Analytic mappings: A new approach to quantum field theory in accelerated frames

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A new approach to quantum field theory in two-dimensional accelerated frames is presented, in which classical, quantum, and thermal aspects of the theory are explicitly expressed in terms of analytic mappings. Each analytic function defines an accelerated frame and a quantum production rate associated with it. Conversely, given the quantum production rate, we reconstruct the mapping. Real singularities of the mapping determine the asymptotic regions of the space-time. Each real singularity has an associated temperature. In particular, critical points define event horizons. Temperature appears irrespective of the presence of event horizons. We classify the accelerated frames in terms of the singularities of analytic mappings and their associated quantum production rates and temperatures. It is proved that the only accelerated frame in which particle production takes place in a global thermal equilibrium situation is the Rindler frame. Nevertheless, the Planckian spectrum is not restricted to Rindler observers. Comparison with the existing literature is made. Generalizations of the approach are considered, for instance its extension to four-dimensional curved space-time.

I. INTRODUCTION

Since the discovery by Hawking[1] of the quantum thermal radiance by black holes, particle production by accelerated observers is a subject of great interest to which a considerable amount of work has been devoted (see, for example, Refs. 2–5).

It is by now well known that the vacuum and thermal effects characteristic of Hawking emission are not exclusive to black holes nor to the curved space-time itself. They appear even in flat space-time when quantum fields are described in terms of accelerated coordinates. The extensive literature discussing this problem is essentially centered around one particular example: the Rindler frame. As is known,[6,7] it describes uniform acceleration and has as a vacuum spectrum a purely Planckian formula. Until now no general description of quantum fields in accelerated coordinates involving nonuniform accelerations (or non-Planckian spectra) has been given. On the other hand, no explicit relation between the thermal aspects appearing in this problem and the structure of the space-time has been obtained. It was accepted that the temperature in this context is due to the presence of event horizons. In this paper we give a new approach to the subject. A large set of new results not contained in the earlier literature has been obtained. We deal here with two-dimensional flat space-time and free massless scalar fields. It exhibits the essential features of our approach and covers a rich class of situations. It can be applied to four-dimensional curved space-time and generalized to massive and interacting fields. The grounds of our approach are given in what follows: From analytic mappings considered in Euclidean space we construct a wide class of accelerated frames in which self-adjointness of propagation equations and completeness and orthogonality of their solutions are guaranteed. The Rindler frame is contained here as a particular case. Real singularities of the mappings determine the asymptotic regions of the space-time. In particular, critical points determine event horizons. We formulate free massless scalar quantum field theory in such accelerated frames. A Bogoliubov transformation relates the positive-frequency λ modes with respect to the accelerated time coordinate t' to the positive-frequency k modes with respect to the inertial time t. We explicitly express Bogoliubov coefficients $[A_\lambda(k), B_\lambda(k)]$ and the set of vacuum spectra, i.e., the number of created λ modes per unit frequency $[N(\lambda)]$ and per unit volume $[N_V(\lambda)]$, the energy $[H(x',t')]$ and momentum $[P(x',t')]$ densities and the total vacuum energy $[E]$ in terms of analytic mappings [Eqs. (3.16)–(3.18)]. In this way, relevant properties of the quantum spectra can be obtained by analyzing the properties of the mappings. Conversely, given the vacuum production rate, we reconstruct the mapping [Eq. (3.19)]. All the accelerated frames defined on an orbit of the bilinear transformation group [O(2, 1)] have the same vacuum spectra. It is proved that the only accelerated frame in which there is a global thermal equilibrium situation over the whole space-time is the Rindler frame. However, the Planckian spectrum $N_V(\lambda)$ is not restricted to Rindler observers. A wide class of nonuniformly accelerated frames have been associated with it (see also Refs. 8 and 9).

Despite being in the context of pure (not mixed) quantum states, there are thermal features in the theory. This fact is due to the noninertial character of classical observers. All the accelerated frames constructed by our procedure entail two characteristic parameters ($T_\pm$) which play the role of temperatures. We relate temperature directly to the mapping [Eq. (3.30)]. Each asymptotic region of the space-time has an associated temperature. Temperature appears as a concept irrespective of the presence of event horizons (in contrast, event horizons are relevant to the concept of entropy[10]). The spatial infinity contributes to the vacuum spectrum with the same thermal properties as an event horizon, each asymptotic region contributing in an independent fashion.

We classify the accelerated frames in terms of the singularities of the analytic mappings defining them and their associated vacuum spectra and temperatures (see Table I and p. 14).

We discuss the vacuum-spectra interpretation in terms of the measurements of accelerated detectors. It can be pointed out that the interpretation of a positive-frequency state associated with accelerated time as being a real particle is nontrivial even in the Rindler case. In this case, Unruh has shown that the accelerated modes are real particles in the sense that a quantum detector moving with uniform acceleration detects them.[2] Rather than discuss here whether the detector measures real particles or vacuum field fluctuations (see Ref. 11), we ask ourselves the following question: Can one generalize Unruh-type detectors to include more general accelerated motions? The answer is affirmative. In this sense, Rindler observers (despite having a Killing vector) are not preferred.

In light of our approach we discuss preceding works on the subject. Comparison is made with the approaches which describe particle production by accelerated observers or by black holes by using moving boundaries (or mirrors).[4,12] The particle production described by such approaches is basically a Casimir-type effect. This boundary effect is essentially different from the acceleration effects considered by us despite giving the same result in the Rindler case. Only in this case, the mirror approach gives the same result as that obtained from quantum field theory in accelerated frames.

Finally, we discuss extensions and generalizations of our approach.

(i) Our approach can be extended to consider curvilinear coordinates in the forward light cone.

(ii) We generalize our approach to the case when two analytic mappings are involved. This allows us to describe accelerated frames for which the observers traveling to the right and to the left have different accelerations. The right- and left-going parts of the vacuum spectra are then different. Past and future horizons (or infinities) can be at different temperatures. Moreover, there can be a horizon in the past but no horizon in the future, etc.

(iii) Our approach is not restricted to the two-dimensional case. It naturally extends to four-dimensional curved space-time admitting two Killing vector fields. In this way, thermal aspects and particle production of the Kerr-Newman and Taub-NUT (Newman-Unti-Tamburino) families of metrics (with or without cosmological terms) and the de Sitter solution can be analyzed in the light of the results given here (see Ref. 10).

(iv) Our approach can be generalized to massive and interacting fields (see Refs. 13 and 14).

In this paper, Sec. II deals with the construction of the accelerated frames. Section III deals with the formulation of quantum field theory and its thermal aspects. In Sec. IV we discuss the physical interpretation. In Sec. V we compare with preceding work by other authors. In Sec. VI we outline the generalizations of our approach.

II. ANALYTIC MAPPINGS AND ACCELERATED FRAMES

We begin by considering two-dimensional space-time because it joins an interesting physical structure to its intrinsic simplicity. In such a space-time, we make a formal analytic continuation of the time variable t ($t\to i\tau$), making it imaginary. We look for real analytic functions

$$u=f(u') \tag{2.1}$$

as establishing a mapping between the points $u=x+i\tau$ of the Euclidean plane and the points $u'=x'+i\tau'$ of a transformed one. Mappings represented by Eq. (2.1) are conformal. As is well known, $|f'(u')|$ is the magnification factor of the transformation and $\arg f'(u')$ is the rotation angle. Points where $f'(u')=0$ are critical points. The transformation is not conformal there.

In the primed coordinates the metric takes the form

$$ds^2=|f'(u')|^2(dx'^2+d\tau'^2)\,. \tag{2.2}$$

The real critical points of the conformal transformation determine event horizons in Minkowski space-time. This is so because the two families of characteristic lines $x\pm t=\text{constant}=c$ coalesce upon going to Euclidean space to the points $(c,0)$, i.e., the real axis of the Euclidean plane. In particular, the frontiers of the light cone coalesce to the origin.

In Minkowski space-time, the mapping

$$x\pm t=f(x'\pm t') \tag{2.3}$$

transforms characteristic lines into characteristic lines. Physically, the mapping represents a transformation from an inertial frame (x,t) to an accelerated one (x',t'). It defines x' and t' as even and odd functions of t. Constant values of x' define the world lines of the accelerated observers in the (x,t) plane. The velocity of these observers is

$$v=\frac{f'(x'+t')-f'(x'-t')}{f'(x'+t')+f'(x-t')} . \tag{2.4}$$

The proper acceleration is given by

$$a=\frac{1}{[\Lambda(x',t')]^{1/2}}\partial_{x'}[\ln\Lambda(x',t')] , \tag{2.5}$$

where $\Lambda(x',t')=f'(x'+t')f'(x'-t')$.

This involves a very large class of accelerated motions. In particular, the Rindler frame[6] corresponds to the analytic mapping

$$f(u')=\beta e^{u'/\alpha} \tag{2.6}$$

and describes uniform acceleration. It can be pointed out that bilinear transformations

$$f(u')=\frac{\alpha u'+\beta}{\gamma u'+\delta}\ (\alpha\delta-\beta\gamma\neq 0) \tag{2.7}$$

for α, β, γ, δ real parameters, i.e., the O(2, 1) group, also describe uniform acceleration. However, whereas the function (2.6) maps the half axis $\mathrm{Re}u>0$ onto the full real axis u', this is not so for bilinear transformations. In accelerated frames whose analytic mappings do not cover the full x' axis, self-adjointness of propagation equations, completeness and orthogonality of their solutions cease to hold, unless additional artificial assumptions on the wave functions are imposed.

It will be possible to carry out a field quantization procedure in those accelerated frames defined by analytic functions which map one to one an interval $[u_-,u_+]\in \mathrm{Re}u$ into the whole real u' axis. That is, we require monotonic functions $f(u')$ such that

$$f(\pm\infty)=u_\pm , \tag{2.8}$$

where one of the $u_\pm$ or both u_+ and u_- can be infinite.

For finite $u_\pm$, conditions (2.8) imply

$$[f'(u')]_{u'=\pm\infty}=0 , \tag{2.9}$$

i.e., critical points of f lie at the ends of the real u' axis.

An accelerated frame defined from Eqs. (2.1) and (2.8) cover a bounded region (a rhombus)

$$u_-<|x\pm t|<u_+ \tag{2.10}$$

of Minkowski space-time; $x\pm t=u_-$ and $x\pm t=u_+$ represent two event horizons. These are the boundaries of the space-time domain over which the (x',t') coordinate system is defined. No event occurring outside this domain can causally communicate with the accelerated observers defined in it.

In particular, it can be $u_-=0$ and $u_+=+\infty$, in which case the accelerated frame covers the right-hand wedge of Minkowski space-time. If $u_\pm=\pm\infty$ there are no event horizons (the accelerated frame covers the whole Minkowski space-time).

In terms of the inverse function

$$u'=F(u) \tag{2.11}$$

conditions (2.8) read

$$F(u_\pm)=\pm\infty . \tag{2.12}$$

That is, $F(u)$ have real singularities at $u=u_\pm$. Real singularities besides $u_\pm$ are excluded in the interval $[u_-,u_+]$. $F(u)$ may have complex singularities but they do not influence event horizons in Minkowski space-time. Real singularities lying outside the interval $[u_-,u_+]$ represent events causally disjoint to the region (2.10). It can be noted that a single function $f(u')$ can define different accelerated frames. If $f(u')$ is multivalued, each branch $f^{(i)}(u')$ such that boundary conditions (2.8) are satisfied defines an accelerated frame. The different accelerated frames cover causally disjoined regions

$$u_-^{(i)}<|x\pm t|<u_+^{(i)} \tag{2.13}$$

of Minkowski space-time.

In accelerated frames defined by Eqs. (2.1) and (2.8), self-adjointness of propagation equations and orthogonality and completeness of their solutions are guaranteed.

Physically, conditions (2.8) mean that event horizons move at the speed of light (light rays take an infinite time t' to reach them). For $t'\to\pm\infty$ world lines of accelerated observers tend asymptotically to the characteristic lines $x\pm t=u_-$ and $x\pm t=u_+$, where its velocity given by Eq. (2.4) reaches the values $\pm c$.

It can be pointed out that the presence of event horizons for the accelerated observer causes a hole in its Euclidean space. Accelerated observers having one or two event horizons see a space with topology characterized by Euler number $\chi=0$. Accelerated observers having no event horizons or inertial observers see a space with topology of $\chi=1$. We recall that for a two-dimensional Riemannian manifold, the Euler number is given by[15,5]

$$2\pi\chi[\mathfrak{M}]=\tfrac{1}{2}\int_{\mathfrak{M}}d^2x\sqrt{g}R+\int_{\partial\mathfrak{M}}dy\sqrt{\tilde\gamma}K , \tag{2.14}$$

where g and $\tilde\gamma$ are the determinants of the metrics over the manifold $\mathfrak{M}$ and its boundary $\partial\mathfrak{M}$, respectively. R and K are the scalar curvature and the trace of the extrinsic curvature, respectively. A boundary at u_+ (u_-) contributes with a term of value +1 (−1). A boundary at $u=+\infty$ contributes with +1.

It can be noted that by means of bilinear transformations one can modify the position as well as the number of horizons. For instance, bilinear transformations such that $(\alpha\delta-\beta\gamma)>0$ and $\delta/\gamma>0$ lead from an accelerated frame defined in the interval $[0,\infty]\in \mathrm{Re}u$ to another one defined in the interval $[\beta/\delta, \alpha/\gamma]\in \mathrm{Re}u$. The accelerated frames related by the O(2, 1) transformations have different accelerations, but as we will see in the following section, all of them define the same quantum vacuum spectra

III. QUANTUM FIELDS IN ACCELERATED FRAMES

We proceed now to quantize a free massless scalar field $\hat{\Psi}$ in the accelerated frames described in the preceding section.

A complete set of solutions of the conformally invariant wave equation

$$\Box\hat{\Psi}=0 \tag{3.1}$$

is given by

$$\phi_\lambda(u')=\frac{1}{2(\pi\lambda)^{1/2}}e^{i\lambda u'}\,,\qquad \phi_\lambda^*(v')=\frac{1}{2(\pi\lambda)^{1/2}}e^{-i\lambda v'}\,, \tag{3.2}$$

where $u'=x'-t'$, $v'=x'+t'$, and $\lambda>0$. Because of Eq. (2.12), they are orthogonal with the scalar product

$$\langle\phi_1,\phi_2\rangle=i\int\phi_1^*\overleftrightarrow{\mathrm{j}}^\mu\phi_2 d\Sigma_\mu\,, \tag{3.3}$$

where

$$\overleftrightarrow{\mathrm{j}}^\mu=\sqrt{g}g^{\mu\nu}\overrightarrow{\partial}_\nu-\overleftarrow{\partial}_\nu g^{\mu\nu}\sqrt{g}\,.$$

A Bogoliubov transformation

$$C_\lambda=\int_0^\infty dk[A_\lambda(k)a_k+B_\lambda(k)a_k^\dagger] \tag{3.4}$$

with coefficients

$$B_\lambda(k)=\langle\phi_\lambda,\varphi_k^*\rangle\,,\quad A_\lambda(k)=\langle\phi_\lambda,\varphi_k\rangle \tag{3.5}$$

relates the annihilation operators C_λ of the modes ϕ_λ (positive-frequency modes with respect to the time t') to the annihilation and creation operators a_k, $a_k^\dagger$ of the modes φ_k (positive-frequency modes with respect to the time t). Here

$$\varphi_k(u')=\frac{1}{2(\pi|k|)^{1/2}}e^{iku'},\qquad \varphi_k^*(v')=\frac{1}{2(\pi|k|)^{1/2}}e^{-ikv'}\,. \tag{3.6}$$

$C_\lambda|0'\rangle=0$ for all λ does not define the vacuum state of the theory. This is defined by

$$a_k|0\rangle=0\quad\forall k\,. \tag{3.7}$$

We define

$$N(\lambda,\lambda')=\langle 0|C_{\lambda'}^\dagger C_\lambda|0\rangle=\int_0^\infty dk\,B_\lambda^*(k)B_{\lambda'}(k)\,, \tag{3.8}$$

which we call the production function. For $\lambda=\lambda'$, it gives the number $N(\lambda)$ of quanta of frequency λ in the inertial vacuum state on the total volume. The number $N_V(\lambda)$ of created modes per unit momentum and volume can be obtained by introducing wave packets, that is, wave functions normalized in a unit volume

$$N_V(\lambda)=\lim_{y\to\infty}\int_0^\infty\int_0^\infty d\lambda' d\lambda'' g_y(\lambda,\lambda')g_y^*(\lambda',\lambda'')N(\lambda,\lambda')\,, \tag{3.9}$$

where $g_y(\lambda,\lambda')$ is such that

$$\int_0^\infty d\lambda|g_y(\lambda,\lambda')|^2=1\,.$$

For instance

$$g_y(\lambda,\lambda')=(2y/\pi)^{1/2}\exp[-y(\lambda-\lambda')^2]\,.$$

Besides $N(\lambda,\lambda')$ we can calculate the vacuum energy $[H]$ and momentum $[P]$ densities. These are the vacuum mean values of the $\hat{T}_{00}$ and $\hat{T}_{01}$ components of the energy-momentum tensor

$$\hat{T}_{\mu\nu}=\partial_\mu\hat{\psi}\partial_\nu\hat{\psi}-\tfrac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\hat{\psi}\partial_\sigma\hat{\psi}\,,$$

i.e.,

$$H(x',t')=\tfrac{1}{2}\langle 0|:(\partial_{t'}\hat{\psi})^2+(\partial_{x'}\hat{\psi})^2:|0\rangle\,,$$

$$P(x',t')=-\langle 0|:(\partial_{t'}\hat{\psi})(\partial_{x'}\hat{\psi}):|0\rangle\,,$$

where : : stands for normal ordering with respect to the creation and annihilation operators in the accelerated frames.

From Eqs. (3.2) and (3.8) and by using

$$\hat{\Psi}=\int_0^\infty d\lambda(C_\lambda\phi_\lambda+C_\lambda^\dagger\phi_\lambda^*)\,,$$

it follows that

$$H(x',t')=\tfrac{1}{2}[H(x'-t',0)+H(x'+t',0)]\,, \tag{3.10}$$

$$P(x',t')=\tfrac{1}{2}[H(x'-t',0)-H(x'+t',0)]\,, \tag{3.11}$$

where

$$H(x',0)=\frac{\mathrm{Re}}{\pi}\int_0^\infty\int_0^\infty d\lambda\,d\lambda'(\lambda\lambda')^{1/2}[e^{i(\lambda-\lambda')x'}N(\lambda,\lambda')-e^{i(\lambda+\lambda')x'}R(\lambda,\lambda')]\,. \tag{3.12}$$

Here

$$R(\lambda,\lambda')=\langle 0|C_\lambda C_{\lambda'}|0\rangle=\int_0^\infty dk\,A_\lambda(k)B_{\lambda'}(k)\,.$$

H and P satisfy

$$\partial_{t'} H(x', t') + \partial_{x'} P(x', t') = 0 . \qquad (3.13)$$

Asymptotically, for $x' \to \pm\infty$, the main contribution to $H(x')$ comes from the values $\lambda \sim \lambda'$. $N(\lambda, \lambda')$ and $R(\lambda, \lambda')$ describe interferences between the created modes with different frequencies λ, λ'. These interferences cancel over the whole volume as can be seen from the relation

$$E = \int_{-\infty}^{\infty} H(x')dx' = \int_0^{\infty} \lambda N(\lambda) d\lambda \qquad (3.14)$$

for the total energy E. The total momentum of created modes over the whole space is zero. All these quantities can also be expressed in terms of the mapping $F(u)$. First, we express Bogoliubov coefficients as Laplace transformations of the accelerated wave functions, i.e.,

$$B_\lambda(k) = \frac{1}{2\pi}\left(\frac{k}{\lambda}\right)^{1/2} \mathcal{L}_\lambda(-ik) , \quad A_\lambda(k) = iB_\lambda(-k) ,$$
$$\mathcal{L}_\lambda(s) = \int_0^{\infty} du\, e^{-su - i\lambda F(u)} . \qquad (3.15)$$

For real λ, $B_\lambda(k)$ $[A_\lambda(k)]$ as a function of k is analytic in the region $\mathrm{Im}k < 0$ $[\mathrm{Im}k > 0]$ and in the domain $\mathrm{Im}k/\mathrm{Re}k < \epsilon[\mathrm{Im}k/\mathrm{Re}k > -\epsilon]$ if $e^{i\lambda F(u)}$ is analytic in the region $|\mathrm{Im}u|/\mathrm{Re}u < \epsilon$.

This allows us to express $N(\lambda, \lambda')$, H, P, and E in terms of analytic mappings. We obtain

$$N(\lambda, \lambda') = \frac{1}{(\lambda\lambda')^{1/2}} \frac{1}{(2\pi i)^2} \int_0^{\infty}\int_0^{\infty} \frac{du\, du'}{(u' - u + i\epsilon)^2} \exp[\lambda' F(u' + i\epsilon) - \lambda F(u - i\epsilon)] , \qquad (3.16)$$

$$H(x', 0) = \frac{1}{(2\pi i)^2} \frac{1}{\pi} \int_0^{\infty}\int_0^{\infty} \frac{du\, du'}{(u' - u + i\epsilon)^2} \frac{1}{[F(u) - x' + i\epsilon][F(u') - x' - i\epsilon]}$$
$$+ \frac{1}{(2\pi i)^2} \frac{1}{\pi} \int_0^{\infty}\int_0^{\infty} \frac{du\, du'}{(u - u' + i\epsilon)^2} \frac{1}{[F(u) - x' - i\epsilon][F(u') - x' - i\epsilon]} , \qquad (3.17)$$

$$E = \frac{1}{(2\pi i)^2} \frac{1}{\pi} \int_0^{\infty}\int_0^{\infty} \frac{du\, du'}{(u - u' + i\epsilon)^2} \frac{1}{[F(u) - F(u') + i\epsilon]} . \qquad (3.18)$$

Conversely, given the quantum production rates we reconstruct the mapping, i.e.,

$$f(u') = f(u_0') \exp\left(-4\pi i \,\mathrm{Re} \int_0^{\infty} \frac{d\lambda}{\lambda} \exp\{i\lambda u' [(\lambda\lambda')^{1/2} N(\lambda, \lambda')]_{\lambda'=0}\}\right) , \qquad (3.19)$$

which follows from Eq. (3.20) below. Here $f(u_0')$ is an integration constant which is a scale factor of the transformation. It can be noted that the measure $du\, du'/(u - u' + i\epsilon)^2$ in the above integrals is invariant under bilinear transformations. Under such transformations, $F(u)$ becomes a new function defining a new accelerated frame, whereas $N(\lambda, \lambda')$ remains invariant. This means that all the accelerated frames defined on an orbit of the O(2, 1) group have the same vacuum spectra. Moreover, the following theorem is proven below.

Each one of the following statements implies the two others.

(i) The production function has the form

$$N(\lambda, \lambda') = N_\gamma(\lambda)\, \delta(\lambda - \lambda') .$$

(ii) The Bogoliubov transformation can be decomposed as a two-term one, i.e.,

$$C_\lambda = [1 + N_\gamma(\lambda)]^{1/2} \tilde{C}_{\lambda(+)} - [N_\gamma(\lambda)]^{1/2} \tilde{C}_{\lambda(-)} .$$

(iii) The analytic mapping is

$$F(u) = \frac{1}{2\pi T} \ln \beta u ,$$

where T, β are real constants.

Proof: Eq. (3.16) for $N(\lambda, \lambda')$ is valid for all λ, λ'. In particular for $\lambda = 0$ one gets

$$\int_0^{\infty} du \frac{e^{-i\lambda F(u - i\epsilon)}}{(u - i\epsilon)} = 4\pi^2 [(\lambda\lambda')^{1/2} N(\lambda, \lambda')]_{\lambda'=0} .$$

By Laplace transforming both sides of this equation, one obtains

$$\pi \frac{f'(u')}{f(u)} (-i) \int_{-\infty}^{\infty} d\mathcal{W}\, \mathcal{P}\left(\frac{1}{\mathcal{W} - u'}\right) \frac{f'(\mathcal{W})}{[f(\mathcal{W}) - i\epsilon]}$$
$$= 4\pi^2 \int_0^{\infty} d\lambda [(\lambda\lambda')^{1/2} N(\lambda, \lambda')]_{\lambda'=0}\, e^{i\lambda u'} ,$$

where $\mathcal{P}$ denotes principal value. By using the relation[16]

$$\int_{-\infty}^{\infty} \mathcal{P}\left(\frac{1}{\mathcal{W} - \gamma}\right) \phi(\mathcal{W}) d\mathcal{W} = \int_{-\infty}^{\infty} \frac{\phi(\gamma + \mathcal{W}) - \phi(\gamma - \mathcal{W})}{\mathcal{W}} d\mathcal{W} ,$$

one finds

$$\frac{d}{du'} [\ln f(u')] = 4\pi\, \mathrm{Re} \int_0^{\infty} d\lambda\, e^{i\lambda u'} [(\lambda\lambda')^{1/2} N(\lambda, \lambda')]_{\lambda=0} . \qquad (3.20)$$

If

$$N(\lambda,\lambda')=N_V(\lambda)\,\delta(\lambda-\lambda')\,, \tag{3.21}$$

then Eq. (3.20) gives

$$f(u')=\beta e^{2\pi T u'}\,, \tag{3.22}$$

i.e.,

$$F(u)=\frac{1}{2\pi T}\ln\frac{u}{\beta}\,, \tag{3.23}$$

where

$$T=[\lambda N_V(\lambda)]_{\lambda=0}$$

and β is an integration constant.

Then, if $N(\lambda,\lambda')$ has the form of Eq. (3.21), the only possible $F(u)$ (up a bilinear transformation) is that of Eq. (3.22). The converse is also true, as can be seen from Eq. (3.16). This proves the equivalence of statements (i) and (ii). It can be noted that as a consequence of the O(2, 1) invariance exhibited by the vacuum spectra, $F(u)$ admits a bilinear transformation such that

$$F(u)=\frac{1}{2\pi T}\ln\left(\frac{u-u_-}{u_+-u}\right) \tag{3.24}$$

satisfies the theorem. This generalizes the Rindler frame to the case of two event horizons.

The equivalence of statements (i) and (ii) follows from the relation

$$A_\lambda(k)=\left(\frac{1+N_V(\lambda)}{N_V(\lambda)}\right)^{1/2}B_\lambda(k)\,, \tag{3.25}$$

which is the necessary and sufficient condition for the Bogoliubov transformation being decomposable. Condition (3.25) allows us to define a basis

$$\tilde{C}_{\lambda(+)}=\int_0^\infty \frac{A_\lambda(k)}{[1+N_V(\lambda)]^{1/2}}\,a_k dk\,,$$

$$\tilde{C}_{\lambda(-)}=\int_0^\infty \frac{B_\lambda(k)^*}{[N_V(\lambda)]^{1/2}}\,a_k^\dagger dk\,,$$

such that

$$[\tilde{C}_{\lambda(+)},\tilde{C}_{\lambda'(+)}]=\int_0^\infty A_\lambda^*(k)A_{\lambda'}(k)dk$$
$$=[1+N_V(\lambda)]^{1/2}\delta(\lambda-\lambda')\,, \tag{3.26}$$

$$[\tilde{C}_{\lambda(-)},\tilde{C}_{\lambda'(-)}]=\int_0^\infty B_\lambda^*(k)B_{\lambda'}(k)dk$$
$$=[N_V(\lambda)]^{1/2}\delta(\lambda-\lambda')\,, \tag{3.27}$$

$$[\tilde{C}_{\lambda(+)},\tilde{C}_{\lambda'(-)}]=0\,. \tag{3.28}$$

We see that Eq. (3.27) gives statement (i). This completes the proof of the theorem. Its physical counterpart is that the vacuum energy density is homogeneous [it does not depend on (x',t')]. At each point of the space-time particles traveling to the right and to the left are created in equal total numbers [i.e., $P(x',t')=0$ for all x',t']. It corresponds to a thermal equilibrium situation over the whole space-time.

A consequence of the theorem is that if $N(\lambda,\lambda')$ satisfies statement (i), then $N_V(\lambda)$ is given by

$$N_V(\lambda)=\frac{1}{(e^{\lambda/T}-1)}\,. \tag{3.29}$$

The converse is not true. The presence of logarithmic singularities in $F(u)$ produces Planckian-type spectra in $N_V(\lambda)$ even if $F(u)$ is not a pure logarithmic function. There is a large class of accelerated observers for which $N_V(\lambda)$ is given by Eq. (3.29) but whose $H(x',t')$ is not homogeneous over the whole space-time.[8,9] For these observers, the acceleration is not uniform, but for $x'\to\pm\infty$ the acceleration becomes uniform. Asymptotically, a thermal equilibrium situation is reached.

From Eq. (3.29) we see that the parameter T defined by Eq. (3.23) plays the role of a temperature. The noninertial frames constructed from our procedure entail two characteristic parameters $T_\pm$. In the vacuum spectra of "particle production" these parameters play the role of temperatures. In the limit $u'\to\pm\infty$, the integral (3.20) is dominated by those λ in a neighborhood of $\lambda\to 0$. According to Eq. (3.20) and as a generalization of Eq. (3.23), we give the following definition of temperature associated with each boundary $u'=\pm\infty$:

$$T_\pm=\frac{1}{2\pi}\frac{d}{du'}[\ln f(u')]_{u'=\pm\infty}\,. \tag{3.30}$$

Each asymptotic region $u'=-\infty$ $(+\infty)$, has associated a temperature. If $u'=+\infty$ $(-\infty)$ is a critical point of $f(u')$, T_+ (T_-) is the temperature of that horizon. Otherwise, T_+ (T_-) is the temperature of the infinity. In the Rindler case, $T_+=T_-$. The horizon and infinity have equal temperatures because the behavior of $f(u')$ in such regions is of the same type.

In terms of the inverse function, $T_\pm$ reads

$$T_\pm=\pm\frac{1}{2\pi}[(u_\pm-u)F'(u)]^{-1}_{u=u_\pm}\,. \tag{3.31}$$

If $u_\pm=\pm\infty$, then

$$T_\pm=\pm\frac{1}{2\pi}[uF'(u)]^{-1}_{u=\pm\infty}\,. \tag{3.32}$$

In terms of the production function, $T_\pm$ are given by

$$T_\pm=\lim_{u'\to\pm\infty}\int_0^\infty d\lambda\,\cos\lambda u'[(\lambda\lambda')^{1/2}N(\lambda,\lambda')]_{\lambda'=0}\,. \tag{3.33}$$

In terms of the asymptotic acceleration of the noninertial observers, $T_\pm$ are

$$T_\pm = \frac{1}{2\pi}(\sqrt{\Lambda}a)_{u'=\pm\infty} .$$

All these results allow us to study the quantum spectra themselves in terms of the analytic properties of the mappings. The main conclusions of this analysis are summarized in Table I. Explicit examples and detailed calculations are given in Ref. 9.

From the results given in Table I we distinguish three basically different types of vacuum spectra:

(i) $N_V(\lambda) = \frac{1}{2}\left(\frac{1}{e^{\lambda/T_-}-1} + \frac{1}{e^{\lambda/T_+}-1}\right)$;

(ii) $N_V(\lambda)=0$, in this case $N(\lambda)$ is finite and nonzero and it is nonthermal;

(iii) $N_V(\lambda)=\infty$.

Each one of these spectra $N_V(\lambda)$ characterizes a class of accelerated frames having the same asymptotic properties and temperatures. $N_V(\lambda)$ reflects the asymptotic properties of the acceleration but not their detailed behavior. This is taken into account by $N(\lambda,\lambda')$ and the local densities H and P. The class (i) corresponds to accelerated frames with logarithmic singularities for both asymptotic regions. For the type (ii), both singularities are of the power or essential type. (iii) corresponds to log-log (or weaker) singularities. Each one of these three classes [even the class (i) when $T_+=T_-$] involves nonuniform accelerations and one, two, or no event horizons. The Rindler frame belongs to the class (i).

By combining different types of singularities one can construct accelerated frames for which T_+ is finite but $T_-=0$ and vice versa (see Ref. 9).

IV. PHYSICAL INTERPRETATION

To begin with we recall that we apply in the accelerated frames the laws of quantum mechanics as they are formulated in the inertial frames. This is a basic assumption for the entire subject.

If one considers a classical detector, then the eigenvalues of the operators and the mean values obtained in our formalism are the detected magnitudes.

If one considers a quantum detector following a general accelerated motion, as considered here, it must necessarily be a relativistic detector. In particular, if the acceleration is uniform, the nonrelativistic approximation can be used to describe the detector.[2] The general relativistic detector can be constructed by two quantum detector fields φ_D and ϕ_D of masses μ and M, respectively, interacting with the field Ψ to be detected through a small coupling $\epsilon\varphi_D\phi_D\Psi$. The transition $\phi_D \to \varphi_D$ shows that the Ψ quanta have been detected. To obtain this transition rate we need to generalize our formalism to interacting fields. For this generalization, see Refs. 13 and 14. For nonuniform acceleration (and massive detector fields), besides the vacuum effects due to $|0\rangle \neq |0'\rangle$, there are further contributions because $|0'^{\text{in}}\rangle \neq |0'^{\text{out}}\rangle$. These two effects interfere coherently. A detailed calculation to be reported in another paper[17] shows that the transition rate of the detector is entirely determined by the contribution of the asymptotic regions of the accelerated trajectories. We consider a measurement in the past (future) asymptotic region $[-\infty, t_0']$ ($[t_0', +\infty]$) where the characteristic time of the interaction $\ll t_0' \ll$ the characteristic time of the variation of the acceleration.

By using the in (out) formalism to first order in ϵ we obtain[17]

$$p_D = \frac{16}{\pi}\epsilon^2 N_V(\lambda_1-\lambda_2)\sigma_D(\lambda_1,\lambda_2) , \tag{4.1}$$

i.e., the $[p_D]$ transition rate is proportional to $N_V(\lambda)$ multiplied by the form factor $[\sigma_D]$ of the detector. If the detector field is massless, then $|0'^{\text{in}}\rangle = |0'^{\text{out}}\rangle$ and Eq. (4.1) holds even for a detection from $t'=-\infty$ to $t'=+\infty$.

Finally, it can be pointed out that because of the Heisenberg principle in quantum mechanics, momentum detectors must be spatially extended. To measure a momentum k with an uncertainty Δk, the detector must have a size $\gtrsim \hbar/\Delta k$. That is, it is an extended detector. Their different parts will have different world lines and hence different accelerations. In particular, in our formalism, the acceleration is given by Eq. (2.5).

V. COMPARISON WITH PRECEDING WORKS

Several authors [see, for example, Fulling and Davies (FD)] have discussed the problem of particle production by accelerated observers (or by black holes) in connection with quantum radiation by moving mirrors. However, there are essential differences between this approach and ours.

In their mirror approach, FD consider a transformation like Eq. (2.1) to relate the trajectory of a stationary mirror to that of an accelerated one. FD did not consider the boundary conditions on that transformation in order to construct accelerated frames suitable to formulate quantum field theory. Instead, they imposed total reflection boundary conditions on the field ($\hat{\Psi}=0$ on the mirror). This leads them to a conceptually different problem from quantum field theory in an accelerated frame. The corresponding basis of positive-frequency solutions are different. The FD Bogoliubov coefficients take into account the presence of the mirror. The vacuum expectation value $\langle T\rangle$ of the energy-momentum tensor they calculate is a Casimir-type effect, which is essentially dif-

TABLE I. Given an analytic real function $F(u)$, monotonic in the interval $[u_-, u_+]$, such that $F(u_\pm)=\pm\infty$, it determines particle production as follows.

Real singularities at $u=u_\pm$ determine:	temperatures $T_\pm$ (asymptotic values of acceleration)	$B_{\lambda k}$ $(k\to 0)$ $B_{\lambda k}$ $(k\to\infty)$	$N_V(\lambda)$	$\lim_{x'\to\pm\infty}\begin{pmatrix}H(x',t')\\P(x',t')\end{pmatrix}$	boundness or unboundness of $N(\lambda)$ and E
Complex singularities and detailed behavior of $F(u)$ determine:	acceleration $a(x',t')$ $\mathbf{V}(x',t')$	$B_{\lambda k}, A_{\lambda k}$ $\mathbf{V}(\lambda, k)$	$N(\lambda,\lambda')$ $\mathbf{V}(\lambda,\lambda')$	$\mathbf{V}(x',t')\begin{pmatrix}H(x',t')\\P(x',t')\end{pmatrix}$	E (if it is bounded)
Power and/or essential singularities at $u=u_-$ $(u=u_+)$ determine:	zero temperature T_+ (T_-)	an exponentially decreasing contribution to $B_{\lambda k}$ $(k\to\infty)$ and a contribution $O(\sqrt{k})$ to $B_{\lambda k}$ $(k\to 0)$	zero contribution to $N_V(\lambda)$	zero contribution to $\lim_{x'\to\pm\infty}\begin{pmatrix}H(x',t')\\P(x',t')\end{pmatrix}$	finite contribution to $N(\lambda)$ and E
Each logarithmic singularity at $u=u_+$ $(u=u_-)$ determines:	finite temperature T_+ (T_-)	a contribution $-\frac{\sqrt{\lambda}}{(2\pi)^2 T_\pm}e^{-\lambda/4T_\pm}\Gamma\left(-\frac{i}{2}\frac{\lambda}{\pi T_\pm}\right)\frac{k^{-i\lambda/(2\pi T\pm)}}{\sqrt{k}}$ to $B_{\lambda k}$ $(k\to\infty)$ and a contribution $O(\sqrt{k})$ to $B_{\lambda k}$ $(k\to 0)$	a contribution $\frac{1}{2}\frac{1}{(e^{\lambda/T}-1)}$ to $N_V(\lambda)$	a contribution $\frac{1}{6}(T_\pm)^2$ to $\lim_{x'\to\infty} H(x',t')$ and zero contribution to $\lim_{x'\to\pm\infty} P(x',t')$	an infinite contribution proportional to $V=2\pi\delta(0)$ to $N(\lambda)$ and E
		(where + stands for the contribution of $u=u_+$ and − for that of $u=u_-$). If $u_+=+\infty$, it contributes to $B_{\lambda k}$ $(k\to 0)$ with $-\frac{\sqrt{\lambda}}{(2\pi)^2 T_+}e^{-\lambda/4T_+}\Gamma\left(-\frac{i\lambda}{2\pi T_+}\right)\frac{k^{-i\lambda/2\pi T_+}}{\sqrt{k}}$. The other contributions remain the same.			
Each log-log singularity at $u=u_+$ $(u=u_-)$ determines:	infinite temperature T_+ (T_-)	a contribution $\frac{1}{(\lambda k)^{1/2}}(\ln k)^{-i\lambda}$ to $B_{\lambda k}$ $(k\to 0)$ and a contribution $O(\sqrt{k})$ to $B_{\lambda k}$ $(k\to 0)$ If $u_+=+\infty$, it contributes to $B_{\lambda k}$ $(k\to 0)$ with $\frac{1}{(\lambda k)^{1/2}}(\ln k)^{-i\lambda}$. The other contributions remain the same.	$N_V(\lambda=\infty$	infinite $\begin{pmatrix}H(x',t')\\P(x',t')\end{pmatrix}$	an infinite contribution proportional to $\frac{1}{4\pi^2}\ln(\Lambda_c)$ to $N(\lambda)$, where Λ_c is a frequency cutoff
The weaker the singularity is at u_- (u_+), present in $F(u)$:	The higher is its temperature and the larger is its contribution to the creation of particles.				

ferent from the acceleration effects considered by us. Only in the Rindler case the FD procedure gives the same result as our approach. One can compare the $\langle T\rangle$ of FD and our energy density H [Eq. (3.17)]. In our notation, the FD formula reads

$$\langle T\rangle_{\mathrm{FD}}=\frac{1}{12\pi}f'(u')^{1/2}[f'(u')^{-1/2}]'' ,$$

i.e.,

$$\langle T\rangle_{\mathrm{FD}}=\frac{1}{12\pi F'(u')^3}\left[F'''(u)-\frac{3}{2}\frac{F''(u)}{F'(u)}\right].$$

Then, one can take, for example, any of the $F(u)$ I consider in Ref. 9 and compare the vacuum energy $H(x',t')$ given there with that calculated from the FD formula. For instance, if we take

$$F(u)=\alpha\ln u+\gamma u , \tag{5.1}$$

one gets

$$\langle T\rangle_{\mathrm{FD}}=\frac{1}{12\pi}\frac{\alpha}{(\alpha+\gamma u)^3}\left[2-\frac{3}{2}\frac{\alpha}{(\alpha+\gamma u)}\right],$$

i.e.,

$$\langle T\rangle_{\mathrm{FD}}\big|_{u\to 0}\sim\frac{1}{24\pi\alpha^2}-\frac{\gamma^2}{4\pi\alpha^4}u^2+O(u^3) ,$$

$$\langle T\rangle_{\mathrm{FD}}\big|_{u\to\infty}\sim\frac{\alpha}{6\pi\gamma^3}\frac{1}{u^3}-\frac{19\alpha^2}{24\pi\gamma^4}\frac{1}{u^4}+O\left(\frac{1}{u^5}\right).$$

Whereas in our problem we obtain

$$H(u)_{u\to 0}\sim\frac{1}{24\pi\alpha^2}+\frac{1}{4\pi^3\alpha}\frac{\ln|\ln u^{\alpha}|}{\ln u^{\alpha}}+O\left(\frac{\ln|\ln u^{\alpha}|}{(\ln u^{\alpha})^2}\right),$$

$$H(u)_{u\to\infty}\sim\frac{1}{4\pi^3\alpha}\frac{\ln|\ln u^{\alpha}|}{\ln u^{\alpha}}+O\left(\frac{\ln|\ln u^{\alpha}|}{(\ln u^{\alpha})^2}\right)$$

only for $u\to 0$ ($u'\to-\infty$), the FD formula reproduces the leading term of our result. This is so because for $u\to 0$, the mapping (5.1) becomes that of the Rindler case.

The reason why the FD procedure gives the same results in the Rindler case as in ours is the following: In the Rindler case the relevant distribution in frequencies is $N_V(\lambda)$ [$N(\lambda)$ is infinite]. As we show in Table I, $N_V(\lambda)$ is determined solely by the asymptotic behavior of $B_\lambda(k)$ for $k\to 0$ and $k\to\infty$. These are fixed by the asymptotic values of the accelerated coordinates, independent of the presence of any mirror. Moreover, in the Rindler case the asymptotic behavior of $B_\lambda(k)$ coincides with the exact one.

For any other accelerated motion, the particle production predicted by the FD approach is different from that obtained by quantum field theory in accelerated frames.

VI. EXTENSIONS OF OUR APPROACH

A. Forward light cone

In the presence of event horizons, the accelerated coordinates defined in Sec. II cover the right light cone (or a portion of it) of Minkowski space-time. By considering the analytic continuation ($t=t$, $x=iy$) we can define curvilinear coordinates in the forward light cone such that

$$t\pm x=f(t'\pm x') . \tag{6.1}$$

The quantization procedure described in Sec. III can be extended directly to the forward light cone.

B. Two different mappings

On the other hand, we can generalize our approach to the case when two analytic functions f and g are involved, i.e.,

$$x-t=f(x'-t') ,\qquad x+t=g(x'+t') , \tag{6.2}$$

which satisfy

$$u_\pm=f(\pm\infty) ,\qquad v_\pm=g(\pm\infty) . \tag{6.3}$$

The metric takes the form

$$ds^2=f'(x'-t')g'(x'+t')(dx'^2-dt'^2) .$$

For finite $u_\pm$, $v_\pm$, the coordinates defined by Eqs. (6.2) and (6.3) cover the region

$$u_-<|x-t|<u_+ ,$$

$$v_-<|x+t|<v_+$$

(a parallelogram) in Minkowski space-time. There can be horizons on u but no horizons on v and vice versa, in which cases the coordinates cover an infinite strip at a 45° angle with the x axis. If $u_\pm=\pm\infty$ and $v_\pm=\pm\infty$, there are no horizons.

Future and past boundaries at $u=u_-$ and $v=v_-$ are defined here by different types of singularities of f and g, respectively, and they have different temperatures. Analogously, for future and past boundaries at $v=v_+$ and $u=u_+$ (boundaries can be horizons or infinities), i.e.,

$$T_{u_\pm}=\frac{1}{2\pi}\frac{d}{du'}[\ln f(u')]_{u'=\pm\infty} , \tag{6.4}$$

$$T_{v_\pm}=\frac{1}{2\pi}\frac{d}{dv'}[\ln g(v')]_{v'=\pm\infty} . \tag{6.5}$$

The complete set of orthogonal positive-frequency solutions with respect to the time t' is

$$\vec{\phi}_\lambda(u) = \frac{1}{2(\pi\lambda)^{1/2}} e^{i\lambda F(u)},$$

$$\overleftarrow{\phi}_\lambda(u) = \frac{1}{2(\pi\lambda)^{1/2}} e^{-i\lambda G(v)}, \quad \lambda > 0 .$$

Here G stands for the inverse function of g and the arrows stand for the sense of the particle's motion.

The quantization procedure of Sec. III can be generalized for this case, by observing that now

$$\vec{\phi}_\lambda \neq \overleftarrow{\phi}^*_\lambda .$$

Then, there is one production function

$$\vec{\mathrm{N}}(\lambda, \lambda') = \int_0^\infty \vec{B}^*_\lambda(k)\vec{B}_{\lambda'}(k)dk$$

determined by the mapping F, and another one

$$\overleftarrow{\mathrm{N}}(\lambda, \lambda') = \int_0^\infty \overleftarrow{B}_\lambda(k)^*\overleftarrow{B}_{\lambda'}(k)dk$$

determined by the mapping G. These define two independent frequency distributions $\vec{\mathrm{N}}_\gamma(\lambda)$ and $\overleftarrow{\mathrm{N}}_\gamma(\lambda)$, respectively. The spatiotemporal distributions show essentially the same structure of Eqs. (3.10)–(3.13), i.e.,

$$H = \tfrac{1}{2}[\vec{\mathrm{H}}(x'-t',0) + \overleftarrow{\mathrm{H}}(x'+t',0)]$$

(analogously for P) and

$$E = \int_0^\infty d\lambda\, \lambda\vec{\mathrm{N}}(\lambda) + \int_0^\infty d\lambda\, \lambda\overleftarrow{\mathrm{N}}(\lambda) .$$

C. Massive and interacting fields

The above approach can be also generalized to massive and interacting fields. The Klein-Gordon equation in accelerated coordinates reads

$$[\Box'^2 + m^2 f'(x'+t')f'(x'-t')]\hat{\Psi}(x',t') = 0 . \qquad (6.6)$$

Asymptotically, $f'(u') \to 0$ for $t' \to -\infty\ (+\infty)$. This allows the definition of positive- and negative-frequency in (out) states and their associated operators $C_{\lambda\,\mathrm{in}}$, $C^\dagger_{\lambda\,\mathrm{in}}$ ($C_{\lambda\,\mathrm{out}}$, $C^\dagger_{\lambda\,\mathrm{out}}$). The quantization procedure follows from that of Sec. III. One has

$$C_{\lambda\,\mathrm{in}}|0'_{\mathrm{in}}\rangle = 0 \quad (C_{\lambda\,\mathrm{out}}|0'_{\mathrm{out}}\rangle = 0) .$$

In this case there is also a Bogoliubov transformation between the in and out accelerated states, because the second term of Eq. (6.6) depends in general on t'. Only in the Rindler case, $|0'_{\mathrm{in}}\rangle = |0'_{\mathrm{out}}\rangle$ up to a phase factor.

The interaction-picture formulation as it is known in the inertial frames holds formally in the accelerated ones. The differences come from the renormalization (see Refs. 13 and 14). In the presence of interactions the connected Green's function $\mathcal{G}(x_1, x_2)$, the production function $\mathfrak{N}(\lambda, \lambda')$, and the vacuum energy density $\mathcal{H}(x', t')$ are defined by

$$\mathcal{G}(x_1,x_2) = \left\langle 0 \left| \psi(x_1)\psi(x_2) \exp\left(i\int \mathcal{L}_I[\psi(y)]\,d^2y\right)\right| 0 \right\rangle, \qquad (6.7)$$

$$\mathfrak{N}(\lambda_1,\lambda_2) = \left\langle 0 \left| C^\dagger_{\lambda_1} C_{\lambda_2} \exp\left(i\int \mathcal{L}_I[\psi(y)]d^2y\right)\right| 0 \right\rangle, \qquad (6.8)$$

$$\mathcal{H}(u',v') = \left\langle 0 \left| \left[\frac{(\partial_{u'}\psi)^2 + (\partial_{v'}\psi)^2}{f'(u')f'(v')} + m^2\psi^2 + \frac{g}{4!}\psi^4\right] \times \exp\left(i\int \mathcal{L}_I[\psi(y)]d^2y\right)\right| 0 \right\rangle . \qquad (6.9)$$

The divergences appearing in the accelerated and inertial frames are of the same type and one can subtract the divergences in the sums over the accelerated modes ϕ_λ by following similar methods to those used in the inertial frame for the modes φ_k. In Refs. 13 and 14 we discuss the renormalization prescription and we calculate one-loop corrections to the magnitudes (6.7)–(6.9) for the case $\mathcal{L}_I(\psi) = g\Psi^4$. The conterterms required in the inertial and accelerated frames are of the same type. In the accelerated frame the tadpole graph gives a nonzero finite contribution, i.e., the inertial and accelerated Green's function are different. (We recall that in the absence of interactions, the inertial and accelerated Green's functions are the same.)

D. Four-dimensional curved space-time

Two-dimensional space-time treated above can be considered as embedded in a four-dimensional space-time $\mathfrak{M}$ with metric tensor g. If $\mathfrak{M}$ admits two Killing vector fields, such a two-dimensional manifold can be taken as the fiber of the manifold $\mathfrak{M}$ considered as an appropriate fiber bundle. Then, the maximal analytic extension of $\mathfrak{M}$ can be obtained from a mapping like Eq. (2.1) satisfying boundary conditions (2.8). It transforms from the coordinates (x',t') (in which g has the removable singularities) to the maximal coordinates (x,t). Examples of this situation are the Kerr-Newman (KN) and Taub-NUT (T-NUT) families of metrics with or without cosmological terms, as well as the de Sitter solution. For all these cases, the mapping defining the maximal extension is

$$u = f(u') = e^{u'/\alpha},$$

i.e.,

$$F(u) = \alpha \ln u .$$

For KN, $u = r_K \pm t_K$ are the Kruskal-type (r_K, t_K) coordinates, $u' = r^* \pm t$ are the Schwarzschild-type (r^*, t) coordinates, and

$$\alpha = 2\left(M + \frac{M^2}{(M^2 + l^2 + Q^2)^{1/2}}\right),$$

where M, l, Q are the characteristic parameters of this metric. For T-NUT, $u=\phi_K \pm t_K$, $u'=\psi \pm t^*$, and

$$\alpha = \frac{M(M+1)+2[L^2-M(M^2+L^2)^{1/2}]}{2L(M^2+L^2)^{1/2}}$$

(M, L being the parameters of this metric).

The theorem of Sec. III concerning the logarithmic mapping as well as the conclusions on logarithmic singularites given in Table I hold in these cases. Quantum and thermal aspects of particle production in these metrics can also be analyzed from the geometrical and topological properties of the mapping f. The differential winding number of the mapping gives the temperature of the vacuum spectrum. In this context the meaning of temperature is the following. If we call $\mathfrak{C}$ the classical trajectories of the accelerated observers for the imaginary time, the differential angle rotated by $\mathfrak{C}$ around the real singularities of the mapping measures (in units of 2π) the temperature carried by the singularity (see Ref. 10). For all the above metrics, $T_+ = T_- = 1/2\pi\alpha$.

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