Magnetic monopole in variations on Einstein's nonsymmetric unified field theory

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The Einstein-Kurşunoğlu and the Einstein-Bonnor unified-field-theory equations for the time-independent, spherically symmetric electric and magnetic fields are reduced to an ordinary integrodifferential equation of the type previously solved by numerical methods. It is found that specifying the mass and charge of the electric monopole along with using Dirac's value for the magnetic charge is sufficient to determine the mass of the magnetic monopole in the Einstein-Kurşunoğlu and the Einstein-Bonnor theories. The observed mass of the electron and the large (unobserved) mass of the magnetic monopole do not disagree with these unified field theories.

I. INTRODUCTION

In recent papers the authors have obtained solutions for two variations of Einstein's unified field theory.[1,2] In each case only the electric field term of the metric (g_{23}) was assumed to be present. This paper generalizes the above solutions to the case where both electric and magnetic fields are included. The nonsymmetric metric for this case has been written by Kurşunoğlu[3] as

$$g_{\mu\nu}=\begin{bmatrix} -\dfrac{e^{-u}}{v^2} & 0 & 0 & \dfrac{\tanh\Gamma}{v} \\ 0 & -e^{\rho}\sin\phi & e^{\rho}\cos\phi\sin\theta & 0 \\ 0 & -e^{\rho}\cos\phi\sin\theta & -e^{\rho}\sin\phi\sin^2\theta & 0 \\ -\dfrac{\tanh\Gamma}{v} & 0 & 0 & e^{u}\end{bmatrix}, \tag{1.1}$$

where e^{ρ} and θ are spherical radial and angular coordinates (the second angular coordinate does not appear). ϕ and Γ create the torsions which are related to the radial electric and magnetic fields.

Derivation of the appropriate field equations can be found in Kurşunoğlu's paper.[3] They are as follows:

$$v(ve^{u+\rho}\phi')'+v^2\phi'\rho' e^{u+\rho}\tanh^2\Gamma=X\,, \tag{1.2}$$

$$v(ve^{u+\rho}\rho')'+v^2\rho'^2e^{u+\rho}\tanh^2\Gamma=Y\,, \tag{1.3}$$

$$\rho''+\rho'\frac{v'}{v}+\tfrac{1}{2}(\rho'^2+\phi'^2)-\rho'^2\tanh^2\Gamma=0\,, \tag{1.4}$$

$$e^{\rho}\sinh\Gamma=\lambda^2\,, \tag{1.5}$$

where in the case of Kurşunoğlu's theory

$$X\equiv\kappa^2(e^{\rho}\cos\phi\cosh\Gamma-2Q\sin\phi)-2\cos\phi\,, \tag{1.6}$$

$$Y\equiv\kappa^2[e^{\rho}(1-\sin\phi\cosh\Gamma)-2Q\cos\phi]+2\sin\phi \tag{1.7}$$

and for Bonnor's theory[4]

$$X\equiv p^2(2e^{\rho}\sin\phi\cos\phi-4Q\sin\phi)-2\cos\phi\,, \tag{1.8}$$

$$Y\equiv p^2[e^{\rho}(\cos^2\phi-\sinh^2\Gamma)-4Q\cos\phi]+2\sin\phi\,. \tag{1.9}$$

Q is the constant of integration related to charge and λ^2 is the constant of integration related to the magnetic monopole. We have the five variables u, v, ϕ, Γ, and ρ to be determined by the four equations [(1.2)–(1.5)] plus the free choice of ρ as a function of r.

Kurşunoğlu includes one additional equation, but shows that as a result of the Bianchi relationships one of the equations is redundant. We have therefore omitted it. Using the method of Huerta and Parker[2] we will now proceed to reduce the equations.

II. ANALYSIS

Starting with Eq. (1.4) we write

$$v\rho''+\rho'v'+\frac{v}{2}[\rho'^2(1-2\tanh^2\Gamma)+\phi'^2]=0\,. \tag{2.1}$$

This can be rewritten as

$$(v\rho')'=-\frac{\rho'v}{2}\left(1-2\tanh^2\Gamma+\frac{\phi'^2}{\rho'^2}\right)\rho'\,. \tag{2.2}$$

Integrating and solving for v then gives

$$v=\frac{A}{\rho'}e^{-\gamma}\,, \tag{2.3}$$

where the constant of integration is chosen so that $v\to1$ as $\rho\to\infty$ ($\Gamma\to0$, $\phi'\to0$):

$$\gamma=\tfrac{1}{2}\int_0^r\left[1-2\tanh^2\Gamma+\left(\frac{\phi'}{\rho'}\right)^2\right]\rho'dr\,. \tag{2.4}$$

We now proceed to formally integrate Eqs. (1.2) and (1.3). Writing (1.2) as

$$(ve^{u+\rho}\phi')'+ve^{u+\rho}\phi'(\rho'\tanh^2\Gamma)=\frac{X}{v}$$

and using an integrating factor we obtain

$$(ve^{u+\rho}\phi' e^{\Theta})' = \frac{e^{\Theta}}{v} X . \tag{2.5}$$

In a similar manner Eq. (1.3) becomes

$$(ve^{u+\rho}\rho' e^{\Theta})' = \frac{e^{\Theta}}{v} Y , \tag{2.6}$$

where $\Theta' \equiv \rho' \tanh^2\Gamma$.

The variable u is eliminated by formally integrating (2.5) and (2.6) then forming the ratio

$$\frac{\phi'}{\rho'} = \frac{d\phi}{d\rho} = \frac{\int_0^r \frac{e^{\Theta}}{v} X\,dr + B}{\int_0^r \frac{e^{\Theta}}{v} Y\,dr + C} . \tag{2.7}$$

Finally, Eq. (2.3) is used to eliminate v and Eq. (1.5) to eliminate Γ which leaves an ordinary integrodifferential equation in ϕ as a function of ρ:

$$\frac{d\phi}{d\rho} = \frac{\int_{\rho_0}^{\rho} e^{\Theta+\gamma} X\,d\rho + B}{\int_{\rho_0}^{\rho} e^{\Theta+\gamma} Y\,d\rho + C} . \tag{2.8}$$

The term in the exponential simplifies to the form

$$\Theta + \gamma = \tfrac{1}{2} \int_{\rho_0}^{\rho} \left[1 + \left(\frac{d\phi}{d\rho}\right)^2\right] d\rho . \tag{2.9}$$

The integration constants B and C depend upon the choice of the lower limit of integration. Equation (2.8) represents the formal solution when ϕ is not identically zero. When the charge (Q) is chosen $\neq 0$ and the constants B and C are specified, Eq. (2.8) may be integrated numerically. When Q is zero only the magnetic field remains, Eq. (2.8) vanishes and the solutions found by Kurşunoğlu[5] and Pant[4] are obtained. If λ is chosen to be zero, Eq. (2.8) reduces to the pure charge problem treated in the nonsingular case by Huerta and Parker[6] for $B = C = 0$. If the lower limit is chosen such that the integral can cover "all space," then setting B and C equal to zero is consistent with defining the mass as the volume integral of the generalized field energy density defined in the next section.

III. ENERGY DENSITY

We now wish to find the term corresponding to the field energy density and see what happens when a nonzero λ is introduced. We start from Kurşunoğlu's form of the equation for R_{22}:

$$\tfrac{1}{2} v [ve^{u+\rho}(\rho' \sin\phi - \phi' \cos\phi)]' - \tfrac{1}{2}\phi' v^2 e^{u+\rho}(\phi' \sin\phi + \rho' \cos\phi) - \tfrac{1}{2} v^2 e^{u+\rho}(\rho' \sin\phi - \phi' \cos\phi)\Gamma' \tanh\Gamma = 1 + \frac{\kappa^2}{2} e^{\rho}(\sin\phi - \cosh\Gamma) . \tag{3.1}$$

Introducing the integrating factor e^{Λ}, where

$$\Lambda' = -\frac{\phi' \sin\phi + \rho' \cos\phi}{\rho' \sin\phi - \phi' \cos\phi}\, \phi' , \tag{3.2}$$

gives

$$\left(\frac{ve^{u+\rho}(\rho' \sin\phi - \phi' \cos\phi) e^{\Lambda}}{\cosh\Gamma}\right)' = \frac{2e^{\Lambda}}{v \cosh\Gamma}\left(1 - \frac{\kappa^2}{2} e^{\rho}(\cosh\Gamma - \sin\phi)\right) . \tag{3.3}$$

Integrating, solving for e^u, and splitting the integral into three parts gives

$$e^u = \frac{\cosh\Gamma e^{-\Lambda-\rho}}{v(\rho' \sin\phi - \phi' \cos\phi)} \left(2 \int_0^r \frac{e^{\Lambda}}{v \cosh\Gamma} dr + \kappa^2 \int_0^{\infty} \frac{e^{\Lambda+\rho}}{v \cosh\Gamma} (\sin\phi - \cosh\Gamma) dr - \kappa^2 \int_r^{\infty} \frac{e^{\Lambda+\rho}}{v \cosh\Gamma} (\sin\phi - \cosh\Gamma) dr + c\right) . \tag{3.4}$$

First, examine the case of vanishing electric and magnetic fields specified by the conditions $\phi = \pi/2$, $\Gamma = 0$, $r^2 \equiv e^{\rho}$. We get

$$v_0 = 2\frac{e^{-\rho/2}}{\rho'} = 1 , \quad \cosh\Gamma = 1, \quad \sin\phi = 1, \quad \Lambda = 0$$

and Eq. (3.4) becomes

$$e^{u_0} = 1 + \frac{c}{2r} . \tag{3.5}$$

Note that when we set the integration constant $c = -4M$, where M is the point mass of the Schwarzschild solution, our solution, for the case of vanishing electromagnetic fields, includes the Schwarzschild solution as an exact solution.

Second, let us examine the case of small electric and magnetic fields specified by the conditions $\sin\phi \sim 1 - \psi^2/2$, $\cos\phi \sim \psi$, $\cosh\Gamma \sim 1 + \Gamma^2/2$, $e^{\rho} \sim r^2$.

We find for large r

$$e^{\Lambda} \sim 1, \quad v \sim 1 , \tag{3.6}$$

and using these in (3.4) we get

$$e^{u} \sim 1+\frac{\kappa^2}{2r}\int_0^\infty \frac{e^{\Lambda+\rho}}{v\cosh\Gamma}(\sin\phi-\cosh\Gamma)dr - \frac{\kappa^2}{4r}\int_r^\infty (\psi^2+\Gamma^2)r^2 dr+\frac{c}{2r}. \quad (3.7)$$

The second integral is not approximated because we can only require that the $\cosh\Gamma$ term be ~1 outside a sphere around the origin. If we now assume the electromagnetic fields fall off as inverse squares far from the origin,

$$\psi \sim \frac{2}{\kappa^2}\psi_0/r^2, \quad \Gamma \sim \lambda^2/r^2. \quad (3.8)$$

ψ_0 is the constant of integration which determines the total electric charge as seen far from the origin, then Eq. (3.7) becomes the Reissner-Nordström solution

$$e^{u} \sim 1-\frac{2M}{r}+\frac{Q^2+P^2}{r^2}, \quad (3.9)$$

where

$$-2M \equiv \frac{c}{2}+\frac{\kappa^2}{2}\int_0^\infty \frac{e^{\Lambda+\rho}}{v\cosh\Gamma}(\sin\phi-\cosh\Gamma)dr \quad (3.10)$$

and

$$Q^2+P^2 \equiv \frac{\kappa^2}{4}\left(\frac{4\psi_0^{\,2}}{\kappa^4}+\lambda^4\right). \quad (3.11)$$

At this point we choose to pursue not the most general solution but the physically interesting solution where the particle has no Schwarzschild-type singularity. This is achieved by setting $c=0$. Equation (3.5) shows for $c\neq 0$ the solution retains the Schwarzschild singularity even for vanishing fields. If we choose $c=0$ the long-range mass is the volume integral of the generalized energy density $(H\sqrt{-g})$

$$\int H_K\sqrt{-g}\,d^3x \equiv \frac{\kappa^2}{4}\int_0^\infty \frac{e^{\Lambda+\rho}}{v\cosh\Gamma}(\cosh\Gamma-\sin\phi)dr, \quad (3.12)$$

where

$$\sqrt{-g}=\frac{e^{\rho}\sin\theta}{v\cosh\Gamma}.$$

In the case of Bonnor's theory this generalized energy can be shown to be

$$\int H_B\sqrt{-g}\,d^3x=\frac{p^2}{4}\int_0^\infty \frac{e^{\Lambda+\rho}}{v\cosh\Gamma}\sin\phi(\cos^2\phi+\sinh^2\Gamma)dr. \quad (3.13)$$

IV. MONOPOLE MASSES

We will assume that the mass of the electric and magnetic particles is given by the integral over "all space" of the generalized energy density [Eq. (3.12) or (3.13)]. Far from the origin this generalized energy density goes to

$$\frac{\kappa^2}{4}(\psi^2+\Gamma^2) \quad (4.1)$$

from Eq. (3.7). If we assume κ has the form $\kappa=|\kappa|e^{i\alpha}$, then ψ and Γ must have the forms $\psi=|\psi|e^{-i\alpha}$ and $\Gamma=|\Gamma|e^{-i\alpha}$ when we require the asymptotic contributions to the mass be real and positive.

We now choose to investigate the case where p, κ, ψ, and Γ are pure imaginary. Let $\kappa\equiv ik$, $p=iq$, $\Psi=i\psi$, and $G=i\Gamma$.

A. Electric monopole

When $\lambda^2=0$ and $Q\neq 0$ the integrodifferential equation (2.8) may be linearized[6] when Ψ is small to give

$$\frac{d\Psi}{dr}\sim\frac{2}{r^2}\int_0^r\left(\Psi-\Psi_0+\frac{k^2}{2}r^2\Psi\right)dr \quad (4.2)$$

for both the Bonnor and Kurşunoğlu forms of the theory when $2p^2=\kappa^2$. This has the solution

$$\Psi \sim 2(1-e^{-kr}-kre^{-kr})\Psi_0/(kr)^2, \quad (4.3)$$

where Ψ_0 is the value of Ψ at the origin. This solution has a finite rest mass since the singularity present in the Reissner-Nordström solution has been eliminated. The solution retains a nonzero derivative which forms a cusp about the origin.

In the weak-field approximation the mass of this electric monopole can be found from Eq. (3.12) or Eq. (3.13) to be

$$m \sim \frac{\Psi_0^{\,2}}{2k^2}\int_0^\infty (1-e^{-kr}-kre^{-kr})^2\frac{dr}{r^2} = \Psi_0^{\,2}/(4k). \quad (4.4)$$

For a charge of the order of that of the electron, this mass has been verified by numerical solution of the integrodifferential equation.[2] The constant in Eq. (3.10) has been chosen to require that the monopole mass be given by the integral of the generalized energy density over all space. The charge of the electric monopole is identified from Eq. (3.11) as

$$Q=\Psi_0/k. \quad (4.5)$$

From this we observe that the integration constant Ψ_0 and fundamental constant k are sufficient to specify a particle with arbitrary (nonzero) charge and a positive mass.

B. Magnetic monopole

1. Kurşunoğlu's theory

When $\Psi_0=0$ and $\lambda^2 \equiv i\delta^2 \neq 0$ we find the linearization fails since $i\Gamma \equiv G$ is not bounded near $r=0$. In this case ($\Psi_0=0$) Eq. (3.10) may be integrated exactly and we find for Kurşunoğlu's theory

$$M = -\frac{k^2}{8}\int_0^\infty e^{3\rho/2}[(1-\delta^4 e^{-2\rho})^{1/2}-1]\rho'\,dr$$

$$= \frac{k^2\delta^3}{12}\left[\sqrt{2}K\left(\frac{\pi}{2}\right)-1\right], \quad (4.6)$$

where $K(\pi/4)$ is the complete elliptic integral[7,8] and we have used

$$\cosh^2\Gamma = 1-\delta^4 e^{-2\rho} \quad (4.7)$$

and

$$e^{\rho/2} = (r+1)\delta. \quad (4.8)$$

The choice of form for the variable r is arbitrary but we have chosen a lower limit ($r=0$) where the point has a finite surface area. We, in agreement with Kunstätter and Moffat,[9] exclude the interior of this surface from the physical space. If the mass integral is extended to include the contents of this surface, the mass becomes complex since g changes sign at $r=0$.

2. Bonnor's theory

Setting the electric field to zero in Eq. (3.13) leads to

$$M_B = \frac{p^2}{4}\int_0^\infty \frac{e^\rho}{v\cosh\Gamma}\sinh^2\Gamma\,dr. \quad (4.9)$$

Using Eq. (1.5) and $e^{\rho/2}=(r+1)\delta$ this simplifies to

$$M_B = \frac{q^2}{4}\delta^3\int_0^\infty \frac{dr}{(r+1)^2} = \frac{q^2}{4}\delta^3. \quad (4.10)$$

The point at $r=0$ has finite surface area as in Kunşunoğlu's theory but if we extend the integration to include the "space" inside this area the mass integral for Bonnor's magnetic-monopole diverges.

C. Numerical values

In a world with electric and magnetic monopoles, we have four values and only the universal constant (k or $\sqrt{2}q$) along with two integration constants (Ψ_0 and δ) to fix them. If we ignore the question of applicability and limit the choice by introducing Dirac's[10] formula relating the charges on the monopoles

$$PQ = n\hbar c/2, \quad n = \text{integer}, \quad (4.11)$$

we may calculate the minimum mass on the magnetic monopole when we use the charge and mass of the electron, set $n=1$ and use gravitational units[11]:

$$m = \Psi_0^2/(4k) = m_e = 6.764\times 10^{-56}\text{ cm},$$

$$Q = \Psi_0/k = e = 1.381\times 10^{-34}\text{ cm},$$

then $k = 4m_e/e^2$ and using Eq. (4.11) we obtain

$$P = \frac{k}{2}\delta^2 = \hbar c/(2e) \equiv \alpha^{-1}\frac{e}{2}, \quad \delta^2 = \frac{\alpha^{-1}e^3}{4m_e},$$

$$M_K = \alpha^{-3/2}\sqrt{m_e e}\,\frac{1}{6}\left[\sqrt{2}K\left(\frac{\pi}{4}\right)-1\right] = 1.32\times 10^{-42}\text{ cm},$$

$$M_B = \frac{q^2}{4}\delta^3 = \frac{k^2}{8}\delta^3 = \frac{1}{4}\alpha^{-3/2}\sqrt{m_e e} = 1.22\times 10^{-42}\text{ cm}.$$

If we attempt to reverse the identification of the electric and magnetic fields as has been suggested by Kurşunoğlu and by Moffat, we find

$$M = \frac{k^2\delta^3}{12}\left[\sqrt{2}K\left(\frac{\pi}{4}\right)-1\right] = 0.135k^2\delta^3 = m_e,$$

$$P = \frac{k}{2}\delta^2 = e, \quad k = 6.86 m_e^2/e^3,$$

$$Q = \Psi_0/k = \alpha^{-1}\frac{e}{2}, \quad \Psi_0 = 3.43\left(\frac{m_e}{e}\right)^2\alpha^{-1},$$

$$m = \frac{\Psi_0^2}{4k} = 8047\frac{m_e^2}{e} = 2.66\times 10^{-73}\text{ cm}.$$

V. CONCLUSIONS

Both the Einstein-Bonnor and the Einstein-Kurşunoğlu theories provide a classical theory of electromagnetism and gravitation that is in asymptotic agreement with Maxwell's equations.[12,3] They are also in agreement with general relativity and predict electric and magnetic monopoles with finite self-energies. When Dirac's theory of the magnetic monopole is used to determine the magnetic charge, the mass of the magnetic monopole is extremely high, well beyond the capabilities of present accelerators.

If one attempts to interchange the roles of the electric and magnetic fields as suggested by Kurşunoğlu and by Moffat one finds a magnetic monopole mass much less than the mass of the electron. We reject this interchange since the resulting light monopole has not been observed.

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[12]J. W. Moffat, Phys. Rev. D 15, 3520 (1977). The association becomes exact in the limit $\kappa \to 0$ when the $F^{\eta\nu} \equiv (1/2\kappa)\epsilon^{\eta\nu\alpha\beta} g_{[\alpha\beta]}$ are held constant. $F^{\eta\nu}$ is the electromagnetic field tensor and $\epsilon^{\eta\nu\alpha\beta}$ is the Levi-Civita pseudotensor. Moffat's choice of $F_{\eta\nu} = \kappa g_{[\eta\nu]}$ does not agree with the results in this paper.