.8)$$

and

$$\langle T_r{}^r\rangle=-\frac{16}{3}\,\frac{2\pi\hbar c}{\ell^2\ell^{*2}}\,e^{-\lambda}(R')^2-\left\langle\frac{\mathcal{L}_{\rm HKG}}{\sqrt{|f|}}\right\rangle \equiv-\frac{2\pi\hbar c}{\ell^{*4}}\,\bar p_0\,, \qquad (4.9)$$

where the spherical average of the Lagrange function is explicitly given by

$$\left\langle\frac{\mathcal{L}_{\rm HKG}}{\sqrt{|f|}}\right\rangle=\frac{8}{3\beta^2N^2}\,\frac{2\pi\hbar c}{\ell^2\ell^{*2}}\left[\left(\omega^2e^{-\nu}-\frac{l(l+1)}{\rho^2}\,\beta^2N^2-1\right)R^2-\beta^2N^2e^{-\lambda}(R')^2+\epsilon R^4-\frac{\bar\epsilon}{3}\,R^6\right]. \qquad (4.10)$$

In order to bring the radial Einstein equations in close analogy with those known for a perfect fluid (Ref. 23, p. 244) we may formally introduce in (4.8) the dimensionless proper density $\bar\rho_{00}$ of the scalar field. It turns out to be explicitly given by

$$\bar\rho_{00}=\frac{8}{3\beta^2N^2}\,\frac{\ell^{*2}}{\ell^2}\left\{\left[\omega^2e^{-\nu}+\frac{l(l+1)}{\rho^2}\beta^2N^2+1\right]R^2+\beta^2N^2e^{-\lambda}(R')^2-\epsilon R^4+\frac{\bar\epsilon}{3}R^6\right\}. \qquad (4.11)$$

Furthermore, the dimensionless proper hydrostatic pressure $\bar p_0$ of a scalar field can (implicitly) be defined by (4.9).

By subtracting (4.1) from (4.3) and evaluating the combination

$$\bar{\rho}_{00}+\bar{p}_0=\frac{16}{3\beta^2N^2}\frac{\ell^{*2}}{\ell^2}[\omega^2e^{-\nu}R^2+\beta^2N^2e^{-\lambda}(R')^2] \tag{4.12}$$

we obtain

$$\nu'+\lambda'=(\bar{\rho}_{00}+\bar{p}_0)\rho e^{\lambda}, \tag{4.13}$$

whereas in this notation (4.3) is equivalent to

$$\lambda'=(\bar{\rho}_{00}+\Lambda)\rho e^{\lambda}-\frac{1}{\rho}e^{\lambda}+\frac{1}{\rho}. \tag{4.14}$$

These equations are generalizations of those considered by Kaup[31] for a linear and, e.g., by Kodama *et al.*[42] for a nonlinear Klein-Gordon field.

It should be noted that (4.14) is a linear differential equation in $e^{-\lambda}$. As is well known the general formal solution can be written as

$$e^{-\lambda}=1-\frac{\alpha(\rho)}{\rho}-\frac{\Lambda}{3}\rho^2, \tag{4.15}$$

where

$$\alpha(\rho)=\int_0^\rho \bar{\rho}_{00}x^2dx \tag{4.16}$$

denotes the effective mass.[73] The meaning of the latter terminology will be illuminated if we consider the Einstein equations outside the region where matter fields vanish, i.e., $R=0$ in our case. Then the nonlinear equations (4.13) become linearized and the vacuum Einstein equations admit the [with respect to the canonical metric (3.1)] unique set of exact solutions

$$e^{\nu}=e^{-\lambda}=1-\frac{\alpha}{\rho}-\frac{\Lambda}{3}\rho^2. \tag{4.17}$$

They describe the exterior Schwarzschild-de Sitter geometry for a mass distribution located at the origin. In the case of vanishing cosmological constant, $\alpha\equiv\alpha(\infty)$ is the parameter measuring the gravitational mass αM^* at spatial infinity.

V. GRAVITATIONAL ENERGY OF GEONS WITH QUANTIZED CHARGE

In order to associate some quantum meaning to the time-dependent localized solutions of the HKG equation the Bohr-Sommerfeld quantization rules may be imposed. For a field theory with infinitely many degrees of freedom this semiclassical quantization condition[43] reads

$$c\int_{-\pi/2\Omega}^{\pi/2\Omega}dt\int d^3x\sum_{q=1}^{N}\Pi_{(q)}\partial_0\varphi^{(q)}=\pi k\hbar \tag{5.1}$$

the time integration being performed over the semiperiod π/Ω of the solution. In a curved space-time the canonical conjugate field momenta are defined (see, e.g., Fulling[44]) by

$$\Pi_{(q)}\equiv\frac{\delta\mathfrak{L}_{\rm HKG}}{\delta(\partial_0\varphi_{(q)})}=\frac{\hbar^2}{2\mu}\sqrt{|f|}\,f^{0\mu}\partial_\mu\varphi^{*(q)}. \tag{5.2}$$

In a static background and for the stationary *Ansätze* (3.2) or (3.3) owning the semiperiod π/Ω, where $\Omega=\omega\mu c^2/\hbar$, it is not difficult to see that the Bohr-Sommerfeld condition (5.1) is equivalent (see also Ref. 45, Sec. 3.6) to the charge quantization

$$Q(\varphi)=ke,\quad k=1,2,\ldots, \tag{5.3}$$

where

$$Q(\varphi)=\frac{e\hbar}{2\mu c}i\int d^3x\sqrt{|f|}\,f^{0\mu}[\varphi^*\partial_\mu\varphi-(\partial_\mu\varphi^*)\varphi] \tag{5.4}$$

in a curved space-time is the conserved total charge of the complex scalar fields. The condition (5.3) may also increase the stability of these quantum geons[26] provided that this stabilizing device for nonlinear semiclassical field theories[45] applies also in curved space-time.

By insertion of the *Ansätze* (3.2) or (3.3) we obtain the expression

$$\frac{Q(\varphi)}{e}=\frac{32}{3}\frac{\omega}{\beta N}\frac{\ell^{*2}}{\ell^2}\int_0^\infty d\rho^*F_l^\omega(\rho^*)^2=k. \tag{5.5}$$

For fixed β and preassigned ω this condition normalizes the *a priori* arbitrary Planck length ℓ^* with regard to the coupling constant ℓ of our nonlinear model. On the other hand, if we fix this ratio to be, e.g., $\ell^*/\ell=1$ as we will assume in our numerical calculations, the condition (5.5) determines the physically immaterial initial constant C_∞ appearing in the asymptotic solutions (7.1) discussed in Sec. VII. Our normalization (5.4) is the same as that used by Kaup[31] but deviates from the condition suggested by Feinblum and McKinley.[46]

In a curved space-time the energy concept is known to be rather subtle. Let us recall that for matter fields coupled to gravitation the locally conserved four-momentum is given by

$$P_\mu=\frac{1}{c}\int_{\substack{\text{spacelike}\\ \text{hypersurface}}}(T_\mu{}^\nu+t_\mu^{LL\nu})\sqrt{|f|}\,d\Sigma_\nu \tag{5.6}$$

the integration being performed over a spacelike hypersurface. Differently as in the case of flat Minkowski space, in (5.6) the stress-energy pseudotensor $t^{LL}_{\mu\nu}$ of Landau and Lifshitz[36] (MTW, p. 466) must be included in order to account for the contribution from the gravtiational field.

For a quasistatic isolated system and $\Lambda=0$ Tolman[32] (see also Ref. 23, p. 235) has derived the following equivalent expression for its total energy

$$P_0 = \frac{1}{c} \int (2T_0{}^0 - T_\mu{}^\mu)\sqrt{|f|}\, d^3x. \tag{5.7}$$

This result which is exact in the static case is operationally more useful, in particular in numerical calculations, since the volume integral has to be extended only over the region actually occupied by the matter fields.

In our construction of localized geons we can satisfy the criteria for the applicability of (5.7) if we require the radial part of the scalar field to be exponentially decreasing in the asymptotic region $\rho \to \infty$ (see Sec. VII).

Thereby the gravitational background field (3.1) tends sufficiently fast to that of a Schwarzschild geometry given by (4.17) with $\Lambda = 0$. By construction the total mass of our geon is then known to be αM^* [compare with Eq. (62) of Ref. 41] and the Einstein relation

$$P_0 = \alpha M^* c^2 \tag{5.8}$$

holds in a rest frame.

Using the static background (3.1) and the formal relations (4.8) and (4.9) the Tolman energy (5.7) can be cast into the form

$$P_0 = M^* c^2 \int_0^\infty d\rho^* \rho^2 e^\nu (\bar{p}_{00} + 3\bar{p}_0) \tag{5.9}$$

(Tolman[23], p. 248). With respect to a nonlinear scalar field theory defined by (2.2), the total energy is equivalently given by

$$P_0 = \frac{2\ell^{*3}}{(2\pi)^{1/2}} \frac{\hbar^2}{\mu} \int_0^\infty \langle 2f^{00} |\partial_0 \varphi|^2 - U(|\varphi|^2)\rangle e^{(\nu+\lambda)/2} \rho^2 d\rho \tag{5.10}$$

the integration over angular variables formally being absorbed in the averages defined earlier by (4.6). After inserting the *Ansätze* (3.5) or (3.6) we obtain the more explicit result

$$P_0 = \frac{32}{3} \frac{\mu c^2}{\beta N} \frac{\ell^{*2}}{\ell^2} \times \int_0^\infty d\rho^* \rho^2 e^\nu \left(2\omega^2 e^{-\nu} - 1 + \epsilon R^2 - \frac{\bar{\epsilon}}{3} R^4\right) R^2. \tag{5.11}$$

The model-dependent ratio ℓ^*/ℓ may be eliminated by the previously derived normalization condition (5.5). Then in view of (3.14) the formula

$$\alpha = \frac{k}{\beta N}\left\{\omega - \frac{1}{2\omega} \frac{\int_0^\infty d\rho^* \rho^2 e^\nu R^2 [1 - \epsilon R^2 + (\bar{\epsilon}/3) R^4]}{\int_0^\infty d\rho^* \rho^2 R^2}\right\} > 0 \tag{5.12}$$

finally determines the total gravitational mass αM^* of a scalar geon with quantized charge. It is interesting to note that gravity alone modifies the bare mass μ by $2k\omega$, whereas *a mass renormalization* on this semiclassical level is due to the (nonlinear) self-interaction. The relation (5.12) may be contrasted with the curved space-time definition (see, e.g., Ref. 44)

$$E = \frac{1}{c} \int T_0{}^0 \sqrt{|f|}\, d^3x = \frac{2\ell^{*3}}{(2\pi)^{1/2}} \int d\rho^* \rho^2 e^\nu \langle T_0{}^0\rangle \tag{5.13}$$

of the field energy of the N scalar constituent fields $\varphi^{(q)}$ of the geon without the contributions from the self-consistently generated gravitational field.

Inserting (4.8) together with (4.10) and then substituting the normalization condition (5.5) yields the mass formula

$$\frac{E}{c^2} = k\frac{\mu}{2}\left\{\omega + \frac{1}{\omega}\left(\int_{\rho_0}^\infty d\rho^* \rho^2 R^2\right)^{-1} \times \int_{\rho_0}^\infty d\rho^* \rho^2 e^\nu R^2 \times \left[1 + \frac{l(l+1)}{\rho^2}\beta^2 N^2 + \beta^2 N^2 e^{-\lambda}\left(\frac{R'}{R}\right)^2 - \epsilon R^2 + \frac{\bar{\epsilon}}{3} R^4\right]\right\} \tag{5.14}$$

for $\rho_0 = 0$. As a precautionary measure for the case that this expression diverges at the origin, we have introduced a cut-off length $\rho_0 > 0$ in (5.14), enabling us to study the regularized field energy E_{reg} instead. After subtracting the boundary term

$$E_0 \equiv -\frac{\hbar^2}{2\mu} \int d^3x\, \partial_a(\varphi^* \sqrt{|f|} f^{ab} \partial_b \varphi) \tag{5.15}$$

from (5.13) and then using the normalization condition (5.5) we may alternatively consider what we call the normalized energy

$$E - E_0 = k\mu c^2 \left[\omega + \frac{1}{\omega} \frac{\int_0^\infty d\rho^* \rho^2 e^\nu (\frac{1}{2}\epsilon R^2 - \frac{1}{3}\bar{\epsilon} R^6)}{\int_0^\infty d\rho^* \rho^2 R^2}\right] \tag{5.16}$$

which should be compared with (4.9) of I. Furthermore, it could be physically interesting to study the binding energy

$$E_{\text{bind}} \equiv \omega\mu c^2 - \frac{\alpha}{N} M^* c^2 = (\omega - 2\alpha\beta)\mu c^2 \tag{5.17}$$

of a scalar particle within a geon as a function of ω and β. Following Ref. 31 we may define it as the energy of a free scalar particle from which its energy contribution to the total geon mass at rest is subtracted. A free test particle means free of gravitational and self-interaction. In view of (5.16) the corresponding normalized energy is $\omega\mu c^2$ which is the factor appearing in the phase of the *Ansätze* (3.5) or (3.6).

VI. GEON WITH A CONSTANT BAGLIKE CORE

Before we turn to a numerical analysis of the spherical Einstein-Klein-Gordon system it is instructive to study an exact solvable geon containing a constant scalar core of radius ρ_B. In order to obtain this highly idealized configuration one has to note the fact that the nonlinear equation (3.15) for the radial distribution of the scalar field φ admits for $l=0$, besides zero, the constant solution

$$(R_c^\omega)^2 = \frac{\epsilon}{\bar\epsilon}(\mp)\left(1-\frac{1}{\bar\epsilon}+\frac{1}{\bar\epsilon}e^{-\nu_c}\omega^2\right)^{1/2}$$
$$=\epsilon(\mp)e^{-\nu_c/2}\omega \text{ for } \bar\epsilon=1 \qquad (6.1)$$

also in a curved space-time (3.1), provided that the f_{00} component of the metric tensor is also a constant, i.e.,

$$\nu_c'=0 \Rightarrow e^{\nu_c}=1-\frac{\alpha}{\rho_B}=\text{const.} \qquad (6.2)$$

In the case that this condition holds Anderson's analysis[35] of the classical phase space (R,R') of the HKG equation can be applied to a certain extent. The main result is that stable (particle-like) solutions can exist if the parameter B_{And} employed already in (2.7) satisfies the condition

$$B_{\text{And}} \equiv \tfrac{1}{4}(1-e^{-\nu_c}\omega^2) < \tfrac{3}{16}. \qquad (6.3)$$

Then the stable equilibrium points in the phase space of the asymptotic version of (3.15) correspond to the constant solution (6.1) with the negative sign of the square root. This solution which the geon may adopt in its interior gives rise to a constant scalar density

$$\bar\rho_{00}=\frac{8}{9\beta^2N^2}\,\frac{\ell^{*2}}{\ell^2}[4\omega^2e^{-\nu_c}+2-\epsilon(R_c^\omega)^2](R_c^\omega)^2 \qquad (6.4)$$

as well as to a constant pressure term

$$\bar p_0=\frac{8}{9\beta^2N^2}\,\frac{\ell^{*2}}{\ell^2}[2e^{-\nu_c}\omega^2-2+\epsilon(R_c^\omega)^2](R_c^\omega)^2 \qquad (6.5)$$

If we absorb (6.4) into an effective cosmological

$$\Lambda_{\text{eff}}=\Lambda_{\text{bag}}+\bar\rho_{00} \qquad (6.6)$$

solutions of (4.14) which are regular at the origin read

$$e^{-\lambda}=1-\tfrac{1}{3}\rho^2\Lambda_{\text{eff}}. \qquad (6.7)$$

From $e^{\nu_c}=\text{const}$, i.e., Eq. (6.2), we can infer that the constant radial solutions (6.1) exist only in (a portion of) an Einstein microcosmos (compare with Tolman[23], Secs. 135 and 139). [Note that (2.6) admits also nontrivial radial solutions[47] in an Einstein Universe. These exact solutions, however, are not geon-type solutions, i.e., they do not satisfy the Einstein equations (1.4) at the same time.]

The remaining radial Einstein equation (4.13) yields the equation of state

$$\bar\rho_{00}=2\Lambda_{\text{bag}}-3\bar p_0 \geq 0 \qquad (6.8)$$

for the density $\bar\rho_{00}$ in terms of the hydrostatic pressure $\bar p_0$. After insertion of (6.4) and (6.5) into (6.8) we obtain

$$5\omega^2e^{-\nu_c}-2+\epsilon(R_c^\omega)^2=\frac{3\beta^2N^2\ell^2}{8\ell^{*2}}\Lambda_{\text{bag}}, \qquad (6.9)$$

which, in view of (6.1), determines $\nu_c=\nu_c(\omega)$ as a function of ω and Λ.

Our geon construction may now follow closely those by which Schwarzschild (see Ref. 23, Sec. 96) obtained the exterior and interior solutions for a spherical star consisting of an incompressible perfect fluid of constant proper density ρ_{00}. To this end we may use (6.1) together with the condition (6.2) and the resulting metric function (6.7) as solutions for the interior $0\leq\rho\leq\rho_B$ of a ball. Then the curvature potential (3.13) associated with this metrical background is also a constant, i.e., more precisely

$$V_{\text{eff}}^0=-\tfrac{1}{3}e^{\nu_c}\Lambda_{\text{eff}}. \qquad (6.10)$$

In order to interpret the interior solution (6.1) as a kind of bound state within a negative potential produced by a gluonic bag of tensor forces we have to require $\Lambda_{\text{eff}}>0$.

In view of (6.6) and (6.8) this condition can be satisfied for $\Lambda_{\text{bag}}\geq\frac{3}{2}\bar p_0>0$ only. Such a nonzero bag constant Λ_{bag} is necessary for the interior of the geon in order to compensate for the vacuum pressure (6.5) of the quark-type scalar fields. A similar mechanism has been proposed in the phenomenological MIT bag model of hadrons.[33] [If Λ_{bag} would be zero, the condition (6.8) is the same as that for a random distribution of electromagnetic radiation (Tolman,[23] p. 217) except for the sign.].

Outside the constant baglike core of radius ρ_B we may simply continue with $R=0$ (if in this idealized construction we are contented with solutions which are only piece-wise differentiable and continuous) and obtain for $\Lambda_{\text{ext}}=0$ a Schwarzschild solution (4.17) for

$$\rho\geq\rho_B\equiv\left(\frac{3\alpha}{\Lambda_{\text{eff}}}\right)^{1/3}>\alpha. \qquad (6.11)$$

This crude geon construction allows us to evaluate the total charge (5.4) in closed form.

Insertion of (6.1), (6.2), and (6.7) into (5.5) yields

$$\frac{Q}{e}=\frac{32}{3}\frac{\omega}{\beta N}\frac{\ell^{*2}}{\ell^2}e^{-\nu_c/2}(R_c^{\omega})^2\int_0^{\rho_B}d\rho\,\rho^2(1-\tfrac{1}{3}\rho^2\Lambda_{\text{eff}})^{-1/2}. \tag{6.12}$$

The remaining integration can be performed with the aid of an integral representation [see Eq. (30) of Ref. 47] of the hypergeometric function. The charge quantization (5.3) then leads to the normalization condition

$$k=\frac{32}{9}\frac{\omega}{\beta N}\frac{\ell^{*2}}{\ell^2}(R_c^{\omega})^2e^{-\nu_c/2}\rho_B{}^3\,{}_2F_1(\tfrac{1}{2},\tfrac{3}{2},\tfrac{5}{2},\tfrac{1}{3}\rho_B{}^2\Lambda_{\text{eff}}). \tag{6.13}$$

For $\Lambda_{\text{ext}}=0$ and assuming here for simplicity also $\rho^2=3/\Lambda_{\text{eff}}$ the above result can be used to determine the scalar density (6.4) respecting the condition (6.8) as

$$\bar{\rho}_{00}=\frac{16}{3}\frac{\beta^2N^2}{k^2}\frac{e^{\nu_c}}{\omega^2}. \tag{6.14}$$

In a similar way we may calculate the already normalized expression (5.12) for the gravitational mass αM^*.

Let us consider the case $\bar{\epsilon}=1$. The insertion of (6.1) and furthermore (6.9) into (5.12) yields

$$\alpha=\frac{k}{\beta N}\frac{5\omega^2(\mp)\epsilon e^{\nu_c/2}-e^{\nu_c}}{6\omega}=k\frac{\beta N}{16}\frac{\ell^2}{\ell^{*2}}\frac{\Lambda_{\text{bag}}}{\omega}, \tag{6.15}$$

a result which could also be inferred from the comparison of condition (6.8) with the equivalent expression (5.9) of the Tolman energy.

Since for $\rho>\rho_B$ the space-time geometry is determined by the Schwarzschild solution (4.17) with $\Lambda_{\text{ext}}=0$ and α given by (6.15), an "observer" placed outside the core will regard this gravitational bound state of scalar fields as an object having the mass αM^*. For $\Lambda_{\text{bag}}=0$ the hadronic environment, i.e., the strong curvature generated by the tensor gluons $f_{\mu\nu}$ inside the constant core gives rise to such a strong "Archimedes effect[7]" on the scalar constituent fields, that their binding energy (5.14) becomes equal to the rest mass $\omega\mu c^2$ of a self-interacting quark.

This can be summarized in a Wheeler-type phrase: A constant baglike geon may have "mass without mass" (Ref. 37, p. 25).

VII. LOCALIZED GEON SOLUTIONS FROM NUMERICAL INTEGRATION

According to our introductory remarks we will reserve the term "geon-type solution" (compare Sec. VIII for other notions) for configurations resulting from a self-consistent coupling of fields to gravity in which both the "matter" waves and the metric tensor are (nonsingular and) sufficiently localized. A precise criterion for localization depends crucially on the circumstance of whether or not a cosmological constant Λ is included. For the present we put $\Lambda=0$ and may then require for localized, spherically symmetric scalar geons the following.

A. Asymptotic solutions at spatial infinity

We proceed similarly as in Sec. III C of paper I (Ref. 30) and consider radial solutions which behave asymptotically as

$$R_l^{\omega}(\rho)\simeq C_\infty\rho^{\sigma}\exp\left[-\frac{1}{\beta N}(1-\omega^2)^{1/2}\rho\right]. \tag{7.1}$$

If $|\omega|<1$, such boundary conditions for the numerical integration would necessarily lead to exponentially decreasing Yukawa-type solutions (see also Ref. 48) irrespective of the parameter σ. Since the scalar waves would already be sufficiently localized, the Eqs. (4.13) and (4.14) pass into the Einstein vacuum equations. Given the canonical form (3.1) these yield uniquely the Schwarzschild solutions (4.9) with $\Lambda=0$.

Therefore

$$e^{\nu}\simeq e^{-\lambda}=1-\frac{\alpha}{\rho}+O(\varphi^2) \tag{7.2}$$

will hold. The asymptotic forms (7.1) and (7.2) are then inserted into (3.15) in order to determine σ on the grounds of self-consistency. By equating the coefficients of the $1/\rho$ expansion we obtain

$$\sigma=-1\,(\mp)\,\frac{\alpha(1-2\,\omega^2)}{2\beta N(1-\omega^2)^{1/2}}. \tag{7.3}$$

This result being independent of the quantum number l of angular momentum agrees for the plus sign with that obtained by Kaup.[31]

Let us turn to the following.

B. Asymptotic solutions at the origin

Guided by the constant-core case analyzed in the preceding section we found it reasonable to assume

$$\nu'\simeq 0\Rightarrow e^{\nu}\simeq e^{\nu_c}=\text{const} \tag{7.4}$$

in the vicinity of the origin.

(0_{I}) Suppose we find

$$R'\simeq\bar{C}_0. \tag{7.5}$$

Then from (4.14) and (4.13) it follows that the radial metric function behaves as

$$e^{-\lambda} \simeq 1 - \tfrac{1}{3}\rho^2 \Lambda_{\text{eff}} \tag{7.6}$$

at the origin. This corresponds to the exact result (6.7). Then the radial equation (3.15) takes the asymptotic form

$$\left[\partial_\rho{}^2 + \frac{2}{\rho}\partial_\rho - \frac{l(l+1)}{\rho^2}\right] R = \frac{1}{\beta^2 N^2}\left(\overline{\epsilon}\, R^4 - 2\epsilon R^2 + 1 - e^{-\nu_c}\omega^2\right) R \tag{7.7}$$

[compare with Eq. (3.11) of I]. A familiar argument expanded in I yields

$$R_l^\omega(\rho) \simeq \overline{C}_0 \rho^l \tag{7.8}$$

as asymptotic solutions regular at the origin. The system of approximate solutions (7.4), (7.6), and (7.8) turns out to be consistent.

(0_{II}) Another set of asymptotic solutions can be obtained by proposing instead of (7.5) the trial function

$$R_0^\omega(\rho) \simeq \left(\frac{3}{8}\right)^{1/2} \frac{\ell}{\ell^*} \ln \rho \tag{7.9}$$

for $l = 0$.

Assuming that $e^\lambda \simeq 0$ for $\rho \to 0$ such that

$$\overline{\rho}_{00}\, e^\lambda \simeq \frac{1}{\rho^2} \tag{7.10}$$

we obtain from (4.14) the result

$$e^\lambda \simeq 4 A_0{}^2 \rho^2 \,, \tag{7.11}$$

i.e., the three-geometry is "conical" (Ref. 73, p. 14). This together with (7.4) satisfies also (4.13). Furthermore, the insertion of (7.11) into the radial equation (3.15) yields

$$\frac{R''}{R'} \simeq -\frac{1}{\rho} \,. \tag{7.12}$$

Its integration results in (7.9), the integration constant already being determined by (7.10). The sets (7.4), (7.9), and (7.10) of asymptotic solutions have earlier been discussed by Feinblum and McKinley.[46]

In spite of the fact that in the latter case the radial part of the scalar waves is logarithmically divergent at origin, we should not disregard these solutions. For a more precise reasoning we have to also take the strongly deformed space-time manifold at the origin into account. This effect becomes more transparent if we consider the function $F_0^\omega(\rho^*)$ defined via (3.8) in terms of the tortoise coordinate (3.9). In view of (7.4) and (7.12) the latter acquires the asymptotic behavior

$$\rho^* \simeq \frac{1}{A_0}\, \rho^2 \tag{7.13}$$

near the origin. By applying l'Hospital's rule we find that

$$F_0^\omega(\rho^*) \simeq \left(\frac{3}{8}\right)^{1/2} \frac{\ell}{\ell^*}\,(A_0\rho^*)^{1/2} \ln[A_0\rho^*)^{1/2}] \to -\left(\frac{3}{2}\right)^{1/2} \frac{\ell}{\ell^*}\,\rho \to 0 \tag{7.14}$$

tends to zero for $\rho \to 0$. Therefore the charge integral (5.5) should be *bounded* even at the origin. Since the subsidiary conditions of bounded square integrability and fast decrease at spatial infinity turn out to be fulfilled, solutions with the asymptotics 0_{II} in curved space-time can also be regarded as "eigensolutions," according to the criteria of quantum mechanics (see Sec. 16 of Ref. 38). With respect to the formal Schrödinger-type equation (3.12) the dynamics corresponds to the motion of a particle in a centrally symmetric field characterized by a centripetal potential

$$V_{\text{eff}}(\rho^*) \simeq -\frac{e^{\nu_c}}{4\rho^{*2}} \tag{7.15}$$

near the origin (compare with Ref. 38, p. 109).

With this information at hand we have performed the numerical calculations on a DEC-PDP-10 computer using single precision NAG and IMSL Library subroutines. The evaluation of the functions $R_l^\omega(\rho)$, $e^{\nu(\rho)}$, and $e^{\lambda(\rho)}$ has been done in double-precision mode. For all calculations the free parameters of the model have been fixed according to

$$\Lambda = 0, \quad \overline{\epsilon} = \epsilon = 1, \quad n = 3, \quad k = 1,$$

$$l = 0, \quad \ell^*/\ell = 1, \quad \beta = 0.2 \,.$$

Then for each given ω and n_r the system of ordinary differential equations (3.15), (4.13) and (4.14) has been numerically solved by Runge-Kutta formulas of order 5 and 6 as developed by Verner and coded by Hull and co-workers[49] in the IMSL subroutine DVERK. The global error of the solutions has been estimated to be less than 10^{-4}. Using the asymptotic conditions (7.1) and (7.2) as starting values the integration has been performed going from $\rho_\infty = 30$ backwards to zero. Self-consistent solutions are constrained by two additional conditions. First, they have to fulfill the charge normalization (5.5) and second, they have to reproduce the parameters α chosen for the initlal conditions consistent with the Tolman integral (5.12) for α. This has been achieved by an iterative least-squares fit using the NAG subroutine EØ4FAF which is based on a method due to Peckham.[50] Thus the parameters α and C_∞ have been determined by minimizing the sum of squares

$$\Sigma_j = \left(\frac{k - Q_j}{k}\right)^2 + \left(\frac{\alpha_{j-1} - \alpha_j}{\alpha_{j-1}}\right)^2 , \tag{7.16}$$

where j denotes the jth iteration step, α_{j-1} is the result of the $(j-1)$th step which fixed the initial conditions (7.1) and (7.2) and Q_j, α_j are calculated from (5.5) and (5.12), respectively.

Using appropriate starting values for C_∞ and α, a convergence of (7.16) better than 10^{-8} has been obtained by the method resulting in a relative error in α and Q of less than 10^{-4}. The numerical integration of Q, α, and E_{reg} has been performed by using the NAG Library procedure DØ1GAF which estimates the value of a definite integral (when the function is specified numerically) using the method described by Gill and Miller.[51] The maximal relative error should in no case be larger than 10^{-6}.

As can be deduced from the asymptotic solutions (7.10) and (7.12) of the second set 0_{II}, the energy expression (5.14) contains a term $\ln \rho_0$ which diverges for $\rho_0 \to 0$. Therefore in Fig. 1 we have computed only the finite part of E, i.e., E_{reg}, corresponding to the cut-off parameter $\rho_0 = 0.001$.

So far our method of integrating backward has produced solutions belonging solely to the set 0_{II} of asymptotic solutions at the origin. In this case we found solutions with and without nodes. The number of nodes of the radial part $R_l^\omega(\rho)$ of the wave function for finite values of ρ (excluding the point $\rho = 0$) may be used to define a radial quantum number n_r as in the nonrelativistic Schrödinger theory (Ref. 38, p. 109). From the theory of the hydrogen atom we suspect the relation

$$n_r = n - l - 1 \tag{7.17}$$

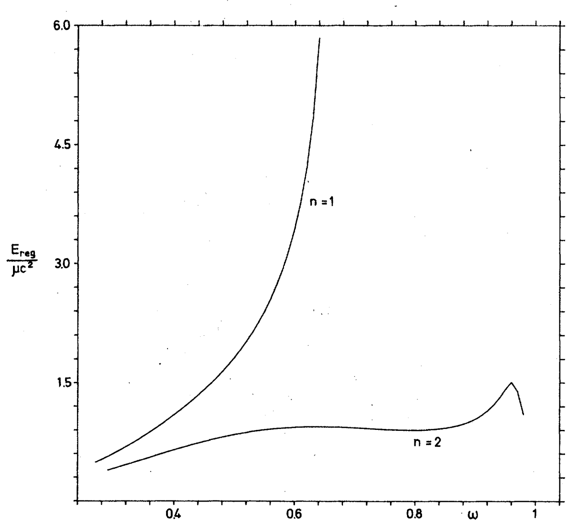

FIG. 1. Regularized field energy of the scalar waves within scalar geons as a function of ω, corresponding to a cutoff length $\rho_0 = 0.001$.

to hold (Ref. 38, p. 123) where n would denote the principal quantum number of the geon.

Figs. 2 and 3 reveal that our numerical results interpolate rather well between the asymptotic solutions at infinity and at the origin (set 0_{II}). In the case without nodes [Fig. 2(a)] the radial scalar function $R_0^\omega(\rho)$ joins smoothly the asymptotic solution (7.9) with the localized solution (7.1). Both metric functions show a Schwarzschild-type behavior for large ρ. For small ρ, e^ν becomes constant [Figs. 2(b) and 3(b)] similar as in the constant-core case, whereas e^λ tends to zero in accordance with (7.11). The latter function develops in between a noticeable peak [Figs. 2(c) and 3(c)] which corresponds to the confining barrier in the effective curvature potential (3.13) [Figs. 2(e) and 3(e)].

An interesting phenomenon can be observed on the level of the Schrödinger-type wave function $F_0^\omega(\rho^*)$ being defined with respect to the "pseudoflat" space-time (3.10). $F_0^\omega(\rho^*)$ is concentrated [see Figs. 2(d) and 3(d)] within the negative well of $V_{\text{eff}}^0(\rho^*)$ with its maximum close to the zero of the potential. For smaller values of ω this maximum is shifted by the barrier of the curvature potential closer to the origin. This seems to indicate a *self-generating* effect of the geometrodynamical confinement mechanism (being here only partial). This confinement scheme and its proposed[13] extension including color may be compared with, e.g., the MIT bag model[33] (see also Hasenfratz and Kuti[52] for a review). There an *ad hoc* introduction of a vacuum pressure term Λ_{bag} is needed to compensate for the outside directed pressure of the quark gas. In contrast to this phenomenological device our approach resembles rather Creutz's[53] reconstruction of a bag model from local nonlinear field theory. Similar to his, the core of our bag is produced by employing the stable (quantum-mechanically meastable solution) of the HKG equation for an extended part of the space. Surrounding this core is a transition region, the skin of the bag, consisting of an exponentially decreasing Yukawa-type radial solution for the scalar field and a Coulomb-type potential for the tensor gluons.

The total gravitational mass (5.12) as measured at infinity exhibits a branching for the zero- and one-node solutions with respect to its ω dependence (Fig. 4). For $n_r = 0$ and low ω we may understand the qualitative behavior of $\alpha(\omega)$ by comparing it with (see Ref. 54)

$$\tilde{\alpha}(\omega) = \frac{k}{\beta N}(\omega - 2\beta N \omega^3) , \tag{7.18}$$

but for higher values of ω Eq. (5.12) tends to

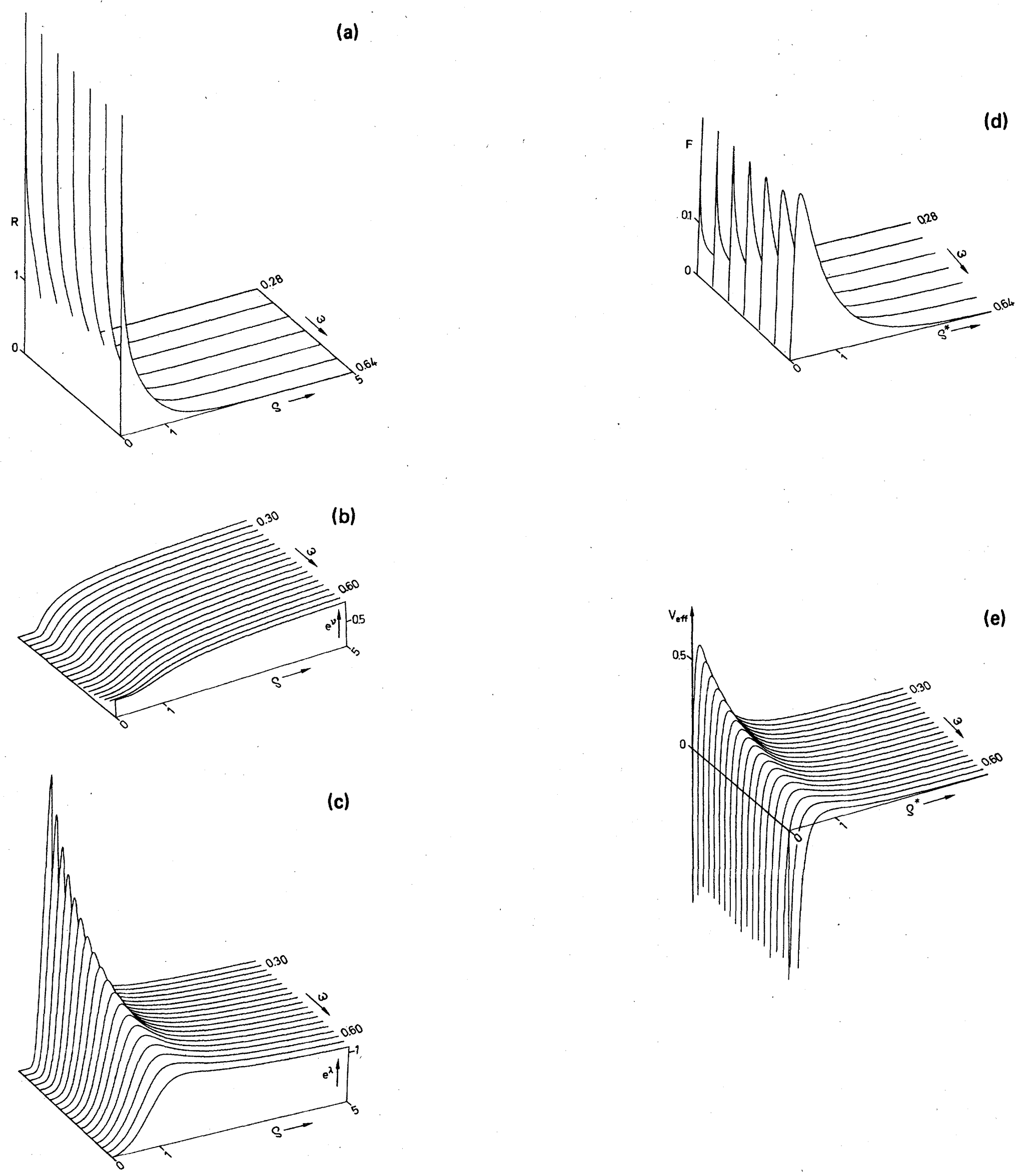

FIG. 2. Scalar geon *without* node. The results of the numerical integration are shown for the case $\ell^*/\ell=1$, $\beta=0.2$, $\Lambda=0$, $\bar{\epsilon}=\epsilon=1$, $n=3$, $k=1$, $l=0$. The solutions which depend on ω and ρ or ρ^*, respectively, are presented as relief. (a) Radial solutions $R_0^{\omega}(\rho)$. (b) Timelike metric function $e^{\nu(\rho)}$. (c) Spacelike metric function $e^{\lambda(\rho)}$. (d) Schrödinger-type wave function $F_0^{\omega}(\rho^*)$ given in terms of the "tortoise" coordinate ρ^*. (e) Effective curvature potential $V^0_{\rm eff}(\rho^*)$ exhibiting a deep well together with a confining barrier. [In Figs. 2(a) and 2(d) only a few solutions are shown in order to avoid a too strong screening in the three-dimensional plot which would otherwise occur. In Figs. 2(d) and 2(e) the curves are plotted up to ρ^* values corresponding to $\rho=5$.]

$$\tilde{\alpha}\simeq\frac{k}{\beta N}\left(\omega-\frac{1}{2\omega}\right). \tag{7.19}$$

The resulting predictions for the zero and the maximum of the Tolman energy, i.e.,

$$\omega_0=(\tfrac{1}{2})^{1/2}=0.71\ . \tag{7.20}$$

and

$$\omega_{\max}=(6\beta N)^{-1/2}=0.53\ , \tag{7.21}$$

$$\alpha(\omega_{\max})=0.59\ , \tag{7.22}$$

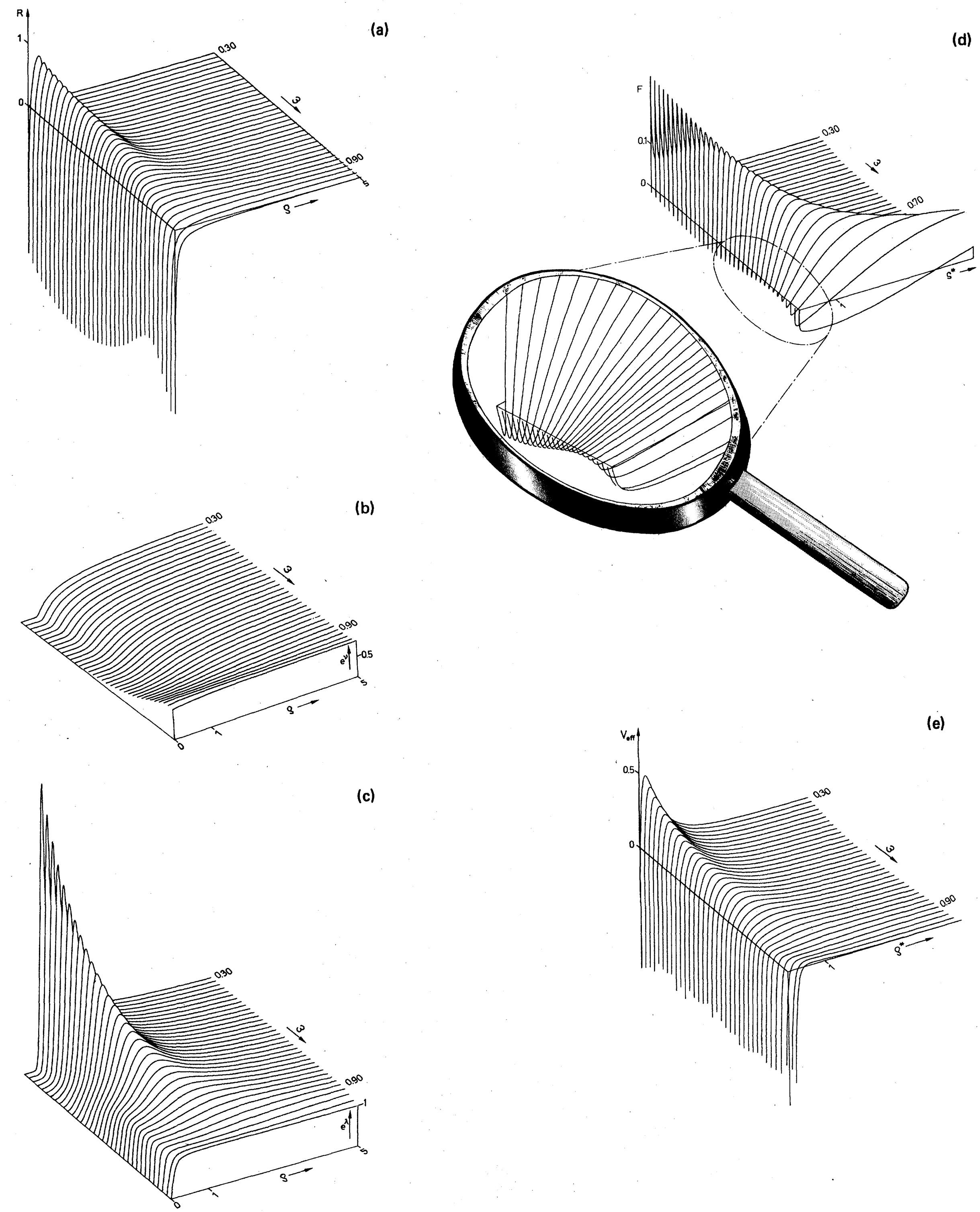

FIG. 3. Scalar geon with *one* node. Same case and presentation as in Fig. 2, except for a broader ω range. (a) Radial solutions $R^{\omega}_{0}(\rho)$ exhibiting one node. (b) Timelike metric function $e^{\nu(\rho)}$. (c) Spacelike metric function $e^{\lambda(\rho)}$. (d) Schrödinger-type wave function $F^{\omega}_{0}(\rho^*)$ having also one node outside the origin [see the magnification ($2\times$ in F and $50\times$ in ρ^*) of part of the relief]. (e) Effective curvature potential $V^{0}_{\rm eff}(\rho^*)$ with a deep well and a confining barrier. [Again in Figs. 3(d) and 3(e) the curves are plotted up to ρ^* values corresponding to $\rho=5$.]

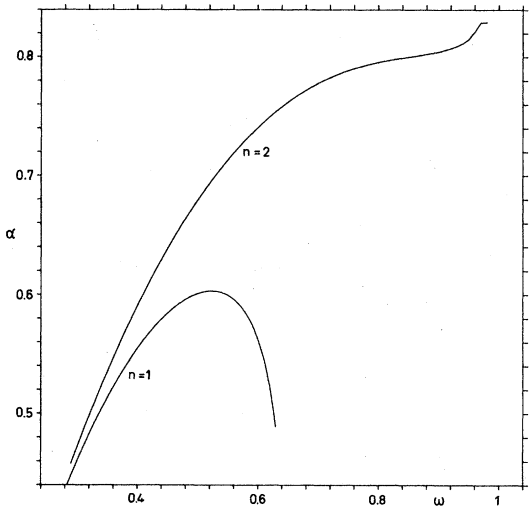

FIG. 4. Tolman energy or Schwarzschild mass αM^* as a function of ω for quantized scalar geons with "principal-quantum number" $n=1,2$.

respectively, agree quite well with our results (Fig. 4).

Already for $n_r=1$ and higher-node solutions we found parts of several sub-branches in $\alpha(\omega)$, i.e., the Tolman energy of these geons deviates in a noticeable way (not shown in Fig . 4). In a preliminary study we could distinguish the corresponding solutions among others by the number of nodes in R''.

Further analysis is in progress in order to understand these highly interesting instances of a possible fine structure in the energy levels of the geons. In view of these rich and prospective structures does there exist the speculative alternative to interpret extended particles as quantum geons[26] capable of internal excitations?

VIII. OTHER GRAVITATIONAL SOLITONS

To some extent Wheeler's[24] geon concept has anticipated the (nonintegrable) soliton solutions[45] of classical nonlinear field theories. As mentioned in the Introduction a geon or gravitational soliton originally was meant to consist of a spherical shell of electromagnetic radiation held together by its own gravitational attraction. In the idealized case of a thin spherical geon the corresponding metric functions have the values $e^{\nu_c}=\frac{1}{9}$ well inside and $e^{\nu}=e^{-\lambda}=1-\alpha/\rho$ well outside the active region. The trapping area for the electromagnetic wave trains has a radius of $\rho_{\text{active}}=\frac{9}{8}\alpha$. This result has been confirmed by applying Ritz variational principles.[55]

Constructions with toroidal or linear electromagnetic waves have been given by Ernst[55] whereas neutrino geons have been analyzed by Brill and Wheeler.[29]

Brill and Hartle[56] could even demonstrate the existence of pure gravitational solitons. By expanding the occurring gravitational waves in terms of tensor spherical harmonics[57] it can be shown that the radial function experiences the same effective potential (3.13) except that an additional factor $\frac{3}{2}$ appears in front of the contribution from the background metric.

In a relevant paper the coupling of linear Klein-Gordon fields to gravity has been numerically studied by Kaup.[31] Moreover, the problem of the stability of the resulting scalar geons with respect to radial perturbations is treated. It is shown that such objects are resistant to gravitational collapse (related works include Refs. 58, 46, and 48). These considerations are on a semiclassical level. However, using a Hartree-Fock approximation for the second quantized two-body problem it can be shown[54] that the same coupled Einstein-Klein-Gordon equations apply.

As an important new ingredient, Kodama *et al.*[42] have been considered a real scalar field with a φ^4 self-interaction as a source for the gravitational field. In this preliminary study the Klein-Gordon operator corresponding to flat space-time is assumed.

A general relativistic Klein-Gordon field with an effective φ^3 self-interaction for an interior ball has also been analyzed by this group.[59] In order to avoid a singular configuration at origin a repulsive (or "ghostlike") scalar field has been chosen as a source of Einstein's equations. In a further step Kodama[60] constructed a spherically symmetric kink-type solution for a repulsive scalar field with φ^4 self-coupling (compare also with Ellis[61]). As is common for kinks, the radial function at spatial infinity is chosen to be a constant characterising this nonlinear model. If we want to transfer the method to our case, we may use instead of (2.3) the symmetric self-interaction

$$\tilde U(|\varphi|^2) \equiv \frac{3}{4^4}\,\ell^4\left(|\varphi|^2-\frac{16\epsilon}{3\ell^2}\,\frac{\mu c}{\hbar}\right)^3$$

$$=U(|\varphi|^2)-\frac{16\epsilon}{9\ell^2}\left(\frac{\mu c}{\hbar}\right)^3 \text{ for } \bar\epsilon=1. \qquad (8.1)$$

The additional constant in (8.1) necessarily eliminates the gravitational source which otherwise would occur for the constant solution

$$\varphi_c^{(N)}=\frac{4}{\ell}\left(\frac{\epsilon\mu c}{3\hbar}\right)^{1/2}e^{i\delta},\quad \varphi^{(q)}=0 \text{ if } q\neq N \qquad (8.2)$$

characterizing the kink solution asymptotically. In flat Minkowski space it is conjecture that (8.2) constitutes a first approximation to the vacuum expectation value $\langle 0|\Phi|0\rangle$ of the corresponding quantum field.

For the general-relativistic kink of Kodama[60] the radial solution becomes zero at a certain radius r_0 at which the background geometry develops a Schwarzschild-type horizon. (Geon-type solutions exhibiting an event horizon may be termed "black solitons."[62]) The boundary condition at r_0, however, allows an extension of these solutions into a three-manifold consisting of two asymptotically Euclidean spaces connected by an Einstein-Rosen bridge.[25] Arguments are given[63] that this extended, nonsingular configuration is stable with respect to radial oscillations.

It should be noted that such solutions cannot be constructed for the wormhole topology $R\times S^1\times S^2$ which would be obtainable by identifying the asymptotically flat regions. The reason simply being that the radial functions of the kink has an opposite sign in the other sheet of the Universe. Since the quantum-mechanical probability density $|\varphi(x)|^2$ is single valued and completely regular also for the wormhole topology, such scalar kinks provide interesting objects for further studies.

Although we have no intention to give a complete review, we would like to mention that other studies on geons involve massless scalar fields,[64–67] coupled Einstein-Maxwell-Klein-Gordon systems,[58,68–70] or even combined Dirac-Einstein-Maxwell field equations.[71]

As a final observation we remark that, according to a result of Brill[72] a massless scalar field can be geometrized in the sense of the already unified field theory or geometrodynamics of Rainich, Misner, and Wheeler.[37] Loosely speaking, this means that the scalar field can be completely read off from the "footprints" it leaves on the geometry.

ACKNOWLEDGMENTS

We would like to thank Professor R. Ruffini and Professor Abdus Salam for useful hints. One of us (E.W.M.) would like to express his sincere gratitude to Professor F. W. Hehl, Professor Abdus Salam, and Professor J. A. Wheeler for encouragement and support, and would also like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste. This work was supported by the Deutsche Forschungsgemeinschaft. The computations were performed at the Rechenzentrum of the Christian-Albrechts-Universität, Kiel.

[1]J. J. J. Kokkedee, *The Quark Model* (Benjamin, New York, 1969).
[2]R. N. Boyd, D. Elmore, A. C. Melissinos, and E. Sugarbaker, Phys. Rev. Lett. 40, 216 (1978).
[3]O. W. Greenberg and C. A. Nelson, Phys. Rep. 32C, 69 (1977).
[4]D. Friedberg, T. D. Lee, and A. Sirlin, Phys. Rev. D 13, 2739 (1976); Nucl. Phys. B115, 1 (1976); B115, 32 (1976).
[5]W. Marciano and H. Pagels, Phys. Rep. 36C, 137 (1978).
[6]A. Salam and J. Strathdee, Phys. Lett. 67B, 429 (1977).
[7]A. Salam and J. Strathdee, Phys. Rev. D 18, 4596 (1978).
[8]C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).
[9]S. Coleman and L. Smarr, Commun. Math. Phys. 56, 1 (1977).
[10]D. Robson, Nucl. Phys. B130, 328 (1977).
[11]E. W. Mielke, Phys. Rev. Lett. 39, 530 (1977); 39, 851 (E) (1977).
[12]E. W. Mielke, in Proceedings of the 1978 Clausthal meeting, Lecture Notes in Physics, edited by H. D. Doebner (Springer, Berlin, to be published).
[13]E. W. Mielke, Int. J. Theor. Phys. 19, 189 (1980).
[14]C. J. Isham, A. Salam, and J. Strathdee, Phys. Rev. D 3, 867 (1971).
[15]C. J. Isham, A. Salam, and J. Strathdee, Phys. Rev. D 8, 2600 (1973).
[16]C. J. Isham, A. Salam, and J. Strathdee, Phys. Rev. D 9, 1702 (1974).
[17]F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. M. Nester, Rev. Mod. Phys. 48, 393 (1976).
[18]H. Weyl, Proc. Nat. Acad. Sci. (USA) 15, 323 (1929).
[19]H. Weyl, Phys. Rev. 77, 699 (1950).
[20]W. Heisenberg, *Introduction to the Unified Field Theory of Elementary Particles* (Wiley, London, 1966).
[21]E. Schrödinger, Sitzungsber. Preuss. Akad. Wiss. Phys.-Math. Kl. XI, 105 (1932).
[22]D. Ivanenko, Sov. Phys. 13, 141 (1938); Suppl. Nuovo Cimento 6, 349 (1957).
[23]R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Clarendon, Oxford, 1934).
[24]J. A. Wheeler, Phys. Rev. 97, 511 (1955).
[25]A. Einstein and N. Rosen, Phys. Rev. 48, 73 (1935).
[26]J. A. Wheeler, Rev. Mod. Phys. 33, 63 (1961).
[27]R. J. Finkelstein, Phys. Rev. 75, 1079 (1949).
[28]R. Finkelstein, R. LeLevier, and M. Ruderman, Phys. Rev. 83, 326 (1951).
[29]D. R. Brill and J. A. Wheeler, Rev. Mod. Phys. 29, 465 (1957); 33, 623 (E) (1961).
[30]W. Deppert and E. W. Mielke, Phys. Rev. D 20, 1303 (1979).
[31]D. J. Kaup, Phys. Rev. 172, 1331 (1968).
[32]R. C. Tolman, Phys. Rev. 35, 875 (1930).

[33]A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, and V. F. Weisskopf, Phys. Rev. D 9, 3471 (1974).
[34]P. Boyanovsky and L. Masperi, Phys. Rev. D 21, 1550 (1980).
[35]D. L. T. Anderson, J. Math. Phys. 12, 945 (1971).
[36]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973) (quoted as MTW).
[37]J. W. Wheeler, *Geometrodynamics* (Academic, New York, 1962) (contains reprints of Refs. 24 and 41).
[38]L. D. Landau and E. M. Lifschitz, *Quantum Mechanics-Non-Relativistic Theory* (Pergamon, London, 1967).
[39]E. W. Mielke, Gen. Relativ. Gravit. 8, 321 (1977).
[40]H. Rund and D. Lovelock, Jber. Deutsch. Math.-Verein 74, 1 (1972).
[41]E. A. Power and J. A. Wheeler, Rev. Mod. Phys. 29, 480 (1957).
[42]T. Kodama, K. C. Chung, and A. F. da F. Teixeira, Nuovo Cimento 46, 206 (1978).
[43]R. Jackiw, Rev. Mod. Phys. 49, 681 (1977).
[44]S. A. Fulling, Phys. Rev. D 7, 2850 (1973).
[45]V. G. Makhankov, Phys. Rep. 35C, 1 (1978).
[46]D. A. Feinblum and W. A. McKinley, Phys. Rev. 168, 1445 (1968).
[47]E. W. Mielke, J. Math. Phys. 21, 543 (1980).
[48]A. F. da F. Teixeira, I. Wolk, and M. M. Som, Phys. Rev. D 12, 319 (1975).
[49]T. E. Hull, W. H. Enright, and K. R. Jackson, Department of Computer Science, University of Toronto, Report No. TR No. 100 1976 (unpublished).
[50]G. Peckham, Computer J. 13, 418 (1970).
[51]P. E. Gill and G. F. Miller, Computer J. 15, 80 (1972).
[52]P. Hasenfratz and J. Kuti, Phys. Rep. 40C, 75 (1978).
[53]M. Creutz, Phys. Rev. D 10, 1749 (1974).
[54]R. Ruffini and S. Bonazzola, Phys. Rev. 187, 1767 (1969).
[55]F. J. Ernst, Jr., Phys. Rev. 105, 1662 (1957); 105, 1665 (1957).
[56]D. R. Brill and J. B. Hartle, Phys. Rev. 135, B271 (1964).
[57]T. Regge and J. A. Wheeler, Phys. Rev. 108, 1063 (1957).
[58]A. Das, J. Math. Phys. 4, 45 (1963).
[59]K. C. Chung, T. Kodama, and A. F. da F. Teixeira, Phys. Rev. D 16, 2412 (1977).
[60]T. Kodama, Phys. Rev. D 18, 3529 (1978).
[61]H. G. Ellis, Gen. Relativ. Gravit. 10, 105 (1979).
[62]A. Salam and J. Strathdee, Phys. Lett. 61B, 375 (1976); 66B, 143 (1977).
[63]T. Kodama, L. C. S. de Oliveira, and F. C. Santos, Phys. Rev. D 19, 3576 (1979).
[64]S. Deser and J. Higbie, Ann. Phys. (N. Y.) 58, 56 (1970).
[65]J. D. Bekenstein, Ann. Phys. (N. Y.) 82, 535 (1974).
[66]M. Gürses, Phys. Rev. D 15, 2731 (1977).
[67]M. Lunetta, I. Wolk, and A. F. da F. Teixeira, Phys. Rev. D 21, 3281 (1980).
[68]K. A. Bronnikov, V. N. Melnikov, G. N. Shikin, and K. P. Staniukovich, Ann. Phys. (N. Y.) 118, 84 (1979).
[69]A. F. da F. Teixeira, I. Wolk, and M. M. Som, J. Phys. A 9, 53 (1976).
[70]A. Banerjee and S. B. Dutta Choudhury, Phys. Rev. D 15, 3062 (1977).
[71]M. J. Hamilton and A. Das, J. Math. Phys. 18, 2026 (1977).
[72]D. Brill, Nuovo Cimento Suppl. 2, 1 (1964).
[73]B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, *Gravitation Theory and Gravitational Collapse* (University of Chicago Press, Chicago, 1965).